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A UNIFORM THEORY FOR SUMS OF MARKOV CHAIN TRANSITION PROBABILITIES

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Necessary and sufficient conditions are given for boundedness of $\sup_n \|\sum_{k=1}^n (P^k(x, \cdot) - P^k(y, \cdot))\|$ and $\sup_n \|\sum_{k=1}^n (P^k(x, \cdot) - \pi\|$, where the norm is total variation and where π is an invariant probability measure. Also conditions for convergence of $\sum_{k=1}^\infty (P^k(x, \cdot) - \pi)$ in norm are given. These results require the study of certain "small sets." Two new types are introduced: uniform sets and strongly uniform sets, and these are related to the sets introduced by Harris in his study of the existence of σ-finite invariant measure.

0. Introduction. The basic technique in the analysis of recurrent Markov chains on a discrete state space is a renewal method, dividing the chain into interblocks between its returns to a designated state (see Blackwell and Freedman (1964), Cogburn (1971) for a discussion of interblocks). When the state space is continuous (nonatomic), one may attempt to approximate this method, replacing the state by a suitably chosen "small set." Chapter 1 of Orey (1971) includes a discussion of two types of small sets that have been important in the development of the theory: the first really general theory for chains on a continuous state space (Doblin (1937), Doob (1953)) makes use of a type of small set called a C-set by Orey, while the type of small set introduced by Harris (1956) in his proof of the existence of σ-finite invariant measure for recurrent chains is called a D-set by Orey and in the following discussion.

The main purpose of the following study is to develop conditions for boundedness and convergence of sums of transition probabilities in total variation norm. These results require the introduction of a new type of small set. They also require finite invariant measure. When the invariant measure is infinite, it is possible to establish some related results for sums of transition probabilities, but not of the uniform type (corresponding to the variational norm) discussed here. See Duflo (1970), Metivier (1969), (1972), Neveu (1971), Orey (1971) for a discussion of results of this type, including convergence of $\sum P^n f$ for certain special functions f.

The interblocks of a Markov chain, between its returns to a specified recurrent set, are themselves a Markov chain. In general these interblocks are not independent unless the specified return set is an atom, but when this set is a D-set the interblock chain is pointwise strong mixing. This fact is sufficient to extend

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many results true for discrete state spaces to the continuous case. Other results (even for discrete state spaces) require some type of probabilistic bound on the return times to the specified set, or, equivalently, on the length of the inter-blocks. This leads to the formulation of new types of small sets. The theory for sums of transition probabilities considered below is a case in point. The type of small set required will be called a "strongly uniform set". These sets have considerably stronger properties than D-sets, and there is another type of small set, called a "uniform set" in the discussion to follow, that serves as a natural intermediary between D-sets and strongly uniform sets. Uniform sets have some interest in themselves and facilitate the study of strongly uniform sets.

Section 1 introduces the basic notation and recalls a few results necessary to the following discussion. Section 2 studies the more obvious relations between D-sets, uniform sets and strongly uniform sets. Section 3 goes more deeply into the properties of strongly uniform sets and a related regularity property of states. Section 4 relates uniform sets to compact sets when the state space is topological. And Section 5 provides the theory for sums of transition probabilities.

The results in Section 5 make it desirable to know when strongly uniform sets exist and how to identify them. This is the main purpose for the discussion in the previous sections. Typically, the readily accessible information about a chain includes little more than the one-step transition probabilities and the invariant measure. When the transition probability satisfies a Feller type of stability condition, the results in Section 4 often make it possible to identify both uniform and strongly uniform sets to the precompacts. See Duflo (1970), Foguel (1968, 1969A) for some related results.

1. Preliminaries. Consider a Markov chain $X_0, X_1, X_2, \cdots$ taking values in a measurable state space $(\mathscr{X}, \mathscr{A})$ with one step transition probability $P(x, A)$ and n step transition probability $P^n(x, A)$.

Let M denote the Banach space of bounded real valued measurable functions on $(\mathscr{X}, \mathscr{A})$ with supremum norm, and let Ψ denote the Banach space of finite signed measures on $(\mathscr{X}, \mathscr{A})$ with total variation norm. The n step transition probability defines an operator on M and on Ψ in the usual way by the equations

$$P^n f(x) = \int f(y) P^n(x, dy)$$
$$\Psi P^n(A) = \int P^n(x, A)\Psi(dx) .$$

The transition probabilities are assumed to satisfy the Chapman–Kolmogorov equation, or, equivalently, to define (discrete parameter) semigroups on M and Ψ.

Frequent use will be made (without further comment) of the following relations, valid for $\mu, \nu \in \Psi$ with $\mu(\mathscr{X}) = \nu(\mathscr{X})$:

$$\begin{aligned} \|\mu - \nu\| &= \sup_{f \in M: \|f\| \leq 1} \left|\int f\,d\mu - \int f\,d\nu\right| \\ &= 2 \sup_{A \in \mathscr{A}} |\mu(A) - \nu(A)| = 2 \sup_{A \in \mathscr{A}} (\mu(A) - \nu(A)) . \end{aligned}$$

A measure π (finite or not) is *invariant* if $\pi = \pi P$. *The symbol π will be used exclusively to denote an invariant measure.*

A probability distribution is determined for the chain $X_0, X_1, X_2, \cdots$ by specifying an *initial distribution*, namely a probability distribution μ for X_0 (of course, $\mu \in \Psi$). This distribution on the chain will be denoted P_μ, and the corresponding expectation E_μ (applied to real valued, measurable functions of the chain). When $\mu = \delta_x$ is the probability degenerate at x, P_μ and E_μ will be denoted P_x and E_x, respectively.

A *stopping time* is a function τ on the chain to the set $\{0, 1, 2, \cdots\} \cup \{\infty\}$ such that $[\tau = n]$ is a set in the σ-field determined by $(X_0, \cdots, X_n)$ for each integer $n \geqq 0$. A particularly important example of stopping times are the return times for a set $A \in \mathscr{A}$: let τ_A be the first $k \geqq 1$ such that $X_k \in A$, if any, and let $\tau_A = \infty$ if no $X_k \notin A$ for $k \geqq 1$. Then τ_A is the *first return to time to A*. This terminology will be used whether the process starts in A or not. Similarly, let $\tau_A{}^{(n)}$ be the first $k > \tau_A{}^{(n-1)}$ such that $X_k \in A$, or ∞ if no such k exists. Then $\tau_A{}^{(n)}$ is the nth *return time to A*. This notation will be used throughout; also the standard notations

$$L(x, A) = P_x[\tau_A < \infty] = P_x(\textstyle\bigcup_{n=1}^\infty [X_n \in A])$$
$$Q(x, A) = P_x(\textstyle\bigcap_{n=1}^\infty [\tau_A{}^{(n)} < \infty]) = P_x([X_n \in A \text{ infinitely often}])$$
$$A^0 = \{x : L(x, A) = 0\}$$

and A^c for $\mathscr{X} - A$.

For any stopping time τ, a "transition probability" P^τ is defined by

$$P^\tau(x, A) = P_x[\tau < \infty, X_\tau \in A] = \textstyle\sum_{n=1}^\infty P_x[\tau = n, X_n \in A] .$$

The strong Markov property is valid for Markov chains and will be used occasionally below. (See, for example, Loève (1963).)

A set $A \in \mathscr{A}$ is *inessential* if $Q(x, A) = 0$ for every $x \in \mathscr{X}$, and otherwise A is *essential*. A countable union of inessential sets is *null*. A null essential set is said to be *improperly essential*. A non-null essential set is *properly essential*, also termed *positive* in what follows.

It is convenient to formulate certain hypotheses in terms of an arbitrary (but fixed) nontrivial, σ-finite measure φ on $(\mathscr{X}, \mathscr{A})$. *The symbol φ will be used exclusively for this measure.*

A chain is *φ-recurrent*, if, for every $A \in \mathscr{A}$ with $\varphi(A) > 0$, $L(x, A) = 1$ for every $x \in \mathscr{X}$. (Such a chain is also said to be recurrent in the sense of Harris. The present terminology follows Orey (1971).

BASIC HYPOTHESIS. *It is assumed throughout this study that the chain is φ-recurrent.*

In Doblin's (1940) terminology, this is equivalent to saying that $\mathscr{X}$ is an *ensemble final*. Harris (1955) shows that a φ-recurrent chain has a σ-finite invariant measure π, unique up to a multiplicative constant. (Harris assumed $\mathscr{A}$ countably generated, and this assumption was eliminated by Jamison and Orey (1967).) The following result is stated for convenience. See Orey (1971) for a discussion.

