The Annals of Probability
1975, Vol. 3, No. 2, 329-337

DECOMPOSITION OF FUNCTIONS OF BOUNDED VARIATION[1]

By Gary L. Grunkemeier

Oregon State University

Cramer's theorem, that a normal distribution function (df) has only normal components, is extended to a case where the components are allowed to be from a subclass (B_1) of the functions of bounded variation other than the class of df's. One feature of B_1 is that it contains more of the df's than the classes for which previous similar extensions have been made; in particular it contains the Poisson df's so that a first extension of Raikov's theorem, that a Poisson df has only Poisson components, in the same direction, is also given.

1. Introduction. Cramér's theorem states that if $F_1 * F_2 = \Phi$ and if F_1 and F_2 are distribution functions (df's) then they must be normal. The class of df's is a subclass of the class BV of functions of bounded variation on the real line, and certain extensions of Cramér's theorem have dealt with decompositions of Φ in which the components are allowed to be from subclasses of BV other than the class of df's.

Linnik and Skitovic (1958) have shown that if functions G_1 and G_2 are in BV, $G_j(x) = 1 - G_j(-x)$ at all continuity points (symmetry), $\int_{-\infty}^{\infty} dG_j(x) = 1$, and

$$\text{(1.1)} \qquad \operatorname{Var} G_j(x, \infty) = O(\exp(-x^{1+\alpha})) ,$$
$$\operatorname{Var} G_j(-\infty, -x) = O(\exp(-x^{1+\alpha})) ,$$

as $x \to \infty$, for some $\alpha > 0$, $j = 1, 2$, and if $G_1 * G_2$ is a normal df, then G_1 and G_2 are both normal df's [$\operatorname{Var} G(a, b)$ is the total variation of $G(x)$ in (a, b)].

Chistyakov (1970) notes that Linnik and Skitovic have not actually generalized Cramér's theorem, even for the symmetric case, since not all df's satisfy the condition (1.1) which they impose on the components. He then offers such a generalization; namely to the class of functions G in BV for which $\int_{-\infty}^{\infty} dG(x) = 1$ and which admit the representation $G(x) = \omega(x) - \sigma(x)$, where ω and σ are both symmetric, σ satisfies (1.1), and $\int_{-\infty}^{\infty} \exp(-yx)\, dG(x) \neq 0$ for all real y.

Among the functions in BV, the df's have the property of monotonicity, which is essential for the proof of certain decomposition theorems. In the absence of this property, a restriction such as condition (1.1) has been needed to get significant results. Condition (1.1) on a function G implies that its Fourier–Stieltjes (F–S) transform is an entire function of finite order $\rho \leqq 1/\alpha + 1$ (see Laha (1964), Lemma 1). This excludes many important classes of df's, in particular Poisson

Received September 28, 1973; revised June 14, 1974.

[1] This paper is based on the author's doctoral dissertation at the Catholic University of America and was prepared with the partial support of NSF grant GP-22585.

AMS 1970 *subject classifications*. Primary 60E05; Secondary 42A72, 42A96.

Key words and phrases. Bounded variation, Cramér's theorem, decomposition, Fourier–Stieltjes transform, Raikov's theorem.

df's, which have entire F–S transforms of infinite order. Thus no analogous extensions have been made of Raikov's theorem that a Poisson df has only Poisson components.

A true generalization of Cramér's theorem would be for a subclass of BV which includes all df's as a proper subclass, and would require a definition which is mild enough to include all monotonic functions, yet strong enough to permit the dropping of all growth conditions such as (1.1). In this paper we introduce a new condition, ultimately positive (u.p.), on the tail behavior of functions in BV which is in some sense a generalization of the property of monotonicity and which all df's satisfy (Definition 2.2). Using this we are able to relax the condition (1.1) to a weaker one (2.1) which implies only that the F–S transform is entire, with no order restriction. Thus we define a new subclass B_1 of BV (Definition 2.3) which, by a result of Ramachandran [(1962) Corollary 2 of Theorem 4.1], contains all df's whose ch.f.'s are entire functions of any order. Then we obtain the following extension of Cramér's theorem: If $G_1 * G_2$ is normal and the components are in B_1 then they must be normal df's (Theorem 3.1). We also extend a theorem of Pólya to the class B_1 (Theorem 4.1). Finally we offer a partial extension of Raikov's theorem to the class B_1, one which requires an additional condition in the hypothesis (Theorem 5.1).

