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A CHARACTERIZATION OF THE KERNEL $\lim_{\lambda\downarrow 0} V_\lambda$ FOR SUB-MARKOVIAN RESOLVENTS (V_λ)[1]

BY J. C. TAYLOR

McGill University

Let (V_λ) be a sub-Markovian resolvent of kernels V_λ on a measurable space $(E, \mathscr{E})$. Assume that $V = \lim_{\lambda\downarrow 0} V_\lambda$ is a proper kernel. The proper kernels V on $(E, \mathscr{E})$ that are of the form $V = \lim_{\lambda\downarrow 0} V_\lambda$, (V_λ) a sub-Markovian resolvent of kernels on $(E, \mathscr{E})$, are proved to be precisely those proper kernels V which satisfy the complete maximum principle and for which the following condition holds: there exists an increasing sequence $(A_n) \subset \mathscr{E}$ with $\bigcup_n A_n = E$ such that (i) $V1_{A_n} < \infty$ for all n; and (ii) if $f \in \mathscr{E}^+$ and $Vf < \infty$ then $\inf_n R_{\mathscr{C}A_n} Vf < \infty$, where $R_B u = \inf\{v \text{ supermedian} \mid u \geqq v \text{ on } B\}$.

Introduction. Let (V_λ) be a sub-Markovian resolvent of kernels V_λ on a measurable space $(E, \mathscr{E})$. Assume that $V = \lim_{\lambda\downarrow 0} V_\lambda$ is a proper kernel. In this article a proof of the following result is given.

THEOREM 1. *The proper kernels V on $(E, \mathscr{E})$ that are of the form $V = \lim_{\lambda\downarrow 0} V_\lambda$, (V_λ) a sub-Markovian resolvent of kernels on $(E, \mathscr{E})$ are precisely those proper kernels V which satisfy the complete maximum principle and for which the following condition holds: there exists an increasing sequence $(A_n) \subset \mathscr{E}$ with $\bigcup_n A_n = E$ such that*

(i) $V1_{A_n} < \infty$ *for all n; and*

(ii) *if $f \in \mathscr{E}^+$ and $Vf < \infty$ then $\inf_n R_{\mathscr{C}A_n} Vf = 0$, where $R_B u = \inf\{v$ supermedian $\mid v \geqq u$ on $B\}$.*

A proof that this condition on V implies that V has the desired form is to be found in [9]. P.-A. Meyer conjectured that this condition was not only sufficient but necessary and suggested how one might use Ray processes and potentials of class (D) to obtain a proof. The author obtained a proof by this method but later received a preprint from F. Hirsch proving a similar result by non-probabilistic methods. In this article a slight adaptation of an idea of Hirsch is used to give a quick proof of the result.

PROOF OF THEOREM 1 (necessity). In view of Proposition 1 in [1], it suffices to show that if $f \in \mathscr{E}^+$ (the nonnegative $\mathscr{E}$-measurable functions) is bounded and strictly positive with Vf bounded (finite will do) then there exists an increasing sequence $(A_n) \subset \mathscr{E}$ with $\bigcup_n A_n = E$ such that (i) and (ii) are satisfied for this function.

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Let (λ_n) be a sequence decreasing to zero. Then, if $A_n' = \{V_{\lambda_n} f \geqq n^{-1}\}$ Hirsch showed in [2] that $\inf_n R_{\mathscr{C}A_n'} Vf = 0$. This follows from the fact that on $\mathscr{C}A_n'$, $Vf = V_{\lambda_n} f + \lambda_n V V_{\lambda_n} f \leqq n^{-1} + u_n$ with $u_n = Vf - V_{\lambda_n} f$.

Now $\bigcup_n A_n' = \{Vf > 0\}$. Let $N = \{Vf = 0\}$. Then, by the complete maximum principle and the fact that f is strictly positive, $V1_N = 0$. Let $A_n = A_{n'} \cup N$. Then, $\bigcup_n A_n = E$ and $V1_{A_n} = V1_{A_n'} \leqq nVV_{\lambda_n} f < \infty$. Clearly, $R_{\mathscr{C}A_n} Vf \leqq R_{\mathscr{C}A_n'} Vf$.

The result of Hirsch. The trick of adding the set N to the sets used by Hirsch shows that Hirsch's theorem is equivalent to the following result.

