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EQUIVALENCE OF INFINITELY DIVISIBLE DISTRIBUTIONS

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If F is an infinitely divisible distribution function without a Gaussian component whose Lévy spectral measure M is absolutely continuous and $M(\mathbb{R}^1\backslash\{0\}) = \infty$, then F is shown to have an a.e. positive density over its support; this support of F is always an interval of the form $(-\infty, \infty)$, $(-\infty, a]$ or $[a, \infty)$. In addition, sufficient conditions are obtained for two infinitely divisible distribution functions without Gaussian components to be absolutely continuous with respect to each other, i.e., equivalent.

1. Summary and introduction. The general problem of concern here is that of absolute continuity of one infinitely divisible distribution function with respect to another. Since, however, two distributions which are each equivalent to Lebesgue measure over $\mathbb{R}^1$ are equivalent to each other, this more specialized problem is investigated as well. By absolute continuity of one distribution function, F_1, with respect to another, F_2, we mean absolute continuity of the Lebesgue–Stieltjes measure determined by F_1 on $(\mathbb{R}^1, \mathscr{B}^1)$ with respect to the same determined by F_2. Equivalence of two measures means absolute continuity of each with respect to the other. A number of particular problems of this sort and related problems occur in fairly recent literature. A. V. Skorokhod has considered in [10] and [11] conditions under which a translate of such a measure is absolutely continuous with respect to the original measure. A related but different problem is that of absolute continuity of probability measures over a function space determined by stochastic processes with independent increments; necessary and sufficient conditions for this were obtained by A. V. Skorokhod in [9]. The results obtained here are for infinitely divisible distribution functions without Gaussian components. Such a distribution has a characteristic function of the form

$$f(u) = \exp\left\{i\gamma u + \int_{-\infty}^{-0} + \int_{+0}^{\infty}\left(e^{iux} - 1 - \frac{iux}{1+x^2}\right)M(dx)\right\}, \tag{1}$$

where γ is a real constant and M is a measure, called the Lévy spectral measure, defined over all Borel subsets of $\mathbb{R}^1\backslash\{0\}$ such that $M(A) < \infty$ for every Borel set A whose closure does not contain 0, and such that $\int_{-1}^{-0} + \int_{+0}^{+1} x^2M(dx) < \infty$.

In Section 2 sufficient conditions are obtained for an infinitely divisible distribution function F without a Gaussion component to be equivalent to Lebesgue measure over $\mathbb{R}^1$ and over intervals of the form $(-\infty, a]$ or $[a, \infty)$ when such an interval is the support of F. In particular, it is shown that if $M(\mathbb{R}^1\backslash\{0\}) = \infty$

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and if M is absolutely continuous with respect to Lebesgue measure over $\mathbb{R}^1\backslash\{0\}$, then the support of F is as just stated, and F is equivalent to Lebesgue measure over its support.

In Section 3, sufficient conditions are found for equivalence of two infinitely divisible distribution functions F_1 and F_2 with no Gaussian components. It is proved that the conditions obtained are equivalent to conditions in a theorem by Gikhman and Skorokhod ([2]; see Theorem 6.3 on page 125).

The following notation and definitions will be used. The same symbol will be used for a distribution function (or any non-decreasing function) and the Lebesgue–Stieltjes measure it determines; i.e., if F is a distribution function, $F(x) = F(-\infty, x)$, and $F(A) = \int_A dF(x)$, this integral being a Lebesgue–Stieltjes integral. If X is a random variable, then F_X denotes the distribution function determined by X, i.e., $F_X(A) = P[X \in A]$. We denote $\mathbb{R}^1 = (-\infty, +\infty)$ and $\mathscr{B}^1$ to be the sigma-algebra of Borel subsets of $\mathbb{R}^1$. The letter Λ will denote Lebesgue measure over $\mathbb{R}^1$. If F and G are distribution functions (or any non-decreasing funtions), we say F is absolutely continuous with respect to G, written $F \ll G$, if $G(A) = 0$ implies $F(A) = 0$. If $F \ll G$ and $G \ll F$, we shall say that F and G are *equivalent* and shall denote this by $F \sim G$. In particular, the above definition must be born in mind when the condition $M \sim \Lambda$ over $\mathbb{R}^1\backslash\{0\}$ is used.

2. Equivalence with Lebesgue measure. In Theorems 1, 2 and 3 in this section, sufficient conditions are established for an infinitely divisible distribution function, F, without a Gaussian component to be equivalent to Lebesgue measure over $\mathbb{R}^1$ and over intervals of the form $(-\infty, a)$ and (a, ∞).

LEMMA 1. *If X and Y are independent random variables with distribution functions F_X and F_Y respectively, if $F_X \ll \Lambda$, if (c, d) is any interval over which the density of F_X is positive* a.e. $[\Lambda]$, *and if $b \in \mathbb{R}^1$ is such that $F_Y([b, b+\varepsilon]) > 0$ for every $\varepsilon > 0$, then the density of F_{X+Y} is positive* a.e. $[\Lambda]$ *over $(b+c, b+d)$.* (Note: c and b may be $-\infty$, d may be $+\infty$. Also, the lemma is clearly true for convolutions of finite measures.)