PROPOSITION 1.1. *A set $A \in \mathscr{A}$ is null if, and only if, $\pi(A) = 0$. Also, the measure φ is dominated by π, so A null implies $\varphi(A) = 0$.*

The chain is *uniformly φ-recurrent* if, for every $A \in \mathscr{A}$ with $\varphi(A) = 0$

$$\sup_{x \in \mathscr{X}} P_x[\tau_A \geqq n] \to 0$$

as $n \to \infty$. See Orey (1971) for a discussion of uniform φ-recurrence, particularly his Proposition 6.1. As he notes, uniform φ-recurrence is closely related to condition (D) of Doob (1953).

The next lemma is basic to several key results in this study.

LEMMA 1.1. *Let the chain be uniformly φ-recurrent and $\varepsilon > 0$. Then there exist constants $a < \infty$, $0 < b < 1$, (depending on ε) such that, for every $A \in \mathscr{A}$ with $\varphi(A) \geqq \varepsilon$*

$$\sup_{x \in \mathscr{X}} P_x[\tau_A \geqq n] \leqq ab^n .$$

The proof is essentially that of Lemma 5.1 of Chapter V of Doob (1953).

Say that a set $A \in \mathscr{A}$ is *recurrent* if $L(x, A) = 1$ for every $x \in A$. For recurrent sets it is often of interest to consider the *chain on A*, namely the sequence X_0, $X_{\tau_A(1)}, X_{\tau_A(2)}, X_{\tau_A(3)}, \cdots$, where X_0 is restricted to A and the chain on A takes values in the state space $(A, \mathscr{A}_A)$, where $\mathscr{A}_A = \{B \in \mathscr{A}\colon B \subset A\}$. This chain has transition probability $P_{(A)}(x, B) = P^{\tau_A}(x, B)$.

Under the basic hypothesis that the chain is φ-recurrent, a set $A \in \mathscr{A}$ is recurrent if, and only if, $\pi(A) > 0$, which, in turn, is equivalent to A being properly essential. Moreover, if $0 < \pi(A) < \infty$, then the chain on A has the invariant probability measure π_A defined by $\pi_A(B) = \pi(AB)/\pi(A)$, $B \in \mathscr{A}_A$, and the chain on A is π_A recurrent.

Certain results from Doblin's general theory of Markov chains (Doblin (1940)) and its later developments will be used in the sequel and referred to as required. For some comprehensive discussions of this theory see Chung (1964), Jain and Jamison (1967) and, especially, Orey (1971). Also, from a different viewpoint, Foguel (1969B) is of interest.

2. Small sets. A small set is one having some kind of uniform property. Three types are considered here. As in Section 1, φ is an arbitrary nontrivial, σ-finite measure on $\mathscr{A}$ such that the chain is φ-recurrent, and π denotes the σ-finite invariant measure (unique up to a multiplicative constant).

DEFINITION 2.1. A set $A \in \mathscr{A}$ is a *D-set for φ* if the process on A is uniformly φ-recurrent. (This requires $\varphi(A) > 0$.)

DEFINITION 2.2. A set $A \in \mathscr{A}$ is *φ-uniform* if, for every B with $\varphi(B) > 0$,

$$\sup_{x \in A} P_x[\tau_B \geqq n] \to 0$$

as $n \to \infty$.

DEFINITION 2.3. A set $A \in \mathscr{A}$ is *φ-strongly uniform* if, for every B with $\varphi(B) > 0$,

$$\sup_{x \in A} E_x \tau_B < \infty .$$

When these properties hold for φ replaced by π, the state set A will be said simply to be a *D-set*, *uniform* or *strongly uniform* in cases 1, 2 or 3, respectively.

NOTE 2.1. Since $\varphi \ll \pi$, in general the concepts with φ replaced by π appear stronger. In fact, they are equivalent for cases 1 and 2 and also for case 3 if π is finite, as will be seen below.

NOTE 2.2. By Markov's inequality,

$$\sup_{x \in A} P_x[\tau_B \geqq n] \leqq \frac{1}{n} \sup_{x \in A} E_x \tau_B .$$

It follows that any φ-strongly uniform set is φ-uniform. When $\varphi(A) > 0$, it is also true that A being φ-uniform implies A is a D-set for φ (Proposition 2.4 below).

PROPOSITION 2.1. *If A is a D-set for φ then A is a D-set (for π).*

This result is included for completeness. For a discussion see Orey (1971).

PROPOSITION 2.2. *If A is φ-uniform then A is uniform.*

PROOF. Let $\pi(B) > 0$. Then $P_x[\tau_B \geqq n] \to 0$ as $n \to \infty$ for every x and these functions of x are $\mathscr{A}$ measurable. By Egoroff's theorem, there is a φ-positive set C with $\sup_{x \in C} P_x[\tau_B \geqq n] \to 0$ as $n \to \infty$. But then, using the strong Markov property at time τ_C,

$$\sup_{x \in A} P_x[\tau_B \geqq m + n] \leqq \sup_{x \in A} P_x[\tau_C \geqq m] + \sup_{y \in C} P_y[\tau_B \geqq n] \to 0$$

as $m, n \to \infty$, hence A is uniform. $\square$

COROLLARY 2.1. *If $\varphi(A) > 0$ and, for every $B \subset A$ with $\varphi(B) > 0$,*

$$\sup_{x \in C} P_x[\tau_B \geqq n] \to 0$$

as $n \to \infty$, then C is uniform.

PROOF. Let $\varphi^*(B) = \varphi(AB)$. Then the chain is φ^*-recurrent and C is φ^*-uniform, hence uniform. $\square$

PROPOSITION 3. *If the chain has a D-set for φ, then there exist uniform sets A_1, $A_2, \cdots$ such that $A_1 \subset A_2 \subset \cdots$ and $\mathscr{X} - \bigcup_{n=1}^{\infty} A_n$ is null.*

PROOF. Whether π is finite or σ-finite, there is a finite measure μ equivalent to π.

Let B be a D-set for φ. Then B is recurrent, and, for every $k \geqq 1$, $P_x[\tau_B^{(k)} \geqq n] \to 0$ as $n \to \infty$ for every x, hence almost uniformly $-\mu$. Hence, for each k and $m \geqq 1$, there is a state set $B_{m,k}$ such that $\mu(\mathscr{X} - B_{m,k}) \leqq 1/m2^k$ and $\sup_{x \in B_{m,k}} P_x[\tau_B^{(k)} \geqq n] \to 0$ as $n \to \infty$. Let $B_m = \bigcap_{k=1}^{\infty} B_{m,k}$. Then $\mu(\mathscr{X} - B_m) \leqq 1/m$.

Now for any $C \subset B$ with $\varphi(C) > 0$, applying Lemma 1.1,

$$\begin{aligned} \sup_{x \in B_m} P_x[\tau_C \geqq n] &\leqq \sup_{x \in B_m} P_x[\tau_C \geqq \tau_B^{(k)}] + \sup_{x \in B_m} P_x[\tau_B^{(k)} \geqq n] \\ &\leqq ab^k + \sup_{x \in B_m} P_x[\tau_B^{(k)} \geqq n] , \end{aligned}$$

where $0 < b < 1$. Letting $n \to \infty$ then $k \to \infty$, the right-hand side converges to 0. By Corollary 2.1, each B_m is uniform. The sets $A_n = \bigcup_{k=1}^{n} B_k$ then satisfy the assertions. $\square$

Orey (1959), (1971) shows that, if $\mathscr{A}$ is countably generated, then there exist D-sets $D_1, D_2, \cdots$ such that $D_1 \subset D_2 \subset \cdots$ and $\bigcup D_n = \mathscr{X}$. Thus when $\mathscr{A}$ is countably generated Proposition 2.3 is in force. It is not true, in general, that the uniform A_n can be chosen so $\bigcup A_n = \mathscr{X}$. The following quite artificial example suffices to show why this is true.

EXAMPLE 1. Let $N = \{1, 2, 3, \cdots\}$ and S be the set of nonnegative sequences $s = \{s_n\}$ on N such that $\sum s_n = 1$. Of course, S is a measurable subset of $[0, 1]^N$, with its product Borel field. Let $\mathscr{S}$ be the σ-field of Borel subsets of S. Now let $\mathscr{X} = N \cup S$ and $\mathscr{A}$ consist of all subsets of $\mathscr{X}$ of form $M \cup T$ where $M \subset N$ and $T \in \mathscr{S}$. Note that $\mathscr{A}$ is countably generated. Now define the transition probability by $P(\{s_n\}, \{k\}) = s_k$ for $\{s_n\} \in S$ and $k \in N$; $P(n, \{n-1\}) = 1$ for $n \in N$ with $n \geqq 2$, and let $P(1, \{1\}) = 1$. Then π is degenerate at 1, $\mathscr{X} - \{1\}$ is null and $\{1\}$ is trivially uniform. But the existence of a sequence of uniform A_n such that $\bigcup A_n = \mathscr{X}$ would imply that S can be partitioned into a countable number of subsets $S = \bigcup S_n$ such that the series $\sum s_n$ converges uniformly on each S_n, and this is not possible.