THEOREM 2 (cf. Theorem 1 of [2]). *Let V be a proper kernel on $(E, \mathscr{E})$ that satisfies the complete maximum principle and let $a \in \mathscr{E}^+$ be finite. The following conditions are equivalent* (*where* $M_a g = ag$):

(i) *there is a family of kernels* $(V_\lambda)_{\lambda>0}$ *such that*
  (a) $0 < \lambda < \mu$ *implies* $V_\lambda = V_\mu + (\mu - \lambda) V_\lambda M_a V_\mu$ *and*

$$V_\lambda M_a V_\mu = V_\mu M_a V_\lambda$$

  (b) $V = \lim_{\lambda \downarrow 0} V_\lambda$; *and*

(ii) *there is an increasing sequence* $(A_n) \subset \mathscr{E}$ *with* $\bigcup_n A_n = E$ *such that*
  (a) $V(a1_{A_n}) < \infty\ \forall n$; *and*
  (b) *if* $f \in \mathscr{E}^+$ *and* $Vf < \infty$ *then* $\inf_n R_{\mathscr{C}A_n} Vf = 0$.

Obviously, Hirsch's theorem is more general than Theorem 1 (let $a = 1$) but in fact Theorem 1 and its proof imply Theorem 2.

Let $W = VM_a$ and $W_\lambda = V_\lambda M_a$. Then, if a has no zeros, Theorem 1 applied to W yields Theorem 2. Assume $F = \{a > 0\} \neq E$. Set $\bar{a}(x) = a(x)$ if $x \in F$ and $= 1$ if $x \notin F$. Then $a = 1_F \bar{a}$ and if $\bar{V} = VM_{\bar{a}}$ we have $W = \bar{V} M_F$ ($M_F = 1_F$) and $W_\lambda = \bar{V}_\lambda M_F$. Hence, to deduce Theorem 2 from Theorem 1 it suffices to consider the case where $a = 1_F$, $F \in E$.

OUTLINE OF PROOF OF THEOREM 2 (the case where $a = 1_F$).

(i) $\Rightarrow$ (ii). The argument above that establishes the corresponding implication in Theorem 1 applies virtually without change. Instead of $u_n = \lambda_n V V_{\lambda_n} f$ one has $u_n = \lambda V_n M_F V_{\lambda_n} f$.

(ii) $\Rightarrow$ (i). If $W = VM_F$ then by Theorem 1 there is a sub-Markovian resolvent (W_λ) with $W = \lim_{\lambda \downarrow 0} W_\lambda$.

Define V_λ by setting $V_\lambda f = (I - \lambda W_\lambda) V(f \cdot 1_{\mathscr{C}F}) + W_\lambda f$ for $f \in \mathscr{E}^+$ with $Vf < \infty$. Then it is easy to see that (V_λ) satisfies condition (i)(a) of Theorem 2 and further that $V = V_\lambda + \lambda W_\lambda V$ for all $\lambda > 0$. It remains to show $V = \lim_{\lambda \downarrow 0} V_\lambda$.

Assume $u = Vf$ is bounded. If $x_0 \in E$ and $\varepsilon > 0$ then there is a V-supermedian function s and $t \geqq 1$ with $s(x_0) < \varepsilon$ and $u \leqq s + u1_{A_r}$, where $(A_n) \subset \mathscr{E}$ is the sequence given by condition (ii) of Theorem 2. Note that V-supermedian functions are also W-supermedian.

The estimate (*) in [1] (line 7 of page 89) can be applied with $(W_\lambda{}^n)$ the resolvent corresponding to W^n (instead of V^n as in [1]). This gives, where K_p is defined so that $W(x_0, 1_{A_r \setminus K_p} u) \leqq \varepsilon$ (as in [1]),

$$\begin{aligned}\lambda W_\lambda{}^n(x_0, u) &\leqq 2\lambda\varepsilon + \lambda W_\lambda{}^n(x_0, 1_{K_p} u)\\ &\leqq \lambda[2\varepsilon + \|u\| W(x_0, K_p)]\end{aligned}$$

with p independent of n.

Let m be given. Then $\lambda W_\lambda(x_0, u 1_{K_m}) \leqq \lambda[2\varepsilon + \|u\| W(x_0, K_p)]$. This follows since $\lim_{n\to\infty} \lambda W_\lambda{}^n(x_0, u 1_{K_m}) = \lambda W_\lambda(x_0, u 1_{K_m})$. Hence, $\lambda W_\lambda(x_0, u) \leqq \lambda[2\varepsilon + \|u\| W(x_0, K_p)]$.

Consequently, Vf bounded implies $\lim_{\lambda\to 0} V_\lambda f = Vf$. Hence, (i)(b) holds in Theorem 2.

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Department of Mathematics
McGill University
Montreal, Canada