2. The class B_1.

DEFINITION 2.1. Let B be the class of functions G in BV on the real line for which $G(-\infty) = 0$, $G(\infty) = 1$, and

$$\operatorname{Var} G(-\infty, -x) + \operatorname{Var} G(x, \infty) = O(e^{-rx}) \tag{2.1}$$

holds for all real $r > 0$, as $x \to \infty$.

LEMMA 2.1. *If G is in B, then the* F–S *transform of G*

$$g(z) = \int_{-\infty}^{\infty} e^{izx}\, dG(x) \tag{2.2}$$

exists for all complex z and defines an entire function.

PROOF. Let $G = P - N$, where P and N are bounded, non-decreasing, right-continuous real valued functions such that $P(-\infty) = N(-\infty) = 0$ and $\operatorname{Var} G = \operatorname{Var} P + \operatorname{Var} N$; then; by (2.1),

$$P(-x) + P(\infty) - P(x) = O(e^{-rx}) \quad \text{and}$$
$$N(-x) + N(\infty) - N(x) = O(e^{-rx})$$

both hold for all $r > 0$, as $x \to \infty$. Assume $N(\infty) \neq 0$, since otherwise G is simply a df. Then $P_1(x) = P(x)/P(\infty)$ and $N_1(x) = N(x)/N(\infty)$ are df's,

$$P_1(-x) + 1 - P_1(x) = O(e^{-rx}) \quad \text{and}$$
$$N_1(-x) + 1 - N_1(x) = O(e^{-rx})$$

for all real $r > 0$ as $x \to \infty$, and hence their F–S transforms, $p_1(z)$ and $n_1(z)$

respectively, are entire functions (see Lukacs (1970) page 198). Therefore $g(z) = P(\infty) \cdot p_1(z) - N(\infty) \cdot n_1(z)$ is also an entire function.

LEMMA 2.2. *If G is in B, then for* $\mathrm{Im}(z) > 0$, *the* F–S *transform* (2.2) *of G can be written*

$$g(z) = -iz \int_{-\infty}^{\infty} e^{izx} G(x)\, dx\,; \tag{2.3}$$

and for $\mathrm{Im}(z) < 0$,

$$g(z) = iz \int_{-\infty}^{\infty} e^{izx}[1 - G(x)]\, dx\,. \tag{2.4}$$

PROOF. The F–S transform (2.2) converges for all complex z by Lemma 2.1. The proof is then deduced by straightforward substitutions from the analogous result for Laplace–Stieltjes transforms given by Widder [(1946) Theorems 3c and 3d].

An example of Laha [(1964) page 80] shows that condition (2.1) alone is not sufficient to insure that if the product of two F–S transforms of functions in B are entire then the factors themselves must be entire. This requirement for an additional condition motivates the following definition:

DEFINITION 2.2. A real-valued function f is *ultimately positive* (u.p. or u.p. (x_0)) if there exists an $x_0 < \infty$ such that

(i) $f(x) \geqq 0$ for all $x \geqq x_0$, and
(ii) $f(x_1) > 0$ for at least one finite value $x_1 > x_0$.

In the remainder of the paper we adopt the convention that whenever an upper case letter stands for a function in BV, its lower case counterpart denotes the F–S transform of the function; also z is reserved for the complex variable whose real and imaginary parts are t and y, respectively.

LEMMA 2.3. *If G is in class B and $G(-x)$ is* u.p. $(-a_1)$, *then there exists y_1, $0 < y_1 < \infty$, such that $y > y_1$ implies*

(i) $g(iy) > \exp(-ya_1)$, *and*
(ii) $|g(z)| \leqq A_1|z|g(iy)$ *for some constant* $A_1 > 0$.