PROPOSITION 2.4. *Let $\varphi(A) > 0$. Then A is φ-uniform if, and only if, A is a D-set for φ and*

$$\sup_{x \in A} P_x[\tau_A \geqq n] \to 0$$

as $n \to \infty$.

PROOF 1. ($\Rightarrow$). Let $B \subset A$ and $\varphi(B) > 0$. Then, since $\tau_A{}^{(n)} \geqq n$,

$$\sup_{x \in A} P_x[\tau_B \geqq \tau_A{}^{(n)}] \leqq \sup_{x \in A} P_x[\tau_B \geqq n] \to 0$$

as $n \to \infty$, hence the process on A is uniformly φ-recurrent. □

PROOF 2. ($\Leftarrow$). Let $B \subset A$ and $\varphi(B) > 0$. By Lemma 1.1, there exist $a < \infty$, $0 < b < 1$ such that $P_x[\tau_B \geqq \tau_A{}^{(n)}] \leqq ab^n$ for every n. Also, using Boole's inequality and the strong Markov property,

$$\sup_{x \in A} P_x[\tau_A{}^{(n)} \geqq nk] \leqq n \sup_{x \in A} P_x[\tau_A \geqq k] .$$

But then

$$\sup_{x \in A} P_x[\tau_B \geqq nk] \leqq ab^n + n \sup_{x \in A} P_x[\tau_A \geqq k] \to 0$$

as $k \to \infty$ then $n \to \infty$. By Corollary 2.1, A is uniform. □

A φ-recurrent chain always has a finite period $d \geqq 1$, and when $d = 1$ the chain is said to be aperiodic. When the chain has an invariant probability measure π, Orey (1959), (1971) has proven that

$$\delta(n, x) = \left\| \frac{1}{d} \sum_{k=0}^{d-1} P^{n+k}(x, \cdot) - \pi \right\| \to 0$$

as $n \to \infty$ for every $x \in \mathscr{X}$. Note that, if $\mathscr{A}$ is countably generated, then $\mathscr{A}$ has a countable field $\mathscr{A}_0 \subset \mathscr{A}$ and generating $\mathscr{A}$, and

$$\delta(n, x) = 2 \sup_{A \in \mathscr{A}_0} \left| \frac{1}{d} \sum_{k=0}^{d-1} P^{n+k}(x, A) - \pi(A) \right| .$$

It follows that $\delta(n, \cdot)$ is measurable in x in this case. Moreover, by Egoroff's theorem, $\delta(n, \cdot) \to 0$ π-almost uniformly as $n \to \infty$. In general δ need not be measurable nor converge uniformly on any π-positive set. (See example in Section 3.)

It will be useful to consider also

$$\gamma(n, x) = \left\| \frac{1}{n} \cdot \sum_{k=1}^{n} P^k(x, \cdot) - \pi \right\| .$$

THEOREM 2.1. *Let π be an invariant probability measure. Then the following are equivalent for an $A \in \mathscr{A}$:*

(i) *A is uniform.*
(ii) $\sup_{x \in A} \delta(n, x) \to 0$ *as* $n \to \infty$.
(iii) $\sup_{x \in A} \gamma(n, x) \to 0$ *as* $n \to \infty$.

PROOF 1. $(1 \Rightarrow 2)$. Let A be uniform. Assume first that $\mathscr{A}$ is countably generated. Then there exists a set B with $\pi(B) > 0$ such that $\sup_{x \in B} \delta(n, x) \to 0$ as $n \to \infty$. Now for any $C \in \mathscr{A}$ and $n \geqq m$,

$$\begin{aligned}
&\left| \frac{1}{d} \sum_{k=0}^{d-1} P^{n+k}(x, C) - \pi(C) \right| \\
&\qquad = \left| E_x \left\{ (\chi_{[\tau_B > m]} + \sum_{j=1}^{m} \chi_{[\tau_B = j]}) \left(\frac{1}{d} \sum_{k=n}^{n+d-1} \chi_{[X_k \in C]} - \pi(C) \right) \right\} \right| \\
&\qquad \leqq P_x[\tau_B > m] + E_x \left\{ \sum_{j=1}^{m} \chi_{[\tau_B = j]} \left| \frac{1}{d} \sum_{k=n-j}^{n+d-1-j} P^k(X_j, C) - \pi(C) \right| \right\} \\
&\qquad \leqq P_x[\tau_B > m] + \sup_{y \in B, k \geqq n-m} \delta(k, y) .
\end{aligned}$$

If A is uniform, then

$$\tfrac{1}{2} \sup_{x \in A} \delta(n, x) \leqq \sup_{x \in A} P_x[\tau_B > m] + \sup_{y \in B, k \geqq n-m} \delta(k, y) \to 0$$

as $n \to \infty$ then $m \to \infty$.

If $\mathscr{A}$ is not countably generated, then there exists a countable set $A_0 \subset A$ such that

$$\sup_{x \in A_0} \delta(n, x) = \sup_{x \in A} \delta(n, x)$$

for every n. Furthermore, there exists an admissible countably generated σ-field $\mathscr{A}_1 \subset \mathscr{A}$ such that, for every $x \in A_0$ and n,

$$\delta(n, x) = 2 \sup_{C \in \mathscr{A}_1} \left(\frac{1}{d} \sum_{k=0}^{d-1} P^{n+k}(x, C) - \pi(C) \right) .$$

But then the first part of the proof applies to the chain on $(\mathscr{X}, \mathscr{A}_1)$ and implies that $\sup_{x \in A_0} \delta(n, x) \to 0$.

PROOF 2. $(2 \Rightarrow 3)$. Obvious.

PROOF 3. $(3 \Rightarrow 1)$. Let $\sup_{x \in A} \gamma(n, x) \to 0$. First note that for any $C \in \mathscr{A}$,

$n \geqq 1$,

$$\chi_{[\tau_C \geqq n]} \leqq \frac{1}{n} \sum_{k=1}^{n} \chi_{[X_k \notin C]}$$

so

$$P_x[\tau_C \geqq n] \leqq \frac{1}{n} \sum_{k=1}^{n} (1 - P^k(x, C)) \leqq 1 - \pi(C) + \gamma(n, x) .$$

Now given B with $\varphi(B) > 0$ and $\varepsilon > 0$, let

$$C_m = \{x : P_x[\tau_B \geqq m] \leqq \varepsilon\} .$$

Since the chain is φ-recurrent, $C_m \uparrow \mathscr{X}$ and there exists an m such that $\pi(C_m) \geqq 1 - \varepsilon$. But then

$$\begin{aligned} \sup_{x \in A} P_x[\tau_B \geqq m + n] &\leqq \sup_{x \in A} P_x[\tau_{C_m} \geqq n] + \sup_{y \in C_m} P_y[\tau_B \geqq m] \\ &\leqq \sup_{x \in A} \gamma(n, x) + 2\varepsilon . \end{aligned}$$

The first term converges to 0 as $n \to \infty$ and $\varepsilon > 0$ is arbitrary, so it follows that A is φ-uniform and (by Proposition 2.2) uniform. □

The concluding result of this section shows that φ-strongly uniform sets are of interest only when π is finite.

PROPOSITION 2.5. *If A is a φ-strongly uniform set with $\varphi(A) > 0$, then the invariant measure π is finite.*

PROOF. The standard Harris proof of existence of a σ-finite invariant measure π shows that the chain on a D-set A has an invariant probability measure π_A. (See Orey (1971) for a version not requiring that $\mathscr{A}$ be countably generated.) A σ-finite invariant measure π is then defined on $\mathscr{X}$ by

$$\pi(B) = \int_A \sum_{n=1}^{\infty} P_x[\tau_A \geqq n, X_n \in B]\pi_A(dx) .$$

But then

$$\begin{aligned} \pi(\mathscr{X}) &= \int_A \sum_{n=1}^{\infty} P_x[\tau_A \geqq n]\pi_A(dx) = E_{\pi_A}\tau_A \\ &\leqq \sup_{x \in A} E_x \tau_A < \infty \end{aligned}$$

under the stated hypotheses on A. □

3. Regular states and strongly uniform sets. *Throughout this section the chain is assumed to have invariant probability measure π.* If $\pi(A) > 0$, then by a well-known formula of Kac (1947)

$$\int_A E_x \tau_A \pi(dx) = 1 . \tag{3.1}$$

Later on in this section a generalized form of Kac's formula is established (Lemma 3.4). It follows from (1) that $E_x \tau_A < \infty$ except for x in a null subset of A. In fact, this is true globally.

