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APPLICATIONS OF SPACE-TIME HARMONIC FUNCTIONS TO BRANCHING PROCESSES

BY THOMAS H. SAVITS

University of Pittsburgh

In attempting to determine the growth properties of a branching process (b.p.), a standard method of attack is to look for the appropriate martingale. Here we show that for many b.p., this really corresponds to looking for the harmonic functions associated with the space-time process. As a particular application of the above we show that in the case of the classical Galton–Watson continuous time process with $m < \infty$ there exists constants $c(t)$ such that $Z_t/c(t)$ converges w.p. 1 to a nontrivial random variable.

0. Introduction. In the study of branching processes, one problem of particular interest is the growth behavior of the population. Martingales have proven themselves to be an extremely useful tool in the study of this problem. The reason they are so useful is because of the known convergence theorems in the general theory of martingales. Consequently if one succeeds in finding a martingale, it is usually relatively easy to show convergence. Of course it is generally more difficult to show that the limit random variable is nontrivial. Thus one wants not only to find martingales, but one needs to find the "appropriate" martingale.

The purpose of this paper is to delineate a systematic approach to the problem of finding martingales. The approach is in a sense not new since it is well known that harmonic functions of Markov processes yield martingales, but this method seems to have been overlooked in the theory of branching processes. Furthermore, the "correct" harmonic functions to be used in studying the growth behavior of branching processes are not the ones associated with the branching process itself, but rather those associated with the space-time branching process. This paper will be an attempt to illustrate the utility of this method.

1. General theory. Let $X = (X_t, P_x)$ be a Markov process with state space S. We shall always assume that X has right-continuous paths and that S is a "nice" topological space; i.e., a locally compact second countable Hausdorff topological space. The associated infinitesimal generator is denoted by A and its domain by $\mathscr{D}(A)$, in either the strong or weak sense (cf. Dykin [2]). It follows that if $f \in \mathscr{D}(A)$, then

$$M_t = f(X_t) - f(X_0) - \int_0^t Af(X_u)\,du \tag{1.1}$$

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is a martingale (with respect to each P_x). In particular, if $Af = 0$, then $f(X_t)$ is itself a martingale. Functions $h \in \mathscr{D}(A)$ such that $Ah = 0$ are said to be harmonic (or A-harmonic) functions. Consequently harmonic functions give rise to martingales.

(1.2) REMARK. One could also talk about sub (super)-harmonic functions. These in turn give rise to sub (super)-martingales.

Consider now the associated space-time Markov process $Y = (Y_t, P_{x,\sigma})$ on $E = S \times [0, \infty)$. This is nothing more than the direct product of the given process X and an independent uniform motion process on the half-line $[0, \infty)$ moving to the right with constant speed one. In particular then, $Y_t = (X_t, t + \sigma)$ a.s. $P_{x,\sigma}$. The infinitesimal generator B for Y is

$$B = \frac{\partial}{\partial t} + A \, . \tag{1.3}$$

The associated B-harmonic functions will be called space-time harmonic functions of X; i.e., they are those $H \in \mathscr{D}(B)$ satisfying the backward heat equation $BH = 0$. Note that H is a function of two variables, space and time.

(1.4) REMARK. One may also consider discrete time Markov processes $X = (X_j, P_x)$ on S. In this case the role of the infinitesimal operator is played by

$$Af(x) = E_x[f(X_1)] - f(x) \, . \tag{1.5}$$

Thus it is easily seen that any function h such that the right-hand side of (1.5) makes sense pointwise and satisfies $Ah = 0$ gives rise to a discrete time martingale $h(X_j)$. As in the continuous time case we can analogously consider the corresponding space-time functions. They are those functions H defined on $E = S \times \{0, 1, 2, \cdots\}$ for which the right-hand side of (1.6) makes sense and vanishes pointwise.

$$\begin{aligned} BH(x, j) &= E_{x,j}[H(X_1, 1)] - H(x, j) \\ &= E_x[H(X_1, j + 1)] - H(x, j) \, . \end{aligned} \tag{1.6}$$