PROOF. We denote the density of a random variable W by φ_W. The convolution of F_X and F_Y is known to be absolutely continuous with density $\varphi_{X+Y}(z) = \int_{-\infty}^{\infty} \varphi_X(z-y)F_Y(dy)$. Let $z \in (b+c, b+d)$. Then $b + c < z$ or $z - c > b$, from which we obtain

$$\varphi_{X+Y}(z) \geqq \int_{[b,z-c)} \varphi_X(z-y)F_Y(dy) ,$$

and where, by hypothesis, $F_Y([b, z-c)) > 0$. Also $z < b + d$ or $b > z - d$. Now $\varphi_X(z-y) > 0$ if $c < z - y < d$ or $z - c > y > z - d$, and hence if $z - c > y \geqq b$. Thus the above integral is positive. □

LEMMA 2. *If φ is a nonnegative, Borel measurable, real-valued function over $\mathbb{R}^1$ with finite, positive Lebesgue integral, then the convolution of φ with itself is positive over some open interval.*

PROOF. We shall refer in this proof to the notion of metric density and the following known theorem (see [6] pages 222–224): if A is a Borel measurable subset of $\mathbb{R}^1$ of positive Lebesgue measure, then there is a subset A_0 of A of measure zero such that every $x \in A \backslash A_0$ has metric density 1 with respect to A, i.e., $\lim_{\varepsilon \downarrow 0} (2\varepsilon)^{-1} \Lambda((x - \varepsilon, x + \varepsilon) \cap A) = 1$. Let $S = \{x : \varphi(x) > 0\}$. We may assume without loss of generality that *every* point in S has metric density 1 with respect to S. Thus for arbitrarily small $\varepsilon > 0$ ($\varepsilon < \frac{1}{10}$), there exists an interval (u, v) such that $\Lambda((u, v) \cap S)/(v - u) > 1 - \varepsilon$. Let $T = (u, v) \cap S$. Then $-T = \{t : -t \in T\} \subset (-v, -u)$, and $-T + u + v \subset (u, v)$. Let

$$U = (-T + u + v) \cap T \subset (u, v) .$$

Then $\Lambda(U) > 0$, and U is symmetric with respect to $(u + v)/2$, and every point in U has metric density 1 with respect to U. Thus $U + U = U - U + u + v$. Now, by a known result (see, e.g., [4] page 68) $U - U$ contains an open interval which contains 0. Hence there exists a nonempty open interval $I \subset U + U$. Let φ_U be defined by $\varphi_U(x) = \varphi(x)$ if $x \in U$ and $= 0$ if $x \notin U$, and define $\varphi_0 = \varphi - \varphi_U$. Since $\varphi * \varphi = \varphi_U * \varphi_U + 2\varphi_U * \varphi_0 + \varphi_0 * \varphi_0$, it is sufficient to prove $\varphi_U * \varphi_U(x) > 0$ for all $x \in U + U$. To do this, we note

$$\varphi_U * \varphi_U(a + b) = \int_U \varphi_U(a + b - y)\varphi_U(y)\, dy ,$$

for $a \in U$, $b \in U$. By properties of U, if $0 < \delta < \frac{1}{4}$, there exists an $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$,

$$(*) \qquad \Lambda(b - \varepsilon, b + \varepsilon) \cap U) > 2\varepsilon(1 - \delta) \qquad \text{and}$$

$$(**) \qquad \Lambda((a - \varepsilon, a + \varepsilon) \cap U) > 2\varepsilon(1 - \delta) .$$

Let us denote $C = \{y \in (b - \varepsilon, b + \varepsilon) : \varphi_U(a + b - y)\varphi_U(y) > 0\}$. It is now clearly sufficient only to prove $\Lambda(C) > 0$. To do this, we note that

$$\begin{aligned} \Lambda((b - \varepsilon, b + \varepsilon)\backslash C) \leqq\ & \Lambda(\{y \in (b - \varepsilon, b + \varepsilon) : \varphi_U(a + b - y) = 0\}) \\ & + \Lambda(\{y \in (b - \varepsilon, b + \varepsilon) : \varphi_U(y) = 0\}) . \end{aligned}$$

By (*), the second term on the right is less than $2\delta\varepsilon$. By (**), the first term on the right of the inequality equals

$$\begin{aligned} & \Lambda(\{b - y \in (-\varepsilon, \varepsilon) : \varphi_U(a + b - y) = 0\}) \\ & \qquad = \Lambda(\{t \in (a - \varepsilon, a + \varepsilon) : \varphi_U(t) = 0\}) < 2\delta\varepsilon . \end{aligned}$$

Thus, $\Lambda(C) > 0$. □

THEOREM 1. *If F is an infinitely divisible distribution function with characteristic function $f(u) = \exp\{\int_{0+}^{\infty} (e^{iux} - 1)M(dx)\}$, if $M \ll \Lambda$ over $(0, \infty)$, if $0 < \int_{0+}^{\infty} xM(dx) < \infty$, and if $M((0, \infty)) = \infty$, then $F \sim \Lambda$ over $(0, \infty)$ and $F((0, \infty)) = 1$.*