PROPOSITION 3.1. *If $\pi(A) > 0$, then $E_x \tau_A < \infty$ except for x in a null set.*

PROOF. Let $D = \{x \in A : E_x \tau_A = \infty\}$. Then $\pi(D) = 0$, and the set $C = D^0 D^C$ is stochastically closed with $\pi(C) = 1$ (see Orey (1971), Chung (1964)). Let

$B = AC$. Then, for every $x \in B$, $E_x\tau_B = E_x\tau_A < \infty$. Now let $F = \{x \in \mathscr{X}: E_x\tau_B = \infty\}$. Then $B \cap F = \emptyset$ and if $P_x[\tau_F < \tau_B] > 0$, then $E_x\tau_B = \infty$. Hence $P_x[\tau_F > \tau_B] = 1$ for every $x \in B$, and it follows that, for $n = 1, 2, 3, \cdots$ and every $x \in B$,

$$P_x[\tau_F > n] \geqq P_x[\tau_F > \tau_B^{(n)}] = 1 .$$

But then $L(x, F) = 0$ for $x \in B$. Since $\pi(B) = \pi(A) > 0$, B is nonempty. But then F is null (see Orey (1971) Proposition 8.1, for example), and $E_x\tau_A < \infty$ for $x \notin F$ since $\tau_A \leqq \tau_B$. □

DEFINITION. A state x is *φ-regular* if $E_x\tau_B < \infty$ for every state set B with $\varphi(B) > 0$. When this property holds for φ replaced by π it will be said simply that x is *regular*. Let $\mathscr{R}$ denote the set of regular states.

NOTE 3.1. A state x is φ-regular iff $\{x\}$ is φ-strongly uniform. Also every element of a φ-strongly uniform set must be φ-regular.

NOTE 3.2. If $\pi(\{x\}) > 0$ for some state x, then Proposition 3.1 implies that x is regular. Thus when $\mathscr{X}$ is countable π-a.e. state is regular. But in general no regular state need exist since the null set of x such that $E_x\tau_B = \infty$ (for $\pi(B) > 0$) can vary with the set B.

EXAMPLE. Blackwell (1945) constructed an example to show that an indecomposable $\mathscr{X}$ need have no final subset. (The chain is φ-recurrent for some φ if, and only if, $\mathscr{X}$ is a final set.) He let $\mathscr{X} = [0, \infty)$, $\mathscr{A}$ be the class of all countable subsets of $\mathscr{X}$ and their complements, and $\pi(A) = 0$ or 1 according as A is countable or uncountable. The transition probability, for $n \leqq x < n + 1$, is given by

$$P(x, \{x + 1\}) = 1 - p_n$$
$$P(x, A) = p_n \pi(A) \qquad \text{for } (x + 1) \notin A .$$

Then $P(x, A)$ is $\mathscr{A}$ measurable for every $A \in \mathscr{A}$, π is invariant and, if all $p_n > 0$, then $\mathscr{X}$ is indecomposable. Also the chain is aperiodic ($d = 1$). If $\sum p_n < \infty$, then $P^n(x, \cdot) \not\to \pi$, and, of course, the chain is not φ-recurrent.

For the present purpose assume that $\sum p_n = \infty$ but $p_n \to 0$ as $n \to \infty$. In this case $\mathscr{X}$ is π-recurrent and, for $m \leqq x < m + 1$,

$$\delta(n, x) = ||P^n(x, \cdot) - \pi|| = 2 \prod_{j=m}^{m+n-1} (1 - p_j) .$$

Then $\delta(n, x) \to 0$ as $n \to \infty$ for each x, but $\delta(n, x) \to 1$ as $x \to \infty$ for each n. It follows that there is no measurable, properly essential A such that $\sup_{x \in A} \delta(n, x) \to 0$ as $n \to \infty$. But then Theorem 2.1 implies that the chain has no uniform set A with $\pi(A) > 0$, and by Proposition 2.3 the chain has no D-set. Of course, $\delta(n, \cdot)$ is not $\mathscr{A}$ measurable.

Consider now the special case $p_n = 1/(n + 2)$. For each x, let $A_x = \{x, x + 1, x + 2, \cdots\}$. Then $\pi(A_x^C) = 1$ and, for $n \leqq x < n + 1$

$$E_x\tau_{A_x^C} = 1 + \sum_{k=1}^{\infty} \prod_{j=n}^{n+k-1} (1 - p_j) = 1 + (n + 1) \sum_{k=1}^{\infty} \frac{1}{n + k + 1} = \infty .$$

Thus no x is regular. If, instead, $p_n = (n + 2)^{-\frac{1}{2}}$, then an example is obtained where every x is regular. But for either choice of p_n, the chain has no D-set and no uniform or strongly uniform set A with $\pi(A) > 0$.

An important question remains open: when π is finite and $\mathscr{A}$ is countably generated, can the chain have no regular state? If the answer is negative, then it would follow from Theorem 3.1 below that the chain must have properly essential, strongly uniform sets in this case. (The theorem requires that $\pi(\mathscr{R}) > 0$, but if $\pi(\mathscr{R}) = 0$, then $\mathscr{R}^0\mathscr{R}^C$ would have no regular state.)

LEMMA 3.1. *For $A, B \in \mathscr{A}$, and every $x \in \mathscr{X}$,*

$$E_x\tau_B \leqq E_x\tau_A + \sup_{y\in A} E_y\tau_B .$$

PROOF. Let τ be the first time the process enters B after entering A. Then $\tau \geqq \tau_B$ and, using the strong Markov property,

$$E_x\tau_B \leqq E_x\tau = E_x\tau_A + E_x(E_{X_{\tau_A}}\tau_B) \leqq E_x\tau_A + \sup_{y\in A} E_y\tau_B . \qquad \square$$

LEMMA 3.2. *If B is strongly uniform and*

$$\sup_{x\in A} E_x\tau_B < \infty ,$$

then A is strongly uniform.

This follows immediately from Lemma 3.1.

PROPOSITION 3.2. *If $A \in \mathscr{A}$ is φ-strongly uniform, then A is strongly uniform. If a state x is φ-regular then x is regular.*

PROOF. Let A be φ-strongly uniform and $\pi(B) > 0$. Then $E_x\tau_B = 0$ for x outside a null set C. Since $\varphi(C) = 0$, there exists a set D with $\varphi(D) > 0$ and $\sup_{x\in D} E_x\tau_B < \infty$. But then, if A is φ-strongly uniform, $\sup_{x\in A} E_x\tau_D < \infty$, and by Lemma 3.1, $\sup_{x\in A} E_x\tau_B < \infty$. Hence A is strongly uniform. The assertion for a state x follows directly from this and Note 3.1 above. $\square$

PROPOSITION 3.3. *Let A be properly essential. Then A is strongly uniform if, and only if, A is a D-set and* $\sup_{x\in A} E_x\tau_A < \infty$.

PROOF 1. If A is strongly uniform and properly essential, then $\pi(A) > 0$ and $\sup_{x\in A} E_x\tau_A < \infty$. Also A is uniform and, by Proposition 2.4, is a D-set.

PROOF 2. Let $\pi(B) > 0$. Then $E_x\tau_B < \infty$ for π-a.e. x, hence there exists a set $C \subset A$ with $\pi(C) > 0$ and $\sup_{x\in C} E_x\tau_B < \infty$. Let $\Delta\tau_A^{(n)} = \tau_A^{(n+1)} - \tau_A^{(n)}$ and consider the equality

$$\tau_C = \tau_A + \sum_{n=1}^{\infty} \chi_{[\tau_C > \tau_A^{(n)}]}\Delta\tau_A^{(n)} . \tag{3.2}$$

Now A is a D-set, so applying Lemma 1.1 for the set C, there exist $a < \infty$, $0 < b < 1$ such that

$$\begin{aligned}\sup_{x\in A} E_x\tau_C &\leqq \sup_{x\in A} E_x\tau_A + \sum_{n=1}^{\infty} E_x\{\chi_{[\tau_C > \tau_A^{(n)}]} E_{\chi_{\tau_A^{(n)}}}(\Delta\tau_A^{(n)})\} \\ &\leqq \sup_{x\in A} E_x\tau_A \cdot (1 + \textstyle\sum_{n=1}^{\infty} ab^n) < \infty .\end{aligned}$$

Then, by Lemma 3.1, $\sup_{x\in A} E_x\tau_B < \infty$. $\square$

PROPOSITION 3.4. *If the chain has a properly essential, strongly uniform set A, then the set of regular states is*

$$\mathscr{R} = \{x : E_x \tau_A < \infty\} .$$

Moreover, $\mathscr{R}$ is stochastically closed, $\mathscr{X} - \mathscr{R}$ is null and there exist strongly uniform $A_1, A_2, \cdots$ such that $A_1 \subset A_2 \subset \cdots$ and $\bigcup A_n = \mathscr{R}$.