Let us now consider the case of a branching Markov process (bmp). According to Ikeda, Nagasawa and Watanabe [4], a bmp on S is a Markov process $\mathbf{X} = (\mathbf{X}_t, \mathbf{P}_x)$ on $\hat{\mathbf{S}}$ having the branching property

$$(\mathbf{T}_t \hat{f}) \,|\, S = \mathbf{T}_t \hat{f}$$

for all $f \in \mathbf{B}(S)$ with $\sup\{|f(x)| : x \in S\} \leqq 1$. Here $\mathbf{S} = \bigcup_{n=0}^{\infty} S^n$, where $S^0 = \{\partial\}$, ∂ being an isolated point, S^n is the quotient topological space of the n-fold Cartesian product by the equivalence relation of permutation, $\hat{\mathbf{S}} = \mathbf{S} \cup \{\Delta\}$ is the one-point compactification of $\mathbf{S}$, $\mathbf{T}_t$ is the induced semigroup of $\mathbf{X}$ and

$$\begin{aligned} \hat{f}(\mathbf{x}) &= 1 && \text{if } \ \mathbf{x} = \partial \\ &= f(x_1) f(x_2) \cdots f(x_n) && \text{if } \ \mathbf{x} = [x_1, \cdots, x_n] \in S^n \\ &= 0 && \text{if } \ \mathbf{x} = \Delta \, . \end{aligned} \tag{1.7}$$

We shall also assume that $\mathbf{X}$ is right-continuous and strong Markov.

The space-time harmonic functions for $\mathbf{X}$ are those functions $H \in \mathscr{D}(B)$ satisfying $\mathbf{B}H = 0$ where

(1.8) $$\mathbf{B} = \frac{\partial}{\partial t} + \mathbf{A} \qquad \text{on} \quad \hat{\mathbf{S}} \times [0, \infty) .$$

There are two problems to be contended with here. The first has to do with $\mathbf{A}$. Although $\mathbf{A}$ is a linear (in general unbounded) operator on $\hat{\mathbf{S}}$, its form is rather complicated. This is true even in the case when $\mathbf{X}$ is a (T_t^0, K, π) process, or more particularly an $[X, k, \pi]$ process (cf. [4] III, page 99; see also Remark (1.16) for an intuitive description). Most bmp of interest belong to the above classes and we shall only consider these types here. It turns out, however, that $\mathbf{A}$ reduces when restricted to two special classes of functions. Fortunately, these two classes seem to be rich enough for applications. More will be said about these in a short time.

The second problem has to do with $\mathscr{D}(\mathbf{B})$ or $\mathscr{D}(\mathbf{A})$. Again it seems to be a formidable task to describe $\mathscr{D}(\mathbf{A})$ even in the case when $\mathbf{X}$ is a (T_t^0, K, π) process. Under some regularity assumptions Ikeda, Nagasawa and Watanabe were able to describe two subclasses of $\mathscr{D}(\mathbf{A})$ for an $[X, k, \pi]$ process ([4] III). Again, these subclasses have to do with the two special classes of functions mentioned above.

Rather than concern ourselves directly with these technical problems, we shall adopt the following point of view. In most cases of interest, one can write down the formal generator with relative ease, at least in the case of the so called Λ- and V-harmonic functions (see (1.10) and (1.13)). Next one looks for formal solutions of the equation $\mathbf{B}H = 0$. Then one either shows directly that $H(\mathbf{Y}_t)$ is a martingale or that $H \in \mathscr{D}(\mathbf{B})$.

Let us now discuss the two special classes of functions. The first class consists of those measurable functions H on $\hat{\mathbf{S}} \times [0, \infty)$ satisfying

(1.9) $$\widehat{H_t \mid S} = H_t$$

where $H_t(\mathbf{x}) = H(\mathbf{x}, t)$. Note that we only need know H_t on S to know it on all of $\hat{\mathbf{S}}$. Consequently, if we set $h_t = H_t \mid S$, then $H_t = \hat{h}_t$.

(1.10) DEFINITION. The formal solutions h of the equation $\mathbf{B}\hat{h} = 0$ will be called the Λ-space-time harmonic functions of $\mathbf{X}$, or more briefly the Λ-harmonic functions.