PROOF. It is already known [12] that the hypotheses imply $F \ll \Lambda$, so we need only prove $\Lambda \ll F$ over $(0, \infty)$ and $F((0, \infty)) = 1$. This second conclusion easily follows from the fact that $F \ll \Lambda$ and Theorem 1 in [12]. In order to prove $\Lambda \ll F$, let $\infty = b_1 > a_1 = b_2 > a_2 = \cdots > 0$ be such that $b_n \downarrow 0$ as $n \to \infty$ and $M((a_n, b_n)) > 0$ for all n. Let us denote $\lambda_n = M((a_n, b_n)) > 0$. Then let $f_n(x) = \lambda_n^{-1}M'(x)$ if $x \in (a_n, b_n)$ and $= 0$ if $x \notin (a_n, b_n)$; f_n is clearly a probability density of a distribution function which we denote by F_n. By Lemma 2 there is an interval $(u_n, v_n) \subset (2a_n, 2b_n)$ such that $f_n * f_n(x) > 0$ for all $x \in (u_n, v_n)$. Observe that $0 < u_n < v_n$ and $v_n \to 0$ as $n \to \infty$. We prove the following

CLAIM. Given $x > 0$, *there exist positive integers* k_1, k_2, n_1, n_2 $(n_1 \neq n_2)$ *such that* $k_1 u_{n_1} + k_2 u_{n_2} < x < k_1 v_{n_1} + k_2 v_{n_2}$. Indeed, let n_1 be large enough so that $u_{n_1} < x$. Then there exists k_1 such that $k_1 u_{n_1} < x \leqq (k_1 + 1)u_{n_1}$. If $k_1 v_{n_1} \geqq x$, the claim is proved, since we can select $k_2 = 1$ and n_2 such that $u_{n_2} < x - k_1 u_{n_1}$. In case $k_1 v_{n_1} < x$, let n_2 be such that $u_{n_2} < \min\{x - k_1 v_{n_1}, v_{n_1} - u_{n_1}\}$. Then select k_2 such that $k_1 u_{n_1} + k_2 u_{n_2} < x$ but $k_1 u_{n_1} + (k_2 + 1)u_{n_2} \geqq x$. Then $x < k_1 u_{n_1} + k_2 u_{n_2} + v_{n_1} - u_{n_1} \leqq v_{n_1} + (k_1 - 1)u_{n_1} + k_2 u_{n_2} < k_1 v_{n_1} + k_2 v_{n_2}$, which proves the Claim. Now for two distinct positive integers n_1 and n_2 we may write $F = K * L$, where K and L are distribution functions with corresponding characteristic functions

$$\exp\{\textstyle\int_{a_{n1}}^{b_{n_1}} + \int_{a_{n2}}^{b_{n_2}} (e^{iux} - 1)M(dx)\}$$

and

$$\exp\{\textstyle\sum_{j\neq n_1,n_2} \int_{a_j}^{b_j} (e^{iux} - 1)M(dx)\}\,.$$

As noted before, $L \ll \Lambda$ and $L((0, \infty)) = 1$. But for every pair of positive integers k_1 and k_2 we may write

$$\begin{aligned} K &= \left(\textstyle\sum_{j\geq 0} e^{-\lambda_{n_1}} \dfrac{\lambda_{n_1}^j}{j!} F_{n_1}^{*j}\right) * \left(\textstyle\sum_{k\geq 0} e^{-\lambda_{n_2}} \dfrac{\lambda_{n_2}^k}{k!} F_{n_2}^{*k}\right) \\ &= U + V\,, \end{aligned}$$

where

$$U = e^{(-\lambda_{n_1}+\lambda_{n_2})} \frac{\lambda_{n_1}^{2k_1}\lambda_{n_2}^{2k_2}}{k_1!\; k_2!} F_{n_1}^{*2k_1} * F_{n_2}^{*2k_2}$$

and

$$V = C_1 F_{n_1}^{*2k_1} * H_1 + C_2 F_{n_2}^{*2k_2} * H_2 + C_3 H_1 * H_2\,;$$

in the expression for V, C_1, C_2 and C_3 are constants, and H_1 and H_2 are distribution functions which give zero mass to $(-\infty, 0)$, positive mass to $\{0\}$ and are absolutely continuous with respect to Λ over $(0, \infty)$. Now by Lemma 1, $F_{n_1}^{*2k_1} * F_{n_2}^{*2k_2}$ has a positive density over $(k_1 u_{n_1} + k_2 u_{n_2}, k_1 v_{n_1} + k_2 v_{n_2})$; it is clear, by Lemma 1 and remarks above, that V is absolutely continuous over this same interval, and hence K has a positive density over this interval. Since $\sup\{x: L(x) = 0\} = 0$, it follows from Lemma 1 that F has a positive density over $(k_1 u_{n_1} + k_2 u_{n_2}, k_1 v_{n_1} + k_2 v_{n_2})$. The Claim above insures that

$$(0, \infty) = \textstyle\bigcup_{k_1,k_2,n_1,n_2} (k_1 u_{n_1} + k_2 u_{n_2}, k_1 v_{n_1} + k_2 v_{n_2})\,.$$

