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THE HAUSDORFF DIMENSIONS OF THE GRAPH AND RANGE OF N-PARAMETER BROWNIAN MOTION IN d-SPACE

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The paper extends results of Lévy and Taylor from one-parameter Brownian motion to the multiparameter case. The following theorem is proved. For N-parameter Brownian motion in d-space, the Hausdorff dimensions of the graph and range are a.s. $\min\{2N, N + d/2\}$ and $\min\{2N, d\}$ respectively.

In this paper results of P. Lévy and S. J. Taylor in one-parameter Brownian motion are generalized to the multiparameter case. A needed extension of a result of A. S. Besicovitch and H. D. Ursell is also obtained.

Besicovitch and Ursell [1] showed that the Hausdorff dimension D of the graph of a curve belonging to the Lipschitz δ-class satisfies the inequalities $1 \leqq D \leqq 2 - \delta$, and that all D in this closed interval are possible. Taylor found that the Hausdorff dimension of the graph of 1-parameter Brownian motion is $\frac{3}{2}$ with probability one [6], and that the path of Brownian motion in d-dimensional Euclidean space, $d \geqq 2$, has Hausdorff dimension 2 with probability one [5]. (See also Lévy [3].)

Let $W^{(N,d)}$ denote Lévy's N-parameter Brownian motion with values in d-dimensional Euclidean space; i.e. if $X = W^{(N,d)}$, then $X(t, \omega) = (X_1(t, \omega), \cdots, X_d(t, \omega)) \in \mathbb{R}^d$, where $t = (t_1, \cdots, t_N) \in \mathbb{R}^N$ and the coordinate functions X_i are mutually independent, separable, Gaussian processes with mean zero and covariance

$$E(X_i(s), X_i(t)) = \tfrac{1}{2}[|s| + |t| - |s - t|] .$$

Here $|\cdot|$ is the Euclidean norm.

Orey and Pruitt [4], with a different definition of multiparameter Brownian motion, have obtained, among other results, a fact which is somewhat weaker than our result on the dimension of the range.

THEOREM. *The Hausdorff dimensions of the graph and range of* $W^{(N,d)}$ *are almost surely* $\min\{2N, N + d/2\}$ *and* $\min\{2N, d\}$, *respectively.*

PROOF. The proof is done in three parts.

I. Let $X\colon \mathbb{R}^N \to \mathbb{R}^d$ belong to the Lipschitz δ-class (Lip δ). Then

$$\dim_H(\text{ra } X) \leqq \dim_H(\text{gr } X) \leqq \min\left\{\frac{N}{\delta}, N + (1 - \delta)d\right\},$$

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where $\dim_H(\text{ra } X)$ and $\dim_H(\text{gr } X)$ denote the Hausdorff dimensions of the range and graph of X, respectively. (Since $W^{(N,d)}$ is almost surely in Lip δ for every $\delta < \frac{1}{2}$ [2], we obtain part of the theorem.)

For the proof of I, it is sufficient to consider only the unit cube U^N in $\mathbb{R}^N$ as the domain of x. Divide the domain into h-cubes (cubes with edge of length h). The graph of each h-cube is contained in a set of diameter $K_1 h^\delta$. These sets form a covering of the graph, and the estimate of the N/δ measure of the graph is

$$h^{-N}(K_1 h^\delta)^{N/\delta} = K_2 < \infty .$$

Thus $\dim_H(\text{gr } X) \leqq N/\delta$.

Again divide the domain into h-cubes. The image of each h-cube is contained in a $K_1 h^\delta$-cube. Divide each of these $K_1 h^\delta$-cubes into h-cubes, and we have a covering of the graph by h-cubes. The estimate of the $N + (1 - \delta)d$ measure is

$$h^{-N}[K_1 h^{\delta-1}]^d[(N + d)^{\frac{1}{2}}h]^{N+(1-\delta)d} = K_2 > \infty .$$

So

$$\dim_H(\text{gr } X) \leqq N + (1 - \delta)d .$$

II. Let $X = W^{(N,d)}$, $2N \leqq d$. Then

$$\dim_H(\text{gr } X) \geqq \dim_H(\text{ra } X) \geqq 2N$$

with probability one. Let $\alpha < 2N$. It is sufficient to show that the α capacity of the range of X, $C_\alpha(\text{ra } X)$, is positive. For this it is sufficient to show

$$\int_{U^N} \int_{U^N} |X(s, \omega) - X(t, \omega)|^{-\alpha}\, ds\, dt < \infty ,$$

where U^N is again the unit cube in $\mathbb{R}^N$. Now

$$\int_\Omega |X(t, \omega)|^{-\alpha}\, d\omega = (2\pi|t|)^{-d/2} \int_{\mathbb{R}^d} |u|^{-\alpha} \exp - \left(\frac{u_1^2 + \cdots + u_d^2}{2|t|}\right) du .$$