In the case when $\mathbf{X}$ is an $[X, k, \pi]$ process, the Λ-harmonic functions h are solutions of the equation

(1.11) $$\frac{\partial h}{\partial t} + Ah + k[F(\cdot; h_t) - h] = 0 \qquad \text{on} \quad S \times [0, \infty)$$

where A is the infinitesimal generator of X and F is the operator given by

$$F(x; f) = \int_{\hat{\mathbf{S}}} \pi(x, d\mathbf{y})\hat{f}(\mathbf{y}) .$$

The second class consists of those measurable functions H on $\hat{\mathbf{S}} \times [0, \infty)$ satisfying

$$\widecheck{H_t | S} = H_t \tag{1.12}$$

where

$$\begin{aligned} \check{f}(\mathbf{x}) &= 0 && \text{if} \quad \mathbf{x} = \partial \quad \text{or} \quad \Delta \\ &= f(x_1) + \cdots + f(x_n) && \text{if} \quad \mathbf{x} = [x_1, \cdots, x_n] \in S^n . \end{aligned}$$

If we set $h_t = H_t | S$, then $H_t = \check{h}_t$.

(1.13) DEFINITION. The formal solutions h of the equation $\mathbf{B}\check{h} = 0$ will be called the V-space-time harmonic functions of $\mathbf{X}$, or more briefly the V-harmonic functions.

In the case when $\mathbf{X}$ is an $[X, k, \pi]$ process, the V-harmonic functions are solutions of the equation

$$\frac{\partial h}{\partial t} + Ah + k[G(\cdot; h_t) - h] = 0 \qquad \text{on} \quad S \times [0, \infty) \tag{1.14}$$

where A is the infinitesimal generator of X and G is the operator given by

$$G(x; g) = \int_{\hat{\mathbf{S}}} \pi(x, dy)\check{g}(\mathbf{y}) .$$

(1.15). REMARK. Note that (1.11) is non-linear whereas (1.14) is linear.

It should be mentioned that not all processes of interest can be represented as an $[X, k, \pi]$ process; in particular, the age dependent process with general lifetime distribution cannot. This process is, however, a $(T_t{}^0, K, \pi)$ process.

(1.16) REMARK. Intuitively a $(T_t{}^0, K, \pi)$ process can be described as follows. Let $X^0 = (X_t{}^0, P_x{}^0, \zeta^0)$ be a given Markov process on S with lifetime ζ^0 and let $\pi(x, dy)$ be a given stochastic kernel on $S \times \hat{\mathbf{S}}$. If $\mathbf{X}$ is the associated bmp, then a "parent object" behaves like X^0 on S up to time ζ^0. At that time this parent object dies. If it dies at position $z \in S$, it is replaced with n "newborn progeny" starting at position $\mathbf{y} = [y_1, \cdots, y_n]$ with probability $\pi(z, dy)$. Each newborn object behaves independently of the others and in the same fashion as its ancester. Here $T_t{}^0$ is the induced semigroup of X^0 and K is a stochastic kernel describing the joint distribution of time and position of parental death. Thus for small values of t, we formally have

$$\begin{aligned} \mathbf{T}_t\hat{f}(x) &= \mathbf{E}_x[\hat{f}(\mathbf{X}_t)] = \mathbf{E}_x[\hat{f}(\mathbf{X}_t); t < \zeta^0] + \mathbf{E}_x[\hat{f}(\mathbf{X}_t); t \geqq \zeta^0] \\ &= T_t{}^0 f(x) + P_x{}^0(\zeta^0 \leqq t) \int_{\hat{\mathbf{S}}} \pi(x, dy)\hat{f}(\mathbf{y}) + o(t) \end{aligned}$$

for $x \in S$. Since $P_x{}^0(\zeta^0 > t) = T_t{}^0 1(x)$ and $\mathbf{A} = \lim_{t \downarrow 0} t^{-1}[T_t - I]$ we see that

$$\mathbf{A}\hat{f}(x) = A^0 f(x) - A^0 1(x) \cdot F(x; f)$$

where F is as in (1.11). This formal derivation can be made rigorous provided $1 \in \mathscr{D}(A^0)$ and f is sufficiently nice. (A^0 is the infinitesimal generator of $T_t{}^0$.)

Similarly we see that

$$\mathbf{A}\check{g}(x) = A^0 g(x) - A^0 1(x) \cdot G(x; f)$$

where G is as in (1.14).