Thus F has a positive density over $(0, \infty)$, i.e., $F \sim \Lambda$ over $(0, \infty)$. □

THEOREM 2. *If F is an infinitely divisible distribution function with characteristic* *function* $\exp \int_{0+}^{\infty} (e^{iux} - 1 - iux/(1 + x^2))M(dx)$, *where* $M \ll \Lambda$ *over* $(0, \infty)$ *and* $\int_{0+}^{1} xM(dx) = \infty$, *then* $F \sim \Lambda$ *over* $\mathbb{R}^1$.

PROOF. Let us define

$$\begin{aligned} M_1((x, \infty)) &= M((x, \infty)) && \text{if } x \geqq 1 \\ &= M((1, \infty)) + \int_x^1 tM(dt) && \text{if } 0 < x < 1 . \end{aligned}$$

One easily verifies that M_1 is a Lévy spectral measure for which $\int_{0+}^{1} xM_1(dx) < \infty$. We also observe that $M - M_1$ is a Lévy spectral measure, since $0 < x < 1$ implies $M((x, \infty)) = M((1, \infty)) + \int_x^1 M(dt) \geqq M((1, \infty)) + \int_x^1 tM(dt) = M_1((x, \infty))$, and since by a similar calculation $(M - M_1)\{(x', x'')\} \geqq 0$ for $0 < x' < x'' < 1$, and $\int_0^1 x^2(M - M_1)(dx) < \infty$. Because $\int_{0+}^{1} xM_1(dx) < \infty$, we may consider a distribution function F_1 with characteristic function

$$f_1(u) = \exp \int_{0+}^{\infty} (e^{iux} - 1)M_1(dx) .$$

For every $n \geqq 2$ we consider distribution functions H_n, Γ_n and Δ_n with characteristic functions

$$\exp \int_{(0,1/n]} \left(e^{iux} - 1 - \frac{iux}{1 + x^2}\right) (M - M_1)(dx) ,$$

$$\exp \int_{(1/n,1]} (e^{iux} - 1)(M - M_1)(dx)$$

and $\exp\{iu\alpha_n\}$ respectively, where

$$\alpha_n = -\int_{(1/n,\infty)} \frac{x}{1 + x^2} M(dx) - \int_{(0,1/n]} \frac{x}{1 + x^2} M_1(dx) .$$

Then $F = (\Delta_n * F_1) * (H_n * \Gamma_n)$. We next observe the following:

(i) $\alpha_n \to -\infty$ as $n \to \infty$,

(ii) $H_n \to_c E$ where $E(x) = 0$ if $x < 1$ and $= 1$ if $x \geqq 1$,

(iii) for all large n, $\Gamma_n(\{0\}) > 0$ (since $\Gamma_n(\{0\}) = \exp\{-(M - M_1)((1/n, 1])\} > 0$ for all n), and

(iv) $F_1 \sim \Lambda$ over $(0, \infty)$ (by Theorem 1).

Now let A be any Borel set such that $F(A) = 0$. For $N = 1, 2, \cdots$, let $A_N = A \cap [-N, \infty)$; it is sufficient to prove $\Lambda(A_N) = 0$ for arbitrary N to obtain $\Lambda \ll F$. By (i) above, for fixed N, $\alpha_n < -N - 1$ for all large n. Next, (ii) and (iii) above imply

$$H_n * \Gamma_n((-\infty, 1]) \geqq H_n((-\infty, 1])\Gamma_n(\{0\}) > 0 \tag{2}$$

for all large n. The formula $F(A) \geqq \int_{-\infty}^{\infty} (\Delta_n * F_1)(A_N - x)(H_n * \Gamma_n)(dx) = 0$ yields $\Delta_n * F_1(A_N - x) = 0$ a.e. in x with respect to the measure $H_n * \Gamma_n$. But (2) implies that for every n sufficiently large there exists an $x_n < 1$ such that $\Delta_n * F_1(A_N - x_n) = 0$. Thus for each such n, $A_N - x_n \subset [-N - 1, \infty)$. By (iv) and the fact that $\alpha_n < -N - 1$ for large n we obtain $\Delta_n * F_1 \sim \Lambda$ over $[-N - 1, \infty)$. Thus $\Lambda(A_N - x_n) = 0$, implying $\Lambda(A_N) = 0$, which proves $\Lambda \ll F$. The relation $F \ll \Lambda$ is known in this case; see [13]. □

THEOREM 3. *If F is an infinitely divisible distribution function without Gaussian component, if $M \ll \Lambda$ over $\mathbb{R}^1\backslash\{0\}$, if $M(\mathbb{R}^1\backslash\{0\}) = \infty$, and if $M((-\infty, 0)) > 0$ and $M((0, \infty)) > 0$, then $F \sim \Lambda$ over $\mathbb{R}^1$.*