PROOF. Let $A_n = \{x : E_x \tau_A \leqq n\}$. By Lemma 3.2, each A_n is strongly uniform. By the observations in Note 3.1 above,

$$\mathscr{R} \subset \{x : E_x \tau_A < \infty\} = \bigcup A_n \subset \mathscr{R} .$$

The first and last assertions follow from this. By Proposition 3.1, $\mathscr{X} - \mathscr{R}$ is null. Finally, let $P(x, \mathscr{X} - \mathscr{R}) > 0$. Then, since $A \subset \mathscr{R}$,

$$E_x \tau_A \geqq \textstyle\int_{\mathscr{X} - \mathscr{R}} E_y \tau_A P(x, dy) = \infty$$

and $x \notin \mathscr{R}$, so $\mathscr{R}$ is stochastically closed. □

LEMMA 3.3. *A state set A is strongly uniform if, and only if, there exists a $\delta > 0$ such that*

$$\sup_{x \in A, B \in \mathscr{A} : \pi(B) > 1 - \delta} E_x \tau_B < \infty .$$

PROOF 1. Let the above condition on $\sup E_x \tau_B$ hold. Given a positive C, by Proposition 3.1 there exists a set $B \in \mathscr{A}$ such that $\pi(B) > 1 - \delta$ and $\sup_{x \in B} E_x \tau_C < \infty$. Then, by Lemma 3.1,

$$\sup_{x \in A} E_x \tau_C \leqq \sup_{x \in A} E_x \tau_B + \sup_{y \in B} E_y \tau_C < \infty ,$$

and it follows that A is strongly uniform.

PROOF 2. If the condition on $\sup E_x \tau_B$ does not hold, then there exist $B_n \in \mathscr{A}$ and $x_n \in A$ such that $\pi(B_n) \to 1$ and $E_{x_n} \tau_{B_n} \to \infty$. Without loss of generality the B_n can be chosen so $\pi(B_n) > 1 - 2^{-n}$ for $n = 1, 2, 3, \cdots$. Let $B = \bigcap B_n$. Then $\pi(B) > 0$ and

$$\sup_{x \in A} E_x \tau_B \geqq \sup_n \sup_{x \in A} E_x \tau_{B_n} = \infty$$

so A is not strongly uniform. □

In general, $\mathscr{R}$ need not be measurable. Proposition 3.4 implies that $\mathscr{R}$ is measurable when the chain has a properly essential, strongly uniform set. It is also true that $\mathscr{R}$ must be measurable when $\mathscr{A}$ is countably generated, as will be shown below.

THEOREM 3.1. *Let $\mathscr{A}$ be countably generated. Then $\mathscr{R} \in \mathscr{A}$, $\pi(\mathscr{R}) = 0$ or 1, and if $\pi(R) = 1$ then the chain has properly essential strongly uniform sets, and the conclusions of Proposition 3.4 apply.*

PROOF. Let $\delta > 0$ and

$$f_\delta(x) = \sup_{A \in \mathscr{A} : \pi(A) > 1 - \delta} E_x \tau_A .$$

Let $\mathscr{A}_0$ be a countable field generating $\mathscr{A}$. Then for each $A \in \mathscr{A}$ and $x \in \mathscr{X}$, there exist $A_n \in \mathscr{A}_0$ such that $\pi(A_n) \to \pi(A)$ and $\tau_{A_n} \to \tau_A$ in P_x probability as $n \to \infty$. By Fatou's theorem, $E_x \tau_A \leqq \liminf E_x \tau_{A_n}$. It follows that

$$f_\delta(x) = \sup_{A \in \mathscr{A}_0 : \pi(A) > 1-\delta} E_x \tau_A ,$$

hence f_δ is measurable. Now let $A_{\delta,n} = \{x : f_\delta(x) \leqq n\}$. By Lemma 3.3, the $A_{\delta,n}$ are strongly uniform, and by the same lemma each regular x must be in some $A_{\delta,n}$. Thus $\mathscr{R} = \bigcup_{m,n} A_{1/m,n}$. If all the $A_{\delta,n}$ are null, then $\mathscr{R}$ is null. If $\pi(\mathscr{R}) > 0$, then some $A_{\delta,n}$ must be properly essential. But then Proposition 3.4 applies and $\pi(\mathscr{R}) = 1$.

Let f be a numerical function defined on $\{1, 2, 3, \cdots\}$ and let

$$\Delta f(n) = f(n+1) - f(n) .$$

The next result generalizes a lemma of Kac (1947). Its validity depends only on P_π being an ergodic probability measure (the assumption that $\mathscr{X}$ is φ-recurrent is not required.)

LEMMA 3.4. *Let $f(n) \uparrow$ for $n = 1, 2, \cdots$. Then for every properly essential set A,*

$$\pi(A) E_{\pi_A}(f(\tau_A)) = f(1)\pi(A) + \int_{\mathscr{X}-A} E_x(\Delta f(\tau_A))\pi(dx) . \tag{3.3}$$

PROOF. Let

$$B_0 = [X_0 \in A] ; \qquad B_n = [X_0 \notin A, \tau_A = n] , \qquad n \geqq 1$$

$$C_n = [X_0 \in A, \tau_A = n] , \qquad n \geqq 1 .$$

Then $\bigcup C_n = B_0$ a.s.-P_π since π is ergodic. Since the C_n are disjoint $\sum_1^\infty P_\pi(C_n) = P_\pi(B_0)$. By stationarity, for $k \geqq 2$,

$$P_\pi[\tau_A = k] = P_\pi[X_1 \notin A, \tau_A = k] = P_\pi(B_{k-1}) .$$

The B_n are disjoint, so $P_\pi B_n \to 0$ as $n \to \infty$, and, since $C_k = [\tau_A = k] - B_k$,

$$\sum_{k=n+1}^\infty P_\pi(C_k) = \sum_{k=n+1}^\infty \{P_\pi(B_{k-1}) - P_\pi(B_k)\} = P_\pi(B_n) ,$$

for $n \geqq 1$.

Now let $f(0) = 0$ so $\Delta f(0) = f(1)$. Then the left-hand side of (3.3) is

$$\begin{aligned}
\int_A E_x(f(\tau_A))\pi(dx) &= \sum_{k=1}^\infty f(k) P_\pi(C_k) \\
&= \sum_{k=1}^\infty P_\pi(C_k) \sum_{n=0}^{k-1} \Delta f(n) \\
&= \sum_{n=0}^\infty \Delta f(n) \sum_{k=n+1}^\infty P_\pi(C_k) \\
&= \sum_{n=0}^\infty \Delta f(n) P_\pi(B_n) \\
&= f(1)\pi(A) + \int_{\mathscr{X}-A} E_x(\Delta f(\tau_A))\pi(dx) .
\end{aligned}$$

The reordering of the summation is justified since all terms except possibly $\Delta f(0)$ are nonnegative. □

COROLLARY 3.1. *Let π be an invariant ergodic starting distribution. Then for any state set A with $\pi A > 0$,*

$$E_{\pi_A}(\tau_A) = \frac{1}{\pi A}$$

$$E_{\pi_A}(\tau_A^2) = \frac{1}{\pi A}(2E_\pi(\tau_A) - 1)$$

and, for any $r \geqq 0$, $s \geqq 0$,

$$E_\pi(\tau_A{}^r(\log \tau_A)^s) < \infty \Leftrightarrow E_{\pi_A}(\tau_A{}^{r+1}(\log \tau_A)^s) < \infty .$$

PROOF. The first two assertions are well known and are an immediate consequence of the lemma. For the last assertion, observe that, if $f(n) = n^{r+1}(\log n)^s$, then

$$\Delta f(n) \sim (r+1)n^r(\log n)^s ,$$

and apply the lemma. □

LEMMA 3.5. *Let* $0 \leqq \Delta f \leqq b$. *Let* μ *be a starting distribution such that* $E_\mu f(\tau_A) < \infty$ *for some strongly uniform* A. *Then* $E_\mu f(\tau_B) < \infty$ *for every properly essential* B.