If X^0 is obtained from a conservative Markov process X on S by "killing" it with a nonnegative function k we call the corresponding bmp $\mathbf{X}$ an $[X, k, \pi]$ process. By this we mean that if U is an exponentially distributed random variable independent of the process X and having mean one, we set $X_t^0 = X_t$ for $t < \zeta^0$ where $\zeta^0 = \inf\{t \geqq 0; \int_0^t k(X_u)\,du > U\}$. In this case $A1 = 0$ and it is not hard to see that $A^0 = A - kI$, where A is the infinitesimal generator of X.

2. Examples. Recall that every space-time harmonic function $\check{H}$ gives rise to a martingale $H(\mathbf{Y}_t)$ with respect to each probability measure $\mathbf{P}_{\mathbf{x},\sigma}$. In the following, we shall only be concerned with $\mathbf{x} \in S$ and $\sigma = 0$. Consequently, $H(\mathbf{X}_t, t)$ is a martingale with respect to each $\mathbf{P}_x$, $x \in S$.

A. Galton–Watson process-discrete time. In this case since S consists of a single point we identify $\hat{\mathbf{S}}$ with $\{0, 1, 2, \cdots\} \cup \{\infty\}$ and interpret $\mathbf{X}_j$ as the number of particles at time j. We shall write the $\wedge$- and $\vee$-harmonic functions as $h(j)$ instead of $h(x, j)$, $x \in S$. Let π_n be the probability of producing n offspring, $n = 0, 1, 2, \cdots$, with $\pi_1 = 0$. (It is not necessary to assume $\pi_1 = 0$, but we do so to be consistent with the INW set-up.) The generating function f is given by $f(s) = \sum_{n=0}^{\infty} \pi_n s^n$, $|s| \leqq 1$.

(i) $\vee$-harmonic functions. These harmonic functions satisfy

$$h(j) = G[h(j+1)] = \textstyle\sum_{n=0}^{\infty} n\pi_n h(j+1) = mh(j+1) \tag{2.1}$$

for $j = 0, 1, \cdots$, where $m = \sum_{n=0}^{\infty} n\pi_n$ is the mean number of new-born progeny. Note that if $m = 0$ or ∞, there are no nontrivial solutions of (2.1). If $0 < m < \infty$, then we have $h(j) = h(0)m^{-j}$. There is no loss of generality in assuming $h(0) = 1$. Consequently we obtain the classical result that $H(\mathbf{X}_j, j) = \check{h}(\mathbf{X}_j, j) = m^{-j}\mathbf{X}_j$ is a nonnegative martingale and hence converges w.p. 1 to a random variable W. It is known that W is nontrivial iff $m > 1$ and $\sum_{n=0}^{\infty} (n \log n)\pi_n = \mathbf{E}_1(\mathbf{X}_1 \log \mathbf{X}_1) < \infty$ (cf. [5]).

(ii) $\wedge$-harmonic functions. In this case h satisfies

$$h(j) = F[h(j+1)] = \textstyle\sum_{n=0}^{\infty} \pi_n h^n(j+1) = f[h(j+1)]\,. \tag{2.2}$$

If $\pi_0 \neq 1$, then $f(s)$ is a strictly increasing function on $0 \leqq s \leqq 1$. Let f^{-1} be its inverse. Consequently one solution of (2.2) is given recursively by

$$h(j+1) = f^{-1}[h(j)] \qquad j = 0, 1, \cdots$$

where we take $h(0) = c$, $q < c < 1$ with q being the extinction probability. So $H(\mathbf{X}_j, j) = [h(j)]^{\mathbf{X}_j}$ is a martingale and being nonnegative it converges w.p. 1 to a random variable W. Thus, $\mathbf{X}_j \log h(j) \to \log W$ w.p. 1. This result is fairly recent and is due to Seneta [6] and Heyde [3]; furthermore, they show that W is nondegenerate if $1 < m < \infty$.

In the case $m = \infty$, Seneta [7] has shown that there are no normalizing constants c_n such that $c_n^{-1}\mathbf{X}_n$ converge in distribution to a nondegenerate limit.

B. Multi-type process-discrete time. Here $S = \{a_1, \cdots, a_N\}$ consists of a finite number of types. We identify $\mathbf{S}$ with the nonnegative integer lattice in N-dimensions. Particle "i" splits according to the generating function $f^i(s_1, \cdots, s_N)$. We assume that the associated stochastic kernel π satisfies $\pi(i, S) = 0$, all i.