PROOF. If $\int_{-1}^{-0} + \int_{+0}^{1} |x|M(dx) = \infty$, then by Theorem 2, F is the convolution of two distribution functions of which at least one is equivalent to Λ over $\mathbb{R}^1$, and thus in this case $F \sim \Lambda$ over $\mathbb{R}^1$. Now suppose $\int_{-1}^{-0} + \int_{+0}^{1} |x|M(dx) < \infty$. Then we may write $F = G * H$, where G and H have characteristic functions of the form $\exp\{iu\beta + \int_{-\infty}^{-0} (e^{iux} - 1)M(dx)\}$ and $\exp\{\int_{+0}^{\infty} (e^{iux} - 1)M(dx)\}$ respectively. If $M((0, \infty)) = \infty$, then by Theorem 1, $H \sim \Lambda$ over $(0, \infty)$ and $H((0, \infty)) = 1$. In this case, $M((-\infty, 0))$ can be finite or infinite. If it is infinite, then by Theorem 1, $G \sim \Lambda$ over $(-\infty, \beta)$, and hence by Lemma 1, $F = G * H \sim \Lambda$ over $\mathbb{R}^1$. If $M((-\infty, 0)) < \infty$, then define a distribution function L by $L(A) = \lambda^{-1}M((-\infty, 0) \cap A)$ for every Borel set $A \subset \mathbb{R}^1$, where $\lambda = M((-\infty, 0)) > 0$ by hypothesis. Then G may be written $G(A - \beta) = \sum_{k\geqq 0} e^{-\lambda}(\lambda^k/k!)L^{*k}(A)$. Now $L \ll \Lambda$, and hence $L^{*k} \ll \Lambda$ for all $k \geqq 1$. By Lemma 2, there is an open interval (u, v), where $u < v < 0$, over which L^{*2} has a positive derivative. For every positive integer n, we have by Lemma 1 that L^{*2n} has a positive derivative over (nu, nv). By the above representation, G has a positive derivative over $\bigcup_{n=1}^{\infty} (nu - \beta, nv - \beta)$. Applying Lemma 1, we obtain that $G * H$ has a positive derivative over $\mathbb{R}^1$, i.e., $F \sim \Lambda$ over $\mathbb{R}^1$. A similar argument holds in the case $M((-\infty, 0)) = \infty$. □

3. **Equivalence of infinitely divisible distributions.** Any distribution function convolved with a nondegenerate Gaussian distribution will have a positive density everywhere. Thus, it is equivalent to Lebesgue measure over $\mathbb{R}^1$, and any two such probability measures are equivalent. As in the previous section, we are forced by the above observation to consider only infinitely divisible distribution functions without Gaussian components. Sufficient conditions are obtained in order that two such distribution functions are equivalent. For $j = 1, 2$, let F_j be an infinitely divisible distribution function with characteristic function

$$f_j(u) = \exp\left\{iu\gamma_j + \int_{-\infty}^{-0} + \int_{+0}^{\infty}\left(e^{iux} - 1 - \frac{iux}{1 + x^2}\right) M_j(dx)\right\}, \tag{3}$$

where γ and M are as in (1).

THEOREM 4. *If F_1 and F_2 are as in* (3) *and satisfy the following hypotheses,*

(i) $M_j(\mathbb{R}^1\backslash\{0\}) = \infty$ *for* $j = 1, 2$,
(ii) $M_1 \ll M_2$,
(iii) $\int_{\mathbb{R}^1\backslash\{0\}} (1 - (dM_1/dM_2)^{\frac{1}{2}})^2\, dM_2 < \infty$, *and*
(iv) $\gamma_1 - \gamma_2 = \int_{\mathbb{R}^1\backslash\{0\}} (x/(1 + x^2))(M_1 - M_2)(dx)$,

then $F_1 \ll F_2$. On the other hand, if (i), (ii) *and* (iv) *hold, and if $F_1 \perp F_2$, then* $\int_{\mathbb{R}^1\backslash\{0\}} (1 - (dM_1/dM_2)^{\frac{1}{2}})^2\, dM_2 = \infty$.