PROOF. Assume $y > 0$. Then (2.3) implies

$$g(iy) = y \int_{-\infty}^{a_1} e^{-yx} G(x) dx + y \int_{a_1}^{\infty} e^{-yx} G(x)\, dx\,. \tag{2.5}$$

By condition (i) of the definition of u.p. $(-a_1)$, $G(x) \geqq 0$ for x in $(-\infty, a_1)$. By condition (ii) of the definition of u.p. $(-a_1)$, there exists an x_1, $-\infty < x_1 < a_1$, and a $\delta > 0$ such that $G(x_1) = \delta > 0$, and since $G(x)$ is right-continuous, there exists an $\varepsilon > 0$ such that $G(x) \geqq \delta/2$ for all x in $(x_1, x_1 + \varepsilon)$. Assume $x_1 + \varepsilon \leqq a_1$; then

$$\begin{aligned} y \int_{-\infty}^{a_1} e^{-yx} G(x)\, dx \geqq y \int_{x_1}^{x_1+\varepsilon} e^{-yx} G(x)\, dx &\geqq (\delta/2) \exp[-y(x_1 + \varepsilon)](e^{\varepsilon y} - 1) \\ &\geqq (\delta/2) \exp(-ya_1)(e^{\varepsilon y} - 1)\,. \end{aligned} \tag{2.6}$$

Let $M = \max[|G(x)| : x \text{ in } (a_1, \infty)] < \infty$. Then

$$\left|y \int_{a_1}^{\infty} e^{-yx} G(x)\, dx\right| \leqq M \exp(-ya_1)\,. \tag{2.7}$$

Together (2.5), (2.6) and (2.7) yield

$$g(iy) \geqq \exp(-ya_1)[(\delta/2)(e^{\varepsilon y} - 1) - M], \tag{2.8}$$

which implies (i) of the lemma, since there exists a $y_1 < \infty$ such that $y > y_1$ implies that the coefficient of $\exp(-ya_1)$ in (2.8) is greater than one.

For the proof of (ii), take x_0 and M as above, and again assume $y > 0$. Then (2.3) implies

$$|g(z)| \leqq |z| \int_{-\infty}^{\infty} e^{-yx}|G(x)|\,dx. \tag{2.9}$$

Since for x in $(-\infty, a_1)$, $|G(x)| = G(x)$, and using (2.5),

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-yx}|G(x)|\,dx &= \int_{-\infty}^{\infty} e^{-yx}G(x)\,dx - \int_{a_1}^{\infty} e^{-yx}(G(x) - |G(x)|)\,dx \\ &\leqq \frac{|g(iy)|}{y} + \frac{2M \cdot \exp(-ya_1)}{y}. \end{aligned} \tag{2.10}$$

Using (i) of this lemma, there exists $y_1 < \infty$ such that for $y > y_1$,

$$\int_{-\infty}^{\infty} e^{-yx}|G(x)|\,dx \leqq g(iy)(1 + 2M)/y_1 = g(iy) \cdot A_1, \tag{2.11}$$

which, together with (2.9), proves (ii) of the lemma.

LEMMA 2.4. *If G is in class B and $1 - G(x)$ is* u.p. (a_2), *then there exists y_2, $-\infty < y_2 < 0$, such that $y < y_2$ implies*

(i) $g(iy) > \exp(ya_2)$, *and*
(ii) $|g(z)| \leqq A_2|z|g(iy)$ *for some constant* $A_2 > 0$.

PROOF. Assume $y < 0$. The proof is then completely analogous to the proof of Lemma 2.3, using formula (2.4) in place of (2.3).

DEFINITION 2.3. G is in the class B_1 if G is in B and both $G(-x)$ and $1 - G(x)$ are u.p.

The following theorem is the key upon which the extension of Cramér's theorem will depend.

THEOREM 2.1. *If G is in B_1 there exists a horizontal strip, $y_1 > y > y_2$, containing the real axis, outside of which*

(i) $g(iy) > \exp(-|y|a)$ *and*
(ii) $|g(z)| \leqq A|z|g(iy)$,

where a and A are positive constants.

PROOF. This is a combination of the previous two lemmas. Letting $A = \max(A_1, A_2)$ and $a = \max(|a_1|, |a_2|)$ gives the theorem.

3. Extension of Cramér's theorem.

THEOREM 3.1. *If G_1 and G_2 are in the class B_1 and $G_1 * G_2$ is a normal* df, *then both G_1 and G_2 are normal* df's.