(i) V-harmonic functions. h satisfies the equation

$$h(i, j) = G[i; h(\cdot, j + 1)] \qquad \text{for } i = 1, \cdots, N, j = 0, 1, \cdots.$$

In vector form, we have

$$h'(j) = Mh'(j + 1) \tag{2.3}$$

where $h(j) = (h(1, j), \cdots, h(N, j))$, "$'$" denotes transpose and M is the $N \times N$ matrix of first moments with entries

$$m_{pq} = \frac{\partial f^p}{\partial s_q}(1, \cdots, 1) \qquad 1 \leqq p, q \leqq N.$$

We shall assume that all the m_{pq} are finite.

If M is invertible, then the only finite solutions of (2.3) are given by

$$h'(j) = M^{-j}h'(0) \qquad j = 0, 1, \cdots$$

and $h(0)$ may be arbitrary. Thus,

$$H(\mathbf{X}_j, j) = \mathbf{X}_j \cdot h(j) = \mathbf{X}_j M^{-j} h'(0)$$

is a martingale, where $\mathbf{X}_j = (X_j^1, \cdots, X_j^N)$ with X_j^i being the number of particles of type "i" at time j.

Solutions of (2.3) exist also in the case when M is not invertible. Let λ be any nonzero eigenvalue of M and ξ an associated right-eigenvector. Then $h(j) = \lambda^{-j}\xi$ satisfies (2.3) and

$$H(\mathbf{X}_j, j) = \lambda^{-j}\mathbf{X}_j \cdot \xi$$

is a martingale. If we assume that M is positively regular, then from the Frobenius theory it follows that there is a largest eigenvalue ρ which is simple, real and positive; furthermore there are positive right and left eigenvectors v and u respectively associated to ρ which we take to be normalized such that their inner product $u \cdot v = 1$. The nonnegative martingale $\rho^{-j}\mathbf{X}_j \cdot v$ thus converges w.p. 1 to a random variable W. Clearly W is trivial if $\rho \leqq 1$ since extinction occurs w.p. 1 in this case. Kesten and Stigum [5] proved the beautiful result that W is nondegenerate iff $\rho > 1$ and $\mathbf{E}_i[X_1^k \log X_1^k] < \infty$, $1 \leqq i, k \leqq N$; moreover, they showed that

$$\rho^{-j}\mathbf{X}_j \to Wu \quad \text{w.p. } 1 .$$

(ii) Λ-harmonic functions. h now satisfies

$$\begin{aligned} h(i, j) &= F[i; h(\cdot, j + 1)] \qquad i = 1, \cdots, N; j = 0, 1, \cdots \\ &= f^i[h(1, j + 1), \cdots, h(N, j + 1)] . \end{aligned}$$

In vector form,

$$h(j) = f[h(j+1)]\,. \tag{2.4}$$

C. Galton–Watson process-continuous time. We have the same set-up as in Example A except now each particle waits according to an exponential distribution with parameter b and then splits according to the generating function $f(s) = \sum_{n=0}^{\infty} \pi_n s^n$.

(i) V-harmonic functions. Here we have the differential equation

$$\begin{aligned} \frac{dh}{dt}(t) &= b[h(t) - G(h(t))] \\ &= b[h(t) - mh(t)]\,. \end{aligned} \tag{2.5}$$

We assume that m is finite. Then

$$h(t) = h(0)\exp\{-b(m-1)t\}$$

is the solution. Choosing $h(0) = 1$ and setting $a = b(m-1)$ we get the classical result that

$$H(\mathbf{X}_t, t) = e^{-at}\mathbf{X}_t$$

is a nonnegative martingale and hence converges w.p. 1 to a random variable W. As before W is nontrivial iff $m > 1$ and $\mathbf{E}_1(\mathbf{X}_1 \log \mathbf{X}_1) = \sum_{n=0}^{\infty} \pi_n(n \log n) < \infty$ (see Athreya [1]).

