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CORRECTION NOTE

CORRECTION TO "ON DISTINGUISHING TRANSLATES OF MEASURES"

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The proof of Theorem 3 of [1] contains a gap which we fill with the three Lemmas that follow. We also note that $f_N(x)$ should be changed to $f_N(x + \alpha m)$ on page 1776 of [1], line 21, where $f_N(x)$ is defined.

Our notation is the same as in [1]. The setting is, as before, a general real stochastic process $X = (X(t) \mid t \in T)$. We let S stand for the set of all real valued functions on T and we let $\mathscr{A}$ stand for the σ-field of subsets of S generated by coordinates. For a fixed element $m \in S$ and $\alpha \in R$, the real line, we let P_α stand for the measure induced on $(S, \mathscr{A})$ by the process $(X(t) + \alpha m(t) \mid t \in T)$.

We say that the measures P_α are *simultaneously Borel* distinguishable if there exists an $\mathscr{A}$ measurable function f such that

$$P_\alpha[f = \alpha] = 1 \qquad \text{for all } \alpha \in R .$$

In our earlier paper [1] we called the parameter α "consistently" distinguishable in this case, a terminology which seems at variance with the usual terminology which reserves the term "consistently" for the convergence (a.s.) or in probability of "finitely defined" functionals to the true parameter α.

Lemma 1. *Let $(X_n \mid n \geqq 1)$ be a sequence of independent identically distributed random variables. Suppose $m = (m_n \mid n \geqq 1)$ is a sequence of real numbers such that $\sup_{n \geqq 1} |m_n| = \infty$. Then there exists a sequence f_k of $\mathscr{A}$ measurable linear functionals defined on S such that*

$$f_k(x + \alpha m) \to \alpha \quad \text{a.s.} \quad \forall \alpha .$$

Proof. Pick a subsequence n_k such that $|m_{n_k}| \to \infty$. For

$$x = (x_1, x_2, \cdots) \in S$$

define

$$f_k(x) = x_{n_k}/m_{n_k} .$$

Now

$$(X_{n_k} + \alpha m_{n_k})/m_{n_k} = \alpha + X_{n_k}/m_{n_k} .$$

Also it is clear that

$$X_{n_k}/m_{n_k} \to 0 \quad \text{a.s.} ,$$

taking a subsequence if necessary. It follows that

$$f_k(x + \alpha m) \to \alpha \quad \text{a.s.} \qquad \forall \alpha . \ \square$$

Lemma 2. *Let $(X_n \mid n \geqq 1)$ be a sequence of independent identically distributed* *random variables. Let* $m = (m_n \mid n \geqq 1)$ *be a sequence of real numbers such that* $\sum_1^\infty (m_n)^2 = \infty$. *Then for any compact interval* $[-a, a]$, *there exists a sequence* f_k *of* $\mathscr{A}$ *measurable functionals defined on S such that*

$$f_k(x + \alpha m) \to \alpha \quad \text{a.s.} \qquad \forall \alpha \in [-a, a].$$

PROOF. We assume that $\sup_n |m_n| = K < \infty$, otherwise Lemma 1 applies. Choose $M > 0$ such that $P[|X| < M] \geqq \frac{1}{2}$.

Let

$$\begin{aligned} h(s) &= M + aK && \text{if } s > M + aK \\ &= s && \text{if } |s| \leqq M + aK \\ &= -M - aK && \text{if } s < -M - aK. \end{aligned}$$

Let $b = E(h(X_1))$ and let $c_n^\alpha = E(h(X_n + \alpha m_n))$ for $n \geqq 1$. We can assume that $m_n \geqq 0$ $\forall n$, by the reasoning in [1]. Finally we correct the proof of Theorem 3 in [1] by noting that $0 \leqq \frac{1}{2} m_n \alpha \leqq (c_n^\alpha - b)$ is valid for $\alpha \in [0, a]$. (This inequality is asserted for $\alpha \geqq 0$ in [1] which is false.) For $\alpha \in [-a, 0]$ we have $(c_n^\alpha - b) \leqq \frac{1}{2} m_n \alpha$.

With these restrictions on α, the rest of the inequalities in [1] are valid, and we can finish our proof by following the reasoning in [1]. □

LEMMA 3. *If for every compact interval* $[-\theta, \theta]$ *there exists an* $\mathscr{A}$ *measurable function* b_θ *such that*

$$P_\alpha[b_\theta = \alpha] = 1 \qquad \textit{for} \quad \alpha \in [-\theta, \theta]$$

then the measures P_α *are simultaneously Borel distinguishable.*

PROOF. In this proof θ and k will be positive integers. For $x \in S$, define

$$f(x) = \lim_{\theta \to \infty} b_\theta(x)$$

if this limit exists. If the limit does not exist, define $f(x) = 0$. Clearly f is $\mathscr{A}$ measurable.

We need to check that

$$P_\alpha[f = \alpha] = 1 \qquad \text{for all} \quad \alpha \in R.$$

However for fixed α and $k > \alpha$ we have

$$P_\alpha[b_\theta = f_k = \alpha] = 1 \qquad \text{for all} \quad \theta \geqq k.$$

We conclude that

$$P_\alpha[\lim_{\theta \to \infty} b_\theta = \alpha] = 1. \qquad \square$$

THEOREM. *Let* $X = (X_n \mid n \geqq 1)$ *and* $m = (m_n \mid n \geqq 1)$ *be as in Lemma* 2. *Then the measures* P_α *are simultaneously Borel distinguishable.*

PROOF. Follows from Lemmas 2 and 3. □

REFERENCE

[1] KANTER, M. (1969). On distinguishing translates of measures. *Ann. Math. Statist.* **49** 1773-1777.