PROOF. By the convolution theorem, the product $g_1(t) \cdot g_2(t) = \exp(-t^2/2)$ is the F–S transform or characteristic function (ch.f.) of the normal df. And the identity theorem for entire functions implies that

$$g_1(z) \cdot g_2(z) = \exp(-z^2/2) \tag{3.1}$$

holds for all complex $z = t + iy$. In particular on the imaginary axis,

$$g_1(iy) \cdot g_2(iy) = \exp(y^2/2) . \tag{3.2}$$

By property (i) of Theorem 2.1 applied to g_1 there exists a positive constant a and an interval I_1 such that for y outside of I_1 $g_1(iy) > \exp(-a|y|)$, which, by (3.2), implies that

$$g_2(iy) < \exp(y^2/2) \cdot \exp(a|y|) \tag{3.3}$$

holds for y outside the interval I_1. Then by property (ii) of Theorem 2.1 applied to g_2, there exists an interval I_2 and a positive constant A such that for y outside of I_2

$$|g_2(z)| \leqq A|z|g_2(iy) . \tag{3.4}$$

Then for y outside of $I = I_1 \cup I_2$, (3.3) and (3.4) together imply

$$|g_2(z)| \leqq A|z| \exp(y^2/2 + a|y|) ,$$

which implies that $g_2(z)$ is bounded by $A \cdot r \cdot \exp(ar + r^2/2)$ in the circle $|z| \leqq r$. For y inside of I, $g_2(z)$ is bounded by a constant since $|g_2(z)| \leqq p_2(iy) + n_2(iy) \leqq C_1 + C_2$, where G_2 is the difference of monotone functions P_2 and N_2. The constants C_1 and C_2 are the maximum values of $p_2(iy)$ and $n_2(iy)$, respectively, for y in the interval I. (These maximums are in fact taken at an endpoint of I since p_2 and n_2 are constant multiples of ch.f.'s and are therefore convex on the imaginary axis.) Thus $g_2(z)$ is of finite order $\rho \leqq 2$, and (3.1) shows that g_2 has no zeros; hence, Hadamard's factorization theorem (see Titchmatsh (1939) page 250) implies that

$$g_2(z) = \exp(b_0 + b_1 z + b_2 z^2) . \tag{3.5}$$

Since $g_2(0) = 1$, $b_0 = 0$. The Hermitian property $g_2(t) = g_2(-t)$ for real t implies b_1 is purely imaginary and b_2 is real. Since $g_2(t)$ must be bounded for real t, b_2 cannot be positive. Thus (3.5) becomes $g_2(t) = \exp(ict - dt^2)$, where c is real and d is nonnegative, and g_2 and also g_1 must be ch.f.'s of normal df's.

4. Another property of class B_1. Pólya (1949) showed that a necessary and sufficient condition for a function G in BV to be constant outside of an interval (L, R) is that g should be an entire function of exponential type, and that if (L, R) is the smallest such interval, then

$$L = -\limsup_{y\to\infty} (\ln|g(iy)|/y) , \qquad \text{and} \tag{4.1}$$

$$R = \limsup_{y\to\infty} (\ln|g(-iy)|/y) . \tag{4.2}$$

Ramachandran (1964) has shown that if G is also a df with ch.f. g, then the "lim sup" may be replaced by "lim" in both (4.1) and (4.2). The following theorem shows that this is also permissible when G is in B_1, a result which will be used to prove Theorem 5.1.

THEOREM 4.1. *If G is in B_1 and is constant on $[-\infty, L]$ and if L is the largest number for which this holds, then*

$$L = -\lim_{y\to\infty} (\ln g(iy)/y) . \tag{4.3}$$

Similarly, if G is in B_1 and is constant on $[R, \infty)$, and if R is the smallest number for which this holds, then

$$R = \lim_{y\to\infty} (\ln g(-iy)/y) . \tag{4.4}$$

PROOF. Note that because of (i) of Theorem 2.1, the absolute value signs on g may be omitted. It is sufficient to prove (4.3), the proof of (4.4) being analogous. And because of (4.1), it is sufficient to show that