(ii) Λ-harmonic functions. In this case, h is a solution of

$$\begin{aligned} \frac{dh}{dt} &= b[h(t) - F(h(t))] \\ &= bh(t) - b\textstyle\sum_{n=0}^{\infty} \pi_n h^n(t) \\ &= b[h(t) - f(h(t))]\,. \end{aligned} \tag{2.6}$$

Let $m > 1$ and $q < c < 1$, q being the extinction probability; then $h(t)$ defined implicitly by the relation

$$\int_c^{h(t)} \frac{d\xi}{\xi - f(\xi)} = bt$$

satisfies (2.6). Furthermore $0 \leqq h \leqq 1$. Consequently

$$H(\mathbf{X}_t, t) = [h(t)]^{\mathbf{X}_t}$$

is a nonnegative bounded martingale which therefore converges w.p. 1 to a random variable W. Taking logarithms,

$$\mathbf{X}_t \log h(t) \to \log W \quad \text{w.p. } 1.$$

This is the continuous time analogue of Seneta's result and seems to be new. By applying Seneta's result to a discrete skeleton for $\mathbf{X}_t$ we deduce that W is nondegenerate; i.e., fix some $t_0 > 0$. Then defining $\mathbf{Y}_n = \mathbf{X}_{nt_0}$, we see that $\mathbf{Y} = (\mathbf{Y}_n)$ is a discrete time bmp with generating function $\phi(s, t_0) = \mathbf{E}_1[\hat{\mathbf{s}}(\mathbf{Y}_1)]$. Letting $h(t)$ denote the solution of (2.6) with initial condition $h(0) = c$, $q < c < 1$, one can show that $h(nt_0)$ is a solution of (2.1) for the $\mathbf{Y}$ process. One way to do this is to consider the function $A(s) = \phi[h(s+t), s]$. It follows that $A'(s) \equiv 0$, which implies that $A(s) \equiv A(0)$. Consequently,

$$\phi[h(s+t), s] = h(t) \qquad \text{and so}$$
$$h(nt_0) = \phi[h((n+1)t_0), t_0] .$$

By Seneta's result $[h(nt_0)]^{Y_n} = [h(nt_0)]^{X_n t_0}$ converges to a nontrivial random variable, which must be W.

D. Multi-type process-continuous time. We modify Example B by letting particle "i" wait on exponential time with parameter b_i before splitting according to the generating function $f^i(s_1, \cdots, s_N)$.

(i) V-harmonic functions. We are now interested in solutions of

$$\frac{dh'}{dt} = B(I - M)h' \tag{2.7}$$

(written in vector form), where B is a diagonal matrix with entries b_i and M is the matrix of first moments as before. Letting $C = B(M - I)$ we have that

$$h'(t) = e^{-Ct}h'(0) .$$

Consequently,

$$H(\mathbf{X}_t, t) = \mathbf{X}_t \cdot h(t) = \mathbf{X}_t e^{-Ct}h(0)$$

is a martingale for any choice $h(0)$. In particular, if we choose $h(0)$ to be a right-eigenvector ξ of C with associated eigenvalue λ (i.e., ξ satisfies $C\xi' = \lambda\xi'$), then $e^{-\lambda t}\mathbf{X}_t \cdot \xi$ is a martingale. From this K. Athreya [1] shows under the assumption of positive regularity and non-singularity that $\mathbf{X}_t \cdot e^{-\rho t} \to Wu$ w.p. 1 where W is a nonnegative random variable, ρ is the maximum (real) eigenvalue of C and u is an appropriately normalized left-eigenvector associated with ρ. As in the discrete-time case, W is nondegenerate iff $\rho > 1$ and $\mathbf{E}_i[X_1^k \log X_1^k] < \infty$ all $1 \leqq i, k \leqq N$.

(ii) Λ-harmonic functions. In this case we have the system of equations

$$\frac{dh_i}{dt} = b_i[h_i(t) - f^i(h_1(t), \cdots, h_N(t))] \qquad i = 1, \cdots, N . \tag{2.8}$$

The solutions of (2.4) and (2.8) will be investigated in the future. It is believed that these solutions will provide the analogous Seneta-Heyde results for the multitype process. One can also consider the space-time harmonic functions for an age-dependent process. Some initial work has been undertaken in this direction and the results will be published in a future paper.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF PITTSBURGH
PITTSBURGH, PENNSYLVANIA 15260