PROOF. Let τ be the first time the chain enters B after entering A. Then

$$f(\tau_B) \leqq f(\tau) \leqq f(\tau_A) + b(\tau - \tau_A)$$

and

$$E_\mu f(\tau_B) \leqq E_\mu f(\tau_A) + b \sup_{x \in A} E_x \tau_B < \infty . \qquad \square$$

COROLLARY 3.2. *If* $E_{\pi_A}(\tau_A{}^2) < \infty$ *for some properly essential, strongly uniform* A *then* $E_{\pi_B}(\tau_B{}^2) < \infty$ *for every properly essential* B.

This is an immediate consequence of Corollary 3.1 and Lemma 3.5.

This section is concluded by showing that under a very weak moment condition the chain has properly essential, strongly uniform sets.

THEOREM 3.2. *If the chain has a D-set* A *such that* $E_{\pi_A}(\tau_A \log \tau_A) < \infty$ *(or, equivalently,* $E_\pi(\log \tau_A) < \infty$*), then the chain has properly essential, strongly uniform sets.*

PROOF. Let A be a D-set such that $E_{\pi_A}(\tau_A \log \tau_A) < \infty$. Let $\varepsilon = \pi(A)/2$ and a, b be the constants that appear in Lemma 1.1. Choose $c > 1$ so $bc < 1$ and observe that

$$\log (c\tau_A) \geqq \sum_{n=0}^\infty (\log c)\chi_{[\tau_A \geqq c^n]} . \tag{3.4}$$

(When $c^n \leqq \tau_A < c^{n+1}$, the right-hand side is $(n+1)\log c$.) Using (3.4), the monotone convergence theorem and the stationarity of $\Delta\tau_A{}^{(n)} = \tau_A{}^{(n+1)} - \tau_A{}^{(n)}$, $n = 0, 1, 2, \cdots$, under P_{π_A} (where $\tau_A{}^{(0)} = 0$),

$$\begin{aligned} \infty &> E_{\pi_A}(\tau_A \log c\tau_A) \\ &\geqq \textstyle\sum_{n=0}^\infty (\log c) \int_{[\tau_A \geqq c^n]} \tau_A \, dP_{\pi_A} \\ &= \textstyle\sum_{n=0}^\infty (\log c) \int_{[\Delta\tau_A{}^{(n)} \geqq c^n]} \Delta\tau_A{}^{(n)} \, dP_{\pi_A} \\ &= \int \{\textstyle\sum_{n=0}^\infty (\log c) \int_{[\Delta\tau_A{}^{(n)} \geqq c^n]} \Delta\tau_A{}^{(n)} \, dP_x\}\pi_A(dx) . \end{aligned}$$

Let $f(x)$ be the quantity in brackets in this last expression. Then f is measurable and finite for a.e. x in A, hence there exists a finite constant d and a state set $B \subset A$ such that $\pi(B) \geqq \pi(A)/2$ and $f(x) \leqq d$ for every $x \in B$.

Applying (3.2) and Lemma 1.1, for every $x \in B$,

$$\begin{aligned} E_x \tau_B &= E_x(\tau_A + \textstyle\sum_{n=1}^{\infty} \chi_{[\tau_B > \tau_A^{(n)}]} \Delta\tau_A^{(n)}) \\ &\leqq \frac{f(x)}{\log c} + E_x(\textstyle\sum_{n=1}^{\infty} \chi_{[\tau_B > \tau_A^{(n)}]} c^n) \\ &\leqq \frac{d}{\log c} + \textstyle\sum_{n=1}^{\infty} a(bc)^n < \infty . \end{aligned}$$

Thus $\sup_{x \in B} E_x \tau_B < \infty$, and Proposition 3.3 implies that B is strongly uniform. □

4. Uniform sets and compactness. *In this section $\mathscr{X}$ is a locally compact, separable metric space and $\mathscr{A}$ is the Borel field in $\mathscr{X}$* (the σ-field generated by the open sets).

Let C denote the set of bounded, continuous, real valued functions on $\mathscr{X}$. Of course, C is a closed subspace of the Banach space M. The transition probability P is *stable* on $\mathscr{X}$ if $Pf \in C$ for every $f \in C$. This is equivalent to the statement that the mapping $x \to P(x, \cdot)$ is continuous in distribution.

A finite measure μ on $\mathscr{A}$ is *regular* if $\mu A = \sup \mu F$, the supremum over all closed $F \subset A$, for each $A \in \mathscr{A}$. Of course, every finite measure on the Borel field of a metric space is regular and this property is essential to some of the results that follow. It is easy to generalize this theory to the situation that $\mathscr{X}$ is a T_1, normal topological space provided the transition probabilities are all regular measures on $\mathscr{X}$.

By making a very strong continuity assumption it is easy to establish that compact sets are uniform.

PROPOSITION 4.1. *Let $P(x, A)$ be continuous in x for every Borel set A. Then every compact set is uniform.*

PROOF. The hypothesis implies that $Pf \in C$ for every simple function $f \in M$. Since every function in M is the uniform limit of such simple functions, $Pf \in C$ for every $f \in M$.

Now let A be any positive set and define

$$Tf = P(\chi_{A^c} f) .$$

Then T is a linear operator on M, $Tf \in C$ for all $f \in M$ and

$$P_x[\tau_A > n] = E_x \textstyle\prod_{k=1}^{n} \chi_{A^c}(X_k) = T^n 1(x)$$

is continuous in x. Since $P_x[\tau_A > n] \downarrow 0$ for every x as n increases the convergence is uniform on compacts by Dini's lemma. □

The task now is to weaken the continuity assumption of the preceding result.

LEMMA 4.1. *Let P be stable and let F be a closed set and G be an open set. Then $P(x, G)$, $P_x[\tau_F \geqq n]$ and $E_x \tau_F$ are all lower semicontinuous in x, while $P(x, F)$ and $P_x(\tau_G \geqq n)$ are upper semicontinuous in x.*

PROOF. Since F is closed, there exist $f_k \in C$ such that $0 \leqq f_k \leqq 1$ and $f_k \downarrow \chi_F$ as $k \uparrow \infty$. Then $P(x, F) = \inf_k Pf_k(x)$ is upper semicontinuous. Now let T_k be defined on M by

$$T_k g = P((1 - f_k)g) .$$

Then $T_k g \in C$ for each $g \in C$, so

$$P_x[\tau_F \geqq n] = \sup_k E_x \prod_{j=1}^{n-1} (1 - f_k(X_j)) = \sup_k T_k^n 1(x)$$

is lower semicontinuous. Also $E_x \tau_F = \sum_{n=1}^{\infty} P_x[\tau_F \geqq n]$ is lower semicontinuous. The assertions for G follow by similar arguments. □

PROPOSITION 4.2. *Let P be stable. Then the closure of a uniform set is uniform, and the closure of a strongly uniform set is strongly uniform.*

PROOF. If $\pi(B) > 0$, then, since π is regular, there is a closed $F \subset B$ with $\pi(F) > 0$. Then, letting $\bar{A}$ denote the closure of A and applying Lemma 4.1,

$$\sup_{x \in \bar{A}} P_x[\tau_B \geqq n] \leqq \sup_{x \in \bar{A}} P_x[\tau_F \geqq n] = \sup_{x \in A} P_x[\tau_F \geqq n]$$
$$\sup_{x \in \bar{A}} E_x \tau_B \leqq \sup_{x \in \bar{A}} E_x \tau_F = \sup_{x \in A} E_x \tau_F .$$

The assertions follow directly from these relations. □

LEMMA 4.2. *Let P be stable and A be a given state set. Suppose that for each $\varepsilon > 0$, there is a positive open set G_ε and an n_ε such that*

$$\sup_{x \in G_\varepsilon} P_x[\tau_A \geqq n_\varepsilon] \leqq \varepsilon .$$

Then, for every compact K,

$$\lim_n \sup_{x \in K} P_x[\tau_n \geqq n] = 0 .$$

PROOF. Given $\varepsilon > 0$, let G_ε and n_ε be stated above. Since G_ε is positive, $P_x[\tau_{G_\varepsilon} \geqq m] \downarrow 0$ as m increases for every x. By Lemma 4.1, these probabilities are upper semicontinuous in x, so the convergence must be uniform on compacts by Dini's lemma. But then

$$\sup_{x \in K} P_x[\tau_A \geqq m + n_\varepsilon] \leqq \sup_{x \in K} P_x[\tau_{G_\varepsilon} \geqq m] + \varepsilon .$$