$$L \geqq -\liminf_{y\to\infty} (\ln g(iy)/y) , \tag{4.5}$$

or, that given $\varepsilon > 0$, there exists y_0 such that $y > y_0$ implies

$$g(iy) \geqq \exp[-(L + \varepsilon)y] . \tag{4.6}$$

Let L be, as in the hypothesis, the left extremity of G, and let $\varepsilon > 0$ be given. Since G is in B_1, $G(-x)$ is u.p. $(-x_0)$ for some x_0 which, by the definition of u.p., must satisfy $L < x_0 = L + \delta$ for some $\delta > 0$. Take $\varepsilon_1 = \min(\varepsilon, \delta)$. Then $G(-x)$ is also u.p. $[-(L + \varepsilon_1)]$, and Lemma 2.3 implies there exists a y_0 such that for $y > y_0$,

$$g(iy) > \exp[-y(L + \varepsilon_1)] \geqq \exp[-y(L + \varepsilon)] ,$$

which implies (4.6).

5. Extension of Raikov's theorem.

THEOREM 5.1. *If G_1 and G_2 are in B_1 and are lattice-like with unit span and $G_1 * G_2$ is a Poisson* df, *then both G_1 and G_2 are Poisson* df's.

PROOF. Since

$$g_1(z) \cdot g_2(z) = \exp[\lambda(e^{iz} - 1)] \tag{5.1}$$

for some $\lambda > 0$,

$$g_1(iy) \cdot g_2(iy) = \exp[\lambda(e^{-y} - 1)] .$$

By taking logarithms (principal branch), dividing by y (assumed positive), and letting $y \to \infty$, the right side of the above equation goes to zero; thus

$$\lim_{y\to\infty} \frac{\ln g_1(iy)}{y} = -\lim_{y\to\infty} \frac{\ln g_2(iy)}{y} = B ,$$

say, and by Theorem 4.1 G_1 and G_2 are bounded to the left by $-B$ and B, respectively. We may assume that $B = 0$ since, if not, multiplication of the F–S transforms by nonessential factors will achieve this result. Thus G_1 and G_2 have jumps only at the nonnegative integers and

$$g_1(z) = \sum_{k=0}^{\infty} a_k e^{izk}, \qquad g_2(z) = \sum_{k=0}^{\infty} b_k e^{izk}, \qquad \text{say, where}$$
$$G_1(x) = \sum_{k=0}^{[x]} a_k, \qquad G_2(x) = \sum_{k=0}^{[x]} b_k,$$

and a_k and b_k may be positive, zero, or negative. The variable $w = e^{iz}$ transforms $g_1(z)$ and $g_2(z)$ into

$$h_1(w) = \sum_{k=0}^{\infty} a_k w^k \qquad \text{and} \qquad h_2(w) = \sum_{k=0}^{\infty} b_k w^k,$$

respectively, and from (5.1),

$$h_1(w) \cdot h_2(w) = \exp[\lambda(w-1)]. \tag{5.2}$$

For convenience we define the following notation:

$$T_G(x) = \operatorname{Var} G(-\infty, -x) + \operatorname{Var} G(x, \infty).$$

Then $T_{G_1}(N) = O(\exp(-rN))$ as $N \to \infty$ for any real $r > 0$ by the definition of B_1, and

$$|a_N| \leqq \sum_{k=N}^{\infty} |a_k| = T_{G_1}(N-1)$$

implies that

$$|a_N| = O(\exp[-r(N-1)])$$

as $N \to \infty$ for any $r > 0$. Thus there exists an $M < \infty$ (assume $M > 1$) and a positive integer N_0 such that $N > N_0$ implies

$$|a_N|^{1/N} < M^{1/N} \cdot \exp(r/N - r).$$

Given any $\varepsilon > 0$, take r_0 such that

$$\exp(r_0) > (M^{1/N_0} \cdot e)/\varepsilon.$$

Then for $N > \max(r_0, N_0)$

$$|a_N|^{1/N} < M^{1/N_0} \cdot \exp(1 - r_0) < \varepsilon$$

and thus by Hadamard's formula the radius of convergence of $h_1(w)$ is infinite. Thus $h_1(w)$ and (similarly for) $h_2(w)$ are entire functions and, by (5.2), have no zeros.