PROOF. Let $\infty = \varepsilon_0 > \varepsilon_1 > \varepsilon_2 > \cdots$ be such that $\varepsilon_n \downarrow 0$ as $n \to \infty$, and denote $S_m = \{x : \varepsilon_m \leqq |x| < \varepsilon_{m-1}\}$ for $m = 1, 2 \cdots$. Because of (i) we may assume without loss of generality that $M_j(S_m) > 0$ for $j = 1, 2$ and for all $m \geqq 1$. Let us denote $(S_m)^k$ as the Cartesian product of S_m with itself k times. Considering the sequence of Cartesian products $\{\{k\} \times (S_m)^k, k = D, 1, 2, \cdots\}$ as *disjoint* sets, we denote $\Omega_m = \bigcup_{k=0}^\infty (\{k\} \times (S_m)^k)$. We define a sigma-algebra $\mathfrak{A}_m$ of subsets of Ω_m as the class of all sets of the form $\bigcup_{k=0}^\infty (\{k\} \times A_k)$, where A_k is a Borel subset of $(S_m)^k$. Let $\lambda_{jm} = \int_{S_m} M_j(dx) > 0$ for $j = 1, 2$, and define probability measures G_m and H_m over the Borel subsets A of S_m by $G_m(A) = \lambda_{1m}^{-1} M_1(A)$ and $H_m(A) = \lambda_{2m}^{-1} M_2(A)$. Define measures μ_m and ν_m over $(\Omega_m, \mathfrak{A}_m)$ as follows: for A_k any Borel subset of $(S_m)^k$, $k = 1, 2, \cdots$, $\mu_m(\bigcup_{k=0}^\infty \{k\} \times A_k) = \sum_{k=0}^\infty e^{-\lambda_{1m}}(\lambda_{1m}^k/k!)(G_m)^k(A_k)$, and $\nu_m(\bigcup_{k=0}^\infty \{k\} \times A_k) = \sum_{k=0}^\infty e^{-\lambda_{2m}}(\lambda_{2m}^k/k!)(H_m)^k(A_k)$, where $(G_m)^k$ denotes the product measure of G_m with itself k times over $(S_m)^k$, and $(H_m)^k$ is the same for H_m. Let $\Omega = \times_{m=1}^\infty \Omega_m$, and let $\mathfrak{A}$, μ, ν be the product sigma-algebra of $\{\mathfrak{A}_m\}$ and the product measures of $\{\mu_m\}$ and $\{\nu_m\}$ respectively. With this notation established we proceed with the computation of the inner product $\int_{\Omega_m} (d\mu_m/d\nu_m)^{1/2}\, d\nu_m$.

Since $M_1 \ll M_2$, then $G_m \ll H_m$ over S_m for $m = 1, 2, \cdots$, and thus

$$\frac{d\mu_m}{d\nu_m}(k, x_1, x_2, \cdots, x_k) = e^{-(\lambda_{1m}-\lambda_{2m})}\left(\frac{\lambda_{1m}}{\lambda_{2m}}\right)^k \prod_{s=1}^k \frac{dG_m}{dH_m}(x_s)$$

at the point $(k, x_1, \cdots, x_k) \in \Omega_m$. Hence

$$\begin{aligned}
&\int_{\Omega_m}(d\mu_m/d\nu_m)^{1/2}\, d\nu_m \\
&\quad= \sum_{k=0}^\infty \int_{\{k\}\times(S_m)^k} (d\mu_m/d\nu_m)^{1/2}\, d\nu_m \\
&\quad= \sum_{k=0}^\infty \frac{(\lambda_{1m}\lambda_{2m})^{k/2}}{k!} \exp\{-\tfrac{1}{2}(\lambda_{1m}+\lambda_{2m})\} \int_{(S_m)^k}\left(\prod_{j=1}^k \frac{dG_m}{dH_m}\right)^{1/2} d\prod_{j=1}^k H_m \\
&\quad= \exp\{-\tfrac{1}{2}(\lambda_{1m}+\lambda_{2m})\} \sum_{k=0}^\infty \{(\lambda_{1m}\lambda_{2m})^{1/2} \int_{S_m}(dG_m/dH_m)^{1/2}\, dH_m\}^k/k!\,.
\end{aligned}$$

Since

$$\frac{dG_m}{dH_m} = \left(\frac{\lambda_{1m}}{\lambda_{2m}}\right)^{-1}\frac{dM_1}{dM_2}$$

over S_m, we have $\int_{S_m}(dG_m/dH_m)^{1/2}\, dH_m = (\lambda_{1m}\lambda_{2m})^{-1/2}\int_{S_m}(dM_1/dM_2)^{1/2}\, dM_2$. Hence

$$\begin{aligned}
\int_{\Omega_m}(d\mu_m/d\nu_m)^{1/2}\, d\nu_m &= \exp\{-\tfrac{1}{2}(\lambda_{1m}+\lambda_{2m}) + \int_{S_m}(dM_1/dM_2)^{1/2}\, dM_2\} \\
&= \exp\left\{-\tfrac{1}{2}\int_{S_m}\left(\frac{dM_1}{dM_2} - 2\left(\frac{dM_1}{dM_2}\right)^{1/2} + 1\right) dM_2\right\} \\
&= \exp\left\{-\tfrac{1}{2}\int_{S_m}\left(1 - \left(\frac{dM_1}{dM_2}\right)^{1/2}\right)^2 dM_2\right\}.
\end{aligned}$$

According to Kakutani's theorem (see [5] page 453) $\mu \ll \nu$ if and only if $\prod_{m=1}^\infty \int_{\Omega_m}(d\mu_m/d\nu_m)^{1/2}\, d\nu_m > 0$; otherwise $\mu \perp \nu$. According to the above calculation, this condition says that $\mu \ll \nu$ if and only if

$$\exp\left\{-\tfrac{1}{2}\int\left(1 - \left(\frac{dM_1}{dM_2}\right)^{1/2}\right)^2 dM_2\right\} > 0\,,$$

$$\text{i.e., if and only if}\quad \int\left(1 - \left(\frac{dM_1}{dM_2}\right)^{1/2}\right)^2 dM_2 < \infty\,.$$