The right-hand side must be less than 2ε for m sufficiently large, and the lemma follows since ε is any positive number. □

A state x is a *point of increase* of a measure μ on $\mathscr{A}$ if $\mu G > 0$ for every open G containing x. The *support* of μ is the set of all its points of increase. Now it is well known, and easy to verify, that the support of μ is closed and equals the intersection of all closed sets that carry μ. Since $\mathscr{X}$ is a separable metric space, the topology is countably generated. Hence the support of μ is a countable intersection of closed sets carrying μ, and μ is carried by its support. *Let S denote the support of the stationary distribution π.*

THEOREM 4.1. *Let P be stable and S be a second category subset of $\mathscr{X}$. Then every compact set is uniform.*

PROOF. Let A be positive. Then $\pi(A) > 0$, and since π is regular, there is a closed $F \subset A$ with $\pi(F) > 0$. Now, for $\varepsilon > 0$, let

$$F_{n,\varepsilon} = \{x : P_x[\tau_F \geqq n] \leqq \varepsilon\} .$$

It follows from Lemma 4.1 that the $F_{n,\varepsilon}$ are closed. Since the chain is φ-recurrent $\bigcup_n F_{n,\varepsilon} = \mathscr{X}$. But then $\bigcup_{n=1}^{\infty} (S \cap F_{n,\varepsilon})$ is a covering of S by closed sets. The category assumption then implies that, for some n, $S \cap F_{n,\varepsilon}$ contains a nonempty open set $G_{n,\varepsilon}$.

Since $G_{n,\varepsilon}$ intersects S, $\pi(G_{n,\varepsilon}) > 0$. But $\tau_A \leqq \tau_F$, and it follows from Lemma 4.2 that, for K compact,

$$\sup_{x \in K} P_x[\tau_A \geqq n] \leqq \sup_{x \in K} P_x[\tau_F \geqq n] \to 0$$

as $n \to \infty$. □

COROLLARY 4.1. *Let P be stable and S have nonempty interior. Then every compact set is uniform.*

This is an immediate consequence of the Baire category theorem. (Even if $\mathscr{X}$ is not complete, it is locally compact and regular, hence of second category.)

EXAMPLE. Let $\mathscr{X} = [0, 1]$ with the usual metric and

$$P(x, \{0\}) = 1 - P\left(x, \left\{\frac{x}{1+x}\right\}\right) = x$$

for all $x > 0$, and let $P(0, \{0\}) = 1$. Then $S = \{0\}$ is a first category set. This P is stable and $\mathscr{X}$ is compact, but $\mathscr{X}$ is not uniform since $P_x[\tau_{\{0\}} \geqq n] \to 1$ as positive $x \to 0$ for every n.

LEMMA 4.3. *Let P be stable. Then S is stochastically closed.*

PROOF. Suppose there is an $x \in S$ such that $P(x, S^c) > 0$. Since S^c is open, $P(\cdot, S^c)$ is lower semicontinuous, hence there is an open G containing x such that $P(y, S^c) > 0$ for all y in G. Since x is a point of increase of π,

$$\pi(S^c) \geqq \int_G P(y, S^c)\pi(dy) > 0 .$$

The lemma follows *ab contrario* since π is carried by S. □

COROLLARY 4.2. *Let P be stable. Then every compact subset of S is uniform.*

PROOF. Since S is closed, it is locally compact (and regular), hence of second category in its relative topology. Since S is stochastically closed, the process on S has transition probability $P_{(S)}(x, A) \doteq P(x, A)$ for every $x \in S$ and $A \in \mathscr{A}_S$. Furthermore, by Tietze's extension theorem, any bounded continuous f on S extends to a bounded continuous $\bar{f}$ on $\mathscr{X}$. Then $P_{(S)}f(x) = P\bar{f}(x)$ for every $x \in S$, so $P_{(S)}$ is stable on S and Theorem 4.1 applies to the chain on S. □

A function f in M *vanishes at infinity* if, for every $\varepsilon > 0$, there exists a compact K such that $|f(x)| < \varepsilon$ for $x \notin K$. The next result does not require that P be stable, but a related property "at infinity" is assumed.

THEOREM 4.2. *Let the function $x \to P(x, K)$ vanish at infinity for every compact K. Then every uniform set is precompact.*

PROOF. Let K be compact and $\varepsilon > 0$. Choose compact K_1, K_2 so that $P(x, K) < \varepsilon$ for $x \notin K_1$ and $P(x, K_1) < \varepsilon$ for $x \notin K_2$. Then, for $x \notin (K_1 \cup K_2)$

$$P_x[\tau_K \leqq 2] \leqq P(x, K) + P(x, K_1) + \int_{K_1{}^c} P(y, K)P(x, dy) < 3\varepsilon .$$

It follows that $P_x[\tau_K \leqq 2]$ vanishes at infinity for every compact K, and by similar arguments, this holds for $P_x[\tau_K \leqq n]$ for every n.

Since $\mathscr{X}$ is locally compact and separable, it is σ-compact. Hence their exists a compact K with $\pi(K) > 0$. Now let A be uniform. Then, for n sufficiently large, $\sup_{x \in A} P_x[\tau_K > n] < 1$. But

$$f(x) = P_x[\tau_K \leqq n] = 1 - P_x[\tau_K > n]$$

vanishes at infinity and is bounded below by a positive number on A, hence A must be precompact. □

Under mild conditions, compact sets are uniform. When can one conclude that compacts are strongly uniform? Some condition on mean return times is required, and the following simple result may be useful.

PROPOSITION 4.3. *Let P be stable and suppose there is a compact set K such that $E_x \tau_K$ is bounded for x in compact subsets of S. Then every compact subset of S is strongly uniform.*

This follows immediately from Corollary 4.2 and Propositions 3.3, 2.4 and Lemma 3.2. Note that $E_x \tau_K$ is lower semicontinuous by Lemma 4.1. If it were upper semicontinuous (hence continuous) for some compact K, then the hypothesis on $E_x \tau_K$ would be satisfied.

5. Sums of transition probabilities. Throughout this section it is assumed that π is an invariant probability measure. Let

$$\alpha(x, y) = \sup_n \|\textstyle\sum_{k=1}^n (P^k(x, \cdot) - P^k(y, \cdot))\|$$
$$\beta(x) = \sup_n \|\textstyle\sum_{k=1}^n (P^k(x, \cdot) - \pi)\| .$$

THEOREM 5.1. *Let A be properly essential. Then* $\sup_{x,y \in A} \alpha(x, y) < \infty$ *if, and only if, A is strongly uniform.*

Several lemmas will be established first.

LEMMA 5.1. *If τ is a stopping time, then for every $n \geqq 1$, $f \in M$ and $x \in \mathscr{X}$,*

$$|P^\tau \textstyle\sum_{k=1}^n P^k f(x) - \sum_{k=1}^n P^k f(x)| \leqq 2\|f\| \cdot E_x \tau .$$

PROOF. The bound is trivial if $E_x \tau = \infty$. If $E_x \tau < \infty$, then $P_x[\tau < \infty] = 1$, and

$$\begin{aligned}|P^\tau \textstyle\sum_{k=1}^n P^k f(x) - \sum_{k=1}^n P^k f(x)| &= |E_x\{\textstyle\sum_{k=0}^\infty \chi_{[\tau=k]}(\sum_{j=k+1}^{k+n} f(X_j) - \sum_{j=1}^n f(X_j))\}| \\ &\leqq E_x\{\textstyle\sum_{k=0}^\infty 2k\|f\|\chi_{[\tau=k]}\} = 2\|f\|E_x\tau .\end{aligned}$$

LEMMA 5.2. *For each properly essential, strongly uniform A, there is a constant $b_A < \infty$ such that*

$$|\textstyle\sum_{k=1}^{n} P^k(x, B) - \sum_{k=1}^{n} \pi_A P^k(B)| \leqq b_A$$

for every $x \in A$, $B \in \mathscr{A}$ and integer $n \geqq 1$.

PROOF. If the assertion is not true, then there exist $B_n \in \mathscr{A}$ such that, letting

$$f_n(x) = \chi_A(\textstyle\sum_{k=1}^{n} P^k(x, B_n) - \sum_{k=1}^{n} \pi_A P^k(B_n)),$$

the quantities $||f_n||$ satisfy $\sup_n ||f_n|| = \infty$.