The next step shows that h_1 and h_2 have finite orders at most one. By definition, $w = \exp(iz)$ and $z = t + iy$. If $w = r \cdot \exp(i\theta)$ is the polar representation of w, then $r = \exp(-y)$, $t = \theta$, and $g_j(iy) = h_j(r)$, $j = 1, 2$. Also for nonzero z, $|z| = |\ln w|$ (principal branch). By substitution in Lemma 2.4, there exists an r_0, $1 < r_0 < \infty$, such that $r > r_0$ implies

(i) $h_j(r) > r^{-\alpha_j}$ and
(ii) $|h_j(w)| \leqq A_j |\ln w| h_j(r)$,

$j = 1, 2$. From (5.2), $h_1(r) \cdot h_2(r) = \exp[\lambda(r-1)]$, and, using (i) and (ii),

$$|h_1(w)| \leqq A_1 |\ln w| \exp[\lambda(r-1)] \cdot r^{\alpha_2}.$$

Thus $h_1(w)$ is of finite order at most one, and similarly for $h_2(w)$. Therefore, by Hadamard's factorization theorem, $h_j(w) = \exp(c_j + d_j w)$, that is, $g_j(z) = \exp(c_j + d_j e^{iz})$, and $g_j(0) = 1$ implies $c_j = -d_j$; thus,

$$g_j(z) = \exp[d_j(e^{iz} - 1)] . \tag{5.3}$$

For $z = t$ (real), the Hermitian property implies d_j is real, and to complete the proof we now show by contradiction that d_j cannot be negative. Suppose that d_1, say, is in fact negative. Equation (5.3) implies

$$h_1(w) = \exp[d_1(w - 1)] = \exp(-d_1) \sum_{k=0}^{\infty} (d_1)^k w^k/k! .$$

Thus the coefficients a_k of the power series definition of h_1 must satisfy

$$a_k = \exp(-d_1) \cdot (d_1)^k/k! , \tag{5.4}$$

and hence are positive for even k and negative for odd k, since d_1 is assumed to be negative. Also, for $k > |d_1|$, the coefficients (5.4) are decreasing in absolute value as k increases. Thus each time $N > |d_1|$ is increased by one, the series

$$S(N) = \sum_{k=N+1}^{\infty} a_k$$

changes sign. But $S(N) = 1 - G_1(N)$. Thus $1 - G_1(x)$ continues to oscillate indefinitely about zero as $x \to \infty$ and is therefore not u.p., and G_1 cannot be in B_1. This contradiction implies d_1 is nonnegative, and similarly for d_2; hence G_1 and G_2 are Poisson df's.

Whether the additional hypothesis in this theorem, that the components must be lattice-like with unit span, can be eliminated is undecided. In the case of df's, the components of a lattice-like df with unit span inherit this property; this is no longer true in general when the components are allowed to be from B_1, as many simple examples show. However in the particular case of the Poisson df, we have not found such an example.

Acknowledgment. The author greatly appreciates the valuable advice and sustaining encouragement recieved from Professor Eugene Lukacs.

REFERENCES

[1] Chistyakov, G. P. (1970). A generalization of theorems due to H. Cramér and Yu. V. Linnik-V. P. Skitovic. *Theor. Probability Appl.* **15** 331–336.

[2] Laha, R. G. (1964). On the decomposition of a class of functions of bounded variation. *Canadian J. Math.* **16** 479–484.

[3] Linnik, Yu. V. and Skitovic, V. P. (1958). A further contribution on generalizations of Cramér's theorem (in Russian). *Vestnik Leningrad Univ.* **13** 39–44. Note—for corrected version of Theorem 4 see page 101 of: Linnik, Yu. V. (1960). *Decomposition of Probability Distributions.* Oliver and Boyd, London.

[4] Lukacs, E. (1970). *Characteristic Functions.* Griffin, London.

[5] Pólya, G. (1949). Remarks on characteristic functions. *Proc. Berkeley Symp. Math. Statist. Prob.* ed. J. Neyman. Univ. of California, Berkeley.

[6] Ramachandran, B. (1962). Application of a theorem of Mamay's to a denumerable α-decomposition of the Poisson law. *Publ. Inst. Statist. Univ. Paris.* **13** 13–19.

[7] TITCHMARSH, E. C. (1939). *The Theory of Functions*. Oxford University Press, Oxford.
[8] WIDDER, D. V. (1946). *The Laplace Transform*. Princeton Univ. Press.

DEPARTMENT OF STATISTICS
OREGON STATE UNIVERSITY
CORVALLIS, OREGON