Now define the random variable Z_m on Ω as follows: if $(y_1, y_2, \cdots) \in \Omega$, where $y_m = (k, x_1, \cdots, x_k) \in \{k\} \times (S_m)^k \subset \Omega_m$, then $Z_m(y_1, \cdots, y_m, \cdots) = x_1 + \cdots + x_k$. It is clear that the random variables $\{Z_1, Z_2, \cdots\}$ are independent over $(\Omega, \mathfrak{A}, \mu)$ and over $(\Omega, \mathfrak{A}, \nu)$. The characteristic functions of Z_m are $E_\mu(e^{iuZ_m}) = \exp\{\int_{S_m} (e^{iux} - 1)M_1(dx)\}$ and $E_\nu(e^{iuZ_m}) = \exp\{\int_{S_m} (e^{iux} - 1)M_2(dx)\}$, where E_μ, E_ν are understood to mean expectations with respect to the probabilities μ, ν respectively. Now define $T_n = \gamma_1 + \sum_{k=1}^{n} (Z_k - \int_{S_k} (x/(1 + x^2))M_1(dx))$. Then

$$E_\mu(e^{iuT_n}) = \exp\left\{iu\gamma_1 + \int_{|x|\geq\varepsilon_n}\left(e^{iux} - 1 - \frac{iux}{1 + x^2}\right) M_1(dx)\right\}$$

and

$$E_\nu(e^{iuT_n}) = \exp\left\{iu\gamma_1 - \int_{|x|\geq\varepsilon_n} \frac{iux}{1 + x^2} M_1(dx) + \int_{|x|\geq\varepsilon_n} (e^{iux} - 1)M_2(dx)\right\} .$$

Since $\{E_\mu(e^{iuT_n})\}$ converges as $n \to \infty$ to $\exp\{iu\gamma_1 + \int (e^{iux} - 1 - (iux/(1 + x^2)) \times M_1(dx)\}$, then $\{T_n\}$ coverges in law. Since $\{T_n\}$ is a sequence of partial sums of independent random variables, it follows that $\{T_n\}$ converges a.e. $[\mu]$. Now by hypothesis (iv),

$$\lim_{n\to\infty} \exp\left\{iu(\gamma_2 - \gamma_1) - \int_{|x|\geq\varepsilon_n} \frac{iux}{1 + x^2} (M_2 - M_1)(dx)\right\} = 1 ,$$

and

$$E_\nu(e^{iuT_n}) \exp\left\{iu(\gamma_2 - \gamma_1) - \int_{|x|\geq\varepsilon_n} \frac{iux}{1 + x^2} (M_2 - M_1)(dx)\right\} = \exp\left\{iu\gamma_2 + \int_{|x|\geq\varepsilon_n}\left(e^{iux} - 1 - \frac{iux}{1 + x^2}\right) M_2(dx)\right\} .$$

These last two computations imply

$$\lim_{n\to\infty} E_\nu(e^{iuT_n}) = \exp\left\{iu\gamma_2 + \int \left(e^{iux} - 1 - \frac{iux}{1 + x^2}\right) M_2(dx)\right\} ,$$

and that $\{T_n\}$ converges a.e. $[\nu]$. Define $T = \lim_{n\to\infty} T_n$ over the intersection of the two sets of convergence, and $T = 0$ otherwise. It is readily observed that $\mu T^{-1} = F_1$ and $\nu T^{-1} = F_2$. Since under our hypotheses we obtained $\mu \ll \nu$, it follows by Lemma 1 that $\mu T^{-1} \ll \nu T^{-1}$, i.e., $F_1 \ll F_2$. On the other hand, if $F_1 \perp F_2$, then $\mu \perp \nu$, which implies

$$\int \left(1 - \left(\frac{dM_1}{dM_2}\right)^{\frac{1}{2}}\right)^2 dM_2 = \infty .$$

REMARK 1. It should be noted that hypothesis (i) in Theorem 4 is used only in the second sentence of the proof, and not in its full strength. It should be remarked however that if $M_j(\mathbb{R}^1\backslash\{0\}) < \infty$ for both $j = 1, 2$, then all that is needed is (ii), namely $M_1 \ll M_2$, in order to obtain $F_1 \ll F_2$; the proof for this is very easy and may be found at the end of the proof of Theorem 6.3 in [2].