By changing to spherical coordinates and then letting $r = |t|^{\frac{1}{2}}x$, this becomes

$$K_1|t|^{-d/2} \int_0^\infty r^{d-1-\alpha} \exp(-r^2/2|t|)\, dr = K_1|t|^{-\alpha/2} \int_0^\infty x^{d-1-\alpha}e^{-x^2/2}\, dx .$$

The integral is finite since $\alpha < d$, so

$$\int_\Omega |X(t, \omega)|^{-\alpha}\, d\omega < K|t|^{-\alpha/2} .$$

We have

$$\int_{U^N} \int_{U^N} ds\, dt \int_\Omega |X(s, \omega) - X(t, \omega)|^{-\alpha}\, d\omega \leqq \int_{U^N} \int_{U^N} K|s - t|^{-\alpha/2}\, ds\, dt$$

which is finite since $\alpha/2 < N$. Using Fubini's theorem we conclude that

$$\int_{U^N} \int_{U^N} |X(s, \omega) - X(t, \omega)|^{-\alpha}\, ds\, dt$$

is finite for almost every ω. A similar argument shows $\dim_H(\text{ra } X) \geqq d$ whenever $2N \geqq d$.

III. If $2N > d$, then $\dim_H(\text{gr } X) \geqq N + d/2$ with probability one. Let α be any number satisfying $d < \alpha < N + d/2$. We show $C_\alpha(\text{gr } X) > 0$. Let

$$r(t, \omega) = [|X(t, \omega)|^2 + |t|^2]^{\frac{1}{2}}$$

and

$$f(t, R) = P\{\omega : r(t, \omega) < R\} .$$

If $R \geqq |t| > 0$,

$$\begin{aligned} f(t, R) &= P\{\omega : |X(t, \omega)|^2 < R^2 - |t|^2\} \\ &= (2\pi|t|)^{-d/2} \int_{|u|^2 < R^2 - |t|^2} \exp - \left(\frac{u_1^2 + \cdots + u_d^2}{2|t|}\right) du \\ &= K_1 |t|^{-d/2} \int_0^{(R^2 - |t|^2)^{\frac{1}{2}}} r^{d-1} \exp(-r^2/2|t|)\, dr . \end{aligned}$$

If $|t| \geqq R > 0$, $f(t, R) = 0$. We have

$$\begin{aligned} \int_\Omega |r(t, \omega)|^{-\alpha} d\omega &= \int_0^\infty R^{-\alpha} \frac{\partial f}{\partial R}\, dR \\ &= K_2 |t|^{-d/2} \int_{|t|}^\infty R^{1-\alpha} (R^2 - |t|^2)^{d/2 - 1} \exp - \left(\frac{R^2 - |t|^2}{2|t|}\right) dR . \end{aligned}$$

The substitution $R^2 = |t| x^2 + |t|^2$ gives

$$\begin{aligned} K_3 |t|^{-\alpha/2} & \int_0^\infty (x^2 + |t|)^{-\alpha/2} x^{d-1} \exp(-x^2/2)\, dx \\ & \leqq K_3 |t|^{-\alpha/2} \{\int_0^{|t|^{\frac{1}{2}}} |t|^{-\alpha/2} x^{d-1}\, dx + \int_{|t|^{\frac{1}{2}}}^\infty x^{-\alpha+d-1}\, dx\} \\ & < K |t|^{d/2-\alpha} . \end{aligned}$$

Now

$$\begin{aligned} \int_{U^N} \int_{U^N} ds\, dt \int_\Omega |(s, X(s, \omega)) - (t, X(t, \omega))|^{-\alpha}\, d\omega \\ \leqq \int_{U^N} \int_{U^N} K |s - t|^{-\alpha + d/2}\, ds\, dt . \end{aligned}$$

This integral is finite since $\alpha - d/2 < N$. Again Fubini's theorem shows that

$$\int_{U^N} \int_{U^N} |(s, X(s, \omega)) - (t, X(t, \omega))|^{-\alpha}\, ds\, dt$$

is finite for almost every ω.

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REFERENCES

[1] BESICOVITCH, A. S. and URSELL, H. D. (1937). Sets of fractional dimensions, V: On dimensional members of some continuous curves. *J. London Math. Soc.* **12** 18–25.

[2] LÉVY, P. (1948). *Processus Stochastiques et Mouvement Brownien.* Gauthier-Villars, Paris, 265.

[3] LÉVY, P. (1953). *La mesure de Hausdorff de la courbe du mouvement Brownien. Giorn. Ist. Ital. Attuari* **16** 1–37.

[4] OREY, S. and PRUITT, W. (1973). Sample functions of the N-parameter Wiener process. *Ann. Probability* **1** 138–63.

[5] TAYLOR, S. J. (1953). The Hausdorff α-dimensional measure of Brownian paths in n-space. *Proc. Cambridge Philos. Soc.* **49** 31–9.

[6] TAYLOR, S. J. (1955). Measure of the graph and set of zeros of a Brownian path. *Proc. Cambridge Philos. Soc.* **51** 265–74.

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