If $M_1(\mathbb{R}^1\backslash\{0\}) < \infty$ but $M_2(\mathbb{R}^1\backslash\{0\}) = \infty$, then the conclusion $F_1 \ll F_2$ is not true. This follows from a theorem by Blum and Rosenblatt [1] and also by Hartman and Wintner [3] which states that F_2 is continuous but F_1 is not. If $M_1(\mathbb{R}^1\backslash\{0\}) = \infty$ and $M_2(\mathbb{R}^1\backslash\{0\}) < \infty$, then we can show that condition (iv) is never satisfied. Indeed, since $x^2/2 \leqq (1 - x)^2$ for $x \geqq 4$, we obtain, letting $\rho = dM_1/dM_2$,

$$\int_{\mathbb{R}^1\backslash\{0\}} (1 - \rho^{\frac{1}{2}})^2\, dM_2 \geqq \int_{[\rho \geqq 16]} (1 - \rho^{\frac{1}{2}})^2\, dM_2 \geqq \int_{[\rho \geqq 16]} \tfrac{1}{2}\rho\, dM_2\,.$$

But

$$M_1(\mathbb{R}^1\backslash\{0\}) = \int_{[\rho \leqq 16]} \rho\, dM_2 + \int_{[\rho > 16]} \rho\, dM_2 = \infty\,,$$

and

$$\int_{[\rho \leqq 16]} \rho\, dM_2 \leqq 16 M_2[\rho \leqq 16] \leqq 16 M_2(\mathbb{R}^1\backslash\{0\}) < \infty\,.$$

Hence $\int_{[\rho > 16]} \rho\, dM_2 = \infty$, and consequently $\int_{\mathbb{R}^1\backslash\{0\}} (1 - \rho^{\frac{1}{2}})^2\, dM_2 = \infty$.

REMARK 2. We present here for the sake of completeness a proof that condition (iii) of our Theorem 4 is equivalent to the condition

$$\int_{\mathbb{R}^1\backslash\{0\}} \frac{(\rho - 1)^2}{1 + |\rho - 1|}\, dM_2 < \infty \quad \left(\rho = \frac{dM_1}{dM_2}\right) \tag{4}$$

which occurs in Theorem 6.3 in the paper by Gikhman and Skorokhod [2]; this fact has been noted without proof at the end of Section 2 in [8]. In order to prove this, one first observes that by applying the mean value theorem to the function $(1 + y)^{\frac{1}{2}}$ one obtains the inequality

$$\frac{y^2}{6} \leqq [(1 + y)^{\frac{1}{2}} - 1]^2 \leqq \frac{y^2}{2} \tag{5}$$

for all $y \in [-\frac{1}{2}, \frac{1}{2}]$. Replacing y by $\rho - 1$ in (4), we obtain

$$\begin{aligned} \tfrac{1}{6} \int_{[|\rho-1| < \frac{1}{2}]} (\rho - 1)^2\, dM_2 &\leqq \int_{[|\rho-1| < \frac{1}{2}]} (1 - \rho^{\frac{1}{2}})^2\, dM_2 \\ &\leqq \tfrac{1}{2} \int_{[|\rho-1| < \frac{1}{2}]} (\rho - 1)^2\, dM_2\,. \end{aligned}$$

Hence we have:

(a) $\int_{[|\rho-1| < \frac{1}{2}]} (1 - \rho^{\frac{1}{2}})^2\, dM_2 < \infty$ if and only if $\int_{[|\rho-1| < \frac{1}{2}]} (\rho - 1)^2\, dM_2 < \infty$, which is easily seen to be equivalent to

$$\int_{[|\rho-1| < \frac{1}{2}]} \frac{(\rho - 1)^2}{1 + |\rho - 1|}\, dM_2 < \infty\,.$$

Now, since $(1 - (x + 1)^{\frac{1}{2}})^2 \sim x^2/(1 + |x|)$ (as $x \to \infty$), we have, upon replacing x by $\rho - 1$:

(b) for T sufficiently large, $\int_{[|\rho-1| > T]} (1 - \rho^{\frac{1}{2}})^2\, dM_2 < \infty$ if and only if

$$\int_{[|\rho-1| > T]} \frac{(\rho - 1)^2}{1 + |\rho - 1|}\, dM_2 < \infty\,.$$

Since over $[\frac{1}{2} \leqq |\rho - 1| \leqq T]$, both $(1 - \rho^{\frac{1}{2}})^2$ and $(\rho - 1)^2/(1 + |\rho - 1|)$ are bounded above and bounded away from zero, we obtain:

(c) $\int_{[\frac{1}{2} \leq |\rho - 1| \leq T]} (1 - \rho^{\frac{1}{2}})^2 \, dM_2 < \infty$ if and only if

$$\int_{[\frac{1}{2} \leq |\rho - 1| \leq T]} \frac{(\rho - 1)^2}{1 + |\rho - 1|} \, dM_2 < \infty .$$

(Note: Both conditions are equivalent to $M_2(\{x : \frac{1}{2} \leqq |\rho(x) - 1| \leqq T\}) < \infty$.) The statements (a), (b) and (c) yield the remark.

REMARK 3. It should be pointed out that hypothesis (iv) of Theorem 4 is an assumption on γ_1 and γ_2 and not on M_1 and M_2. More precisely, hypothesis (iii) implies that

$$\int_{R^1 \setminus \{0\}} \frac{|x|}{1 + x^2} \, |M_1 - M_2|(dx) < \infty .$$

This is noted by Charles M. Newman [7] and proved by him in [8].

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