Let

$$\bar{f}_n = \frac{f_n}{||f_n||}$$

$$g_n = \frac{1}{||f_n||}\left(\chi_{B_n} - \frac{1}{n}\textstyle\sum_{k=1}^{n} \pi_A P^k(B_n)\right).$$

Then

$$\bar{f}_n = (\textstyle\sum_{k=1}^{n} P^k g_n)\chi_A,$$

and, since $P_{(A)}(x, \cdot)$ is carried by A,

$$P_{(A)}\bar{f}_n = P_{(A)}(\textstyle\sum_{k=1}^{n} P^k g_n).$$

Now, $P_{(A)} = P^{\tau_A}$, so by Lemma 5.1, for $x \in A$,

$$|P_{(A)}\bar{f}_n(x) - \bar{f}_n(x)| \leqq 2||g_n||E_x\tau_A \leqq \frac{2}{||f_n||} E_x\tau_A.$$

Let

$$\delta_n = \sup_{x\in A} |P_{(A)}\bar{f}_n(x) - \bar{f}_n(x)|.$$

Applying Jensen's inequality

$$\sup_{x\in A} |P_{(A)}^{k+1}\bar{f}_n(x) - P_{(A)}^k\bar{f}_n(x)| \leqq \sup_{x\in A} \int \delta_n P_{(A)}^k(x, dy) \leqq \delta_n$$

for every k. It follows easily that

$$\sup_{x\in A}\left|\frac{1}{m}\textstyle\sum_{k=1}^{m} P_{(A)}^k\bar{f}_n(x) - \bar{f}_n(x)\right| \leqq \frac{(m+1)\delta_n}{2}.$$

Since A is positive and strongly uniform, by Lemma 5.1

$$\inf_n \delta_n \leqq \inf_n \frac{2}{||f_n||}(\sup_{x\in A} E_x\tau_A) = 0.$$

But $\sup_{x\in A} |\bar{f}_n(x)| = ||\bar{f}_n|| = 1$, so

$$\sup_n \sup_{x\in A}\left|\frac{1}{m}\textstyle\sum_{k=1}^{m} P_{(A)}^k\bar{f}_n(x)\right| = 1.$$

Since $\pi_A(\bar{f}_n) = 0$ for each n, it follows that for each m,

$$\sup_{||f||\leq 1, x\in A}\left|\frac{1}{m}\textstyle\sum_{k=1}^{m} P_{(A)}^k f(x) - \pi_A f\right| \geqq 1.$$

But then A is not a D-set (see Theorem 7.1 part (iii) of Orey (1971)). Since Proposition 2.4 asserts that A must be a D-set, the lemma follows *ab contrario.* $\square$

The next lemma is somewhat more general than required for the theorems in this section, but may have some independent interest.

LEMMA 5.3. *Let $\pi(A) > 0$ and μ be a probability distribution on $\mathscr{A}$. Then for every positive integer n*

$$\left|E_\mu \tau_A^{(n)} - \frac{n}{\pi(A)}\right| \leqq \frac{1}{\pi(A)} \sup_{x \in A, m} |\textstyle\sum_{k=1}^m (P^k(x, A) - \mu P^k(A))| .$$

PROOF. Assume the right-hand side of the inequality is finite, since the assertion is trivial otherwise. For each integer l let $l^* = \min\{l, \tau_A^{(n)} - 1\}$. Then, taking $l \geqq n$, and using the fact that $\tau_A^{(n)}$ is a stopping time in the third equality,

$$\begin{aligned} \textstyle\sum_{k=1}^l \mu P^k(A) &= E_\mu(\textstyle\sum_{k=1}^l \chi_A(X_k)) \\ &= E_\mu(\textstyle\sum_{j=n}^l \chi_{[\tau_A^{(n)}=j]} \sum_{k=j}^l \chi_A(X_k)) + E_\mu(\sum_{k=1}^{l^*} \chi_A(X_k)) \\ &= E_\mu(\textstyle\sum_{j=n}^l \chi_{[\tau_A^{(n)}=j]} \sum_{k=0}^{l-j} P^k(X_j, A)) + E_\mu(\sum_{k=1}^{l^*} \chi_A(X_k)) . \end{aligned} \tag{5.1}$$

Now let

$$\begin{aligned} f_l(j) &= \textstyle\sum_{k=l-j+1}^l \mu P^k(A) && \text{for } j \leqq l \\ &= \textstyle\sum_{k=0}^l \mu P^k(A) && \text{for } j > l . \end{aligned}$$

Then, using (5.1) at the third equality,

$$\begin{aligned} E_\mu f_l(\tau_A^{(n)}) &= E_\mu\{\textstyle\sum_{j=n}^l \chi_{[\tau_A^{(n)}=j]} \sum_{k=l-j+1}^l \mu P^k(A)\} + P_\mu[\tau_A^{(n)} > l] \sum_{k=0}^l \mu P^k(A) \\ &= \textstyle\sum_{k=0}^l \mu P^k(A) - E_\mu\{\sum_{j=n}^l \chi_{[\tau_A^{(n)}=j]} \sum_{k=0}^{l-j} \mu P^k(A)\} \\ &= \mu(A) + E_\mu\{\textstyle\sum_{j=n}^l \chi_{[\tau_A^{(n)}=j]} \sum_{k=0}^{l-j} (P^k(X_j, A) - \mu P^k(A))\} \\ &\quad + E_\mu(\textstyle\sum_{k=1}^{l^*} \chi_A(X_k)) \\ &= E_\mu\{\textstyle\sum_{j=n}^l \chi_{[\tau_A^{(n)}=j]} \sum_{k=1}^{l-j} (P^k(X_j, A) - \mu P^k(A))\} \\ &\quad + E_\mu(\textstyle\sum_{k=1}^{l^*} \chi_A(X_k)) + 1 - P_\mu[\tau_A^{(n)} > l](1 - \mu(A)) . \end{aligned}$$

Thus

$$\begin{aligned} &|E_\mu f_l(\tau_A^{(n)}) - E_\mu(\textstyle\sum_{k=1}^{l^*} \chi_A(X_k)) - 1| \\ &\qquad \leqq \sup_{x \in A, m} |\textstyle\sum_{k=1}^m (P^k(x, A) - \mu P^k(A))| + P_\mu[\tau_A^{(n)} > l] . \end{aligned}$$

Let

$$\begin{aligned} g_j &= \frac{1}{j} \textstyle\sum_{l=n}^{n+j-1} f_l \\ Y_j &= 1 + \frac{1}{j} \textstyle\sum_{l=n}^{n+j-1} E_\mu(\sum_{k=1}^{l^*} \chi_A(X_k)) . \end{aligned}$$

Then

$$\begin{aligned} |E_\mu g_j(\tau_A^{(n)} - \gamma_j| &\leqq \sup_{x \in A, m} |\textstyle\sum_{k=1}^m (P^k(x, A) - \mu P^k(A))| \\ &\qquad + \frac{1}{j} \textstyle\sum_{l=n}^{n+j-1} P_\mu[\tau_A^{(n)} > l] . \end{aligned}$$

But, for each fixed k, $g_j(k) \to k\pi(A)$ as $j \to \infty$. Also, $\gamma_j \uparrow n$ as $j \to \infty$ and the last term on the right of the inequality converges to 0. Since $g_j \geqq 0$ for each j, it follows by Fatou's theorem that

$$\pi(A) \cdot E_\mu(\pi_A{}^{(n)}) \leqq \liminf_j E_\mu g_j(\tau_A{}^{(n)}) < \infty .$$

Since $0 \leqq g_j(\tau_A{}^{(n)}) \leqq \tau_A{}^{(n)}$, the lemma follows by the dominated convergence theorem. □

PROOF OF THEOREM 5.1. If A is strongly uniform, then, using Lemma 5.2,

$$\sup_{x,y \in A} \alpha(x, y) = 2 \sup_{x,y \in A, B \in \mathscr{A}, n} |\textstyle\sum_{k=1}^n P^k(x, B) - \sum_{k=1}^n P^k(y, B)| \leqq 4b_A .$$

Conversely, if α is bounded on A, then using the $\gamma(n, x)$ defined in Section 2

$$\sup_{x,y \in A} |\gamma(n, x) - \gamma(n, y)| \leqq \frac{1}{n} \sup_{x,y \in A} \alpha(x, y) \to 0$$

as $n \to \infty$. Since the chain is φ-recurrent, $\gamma(n, x) \to 0$ as $n \to \infty$ for each x. Hence, by Theorem 2.1, A is uniform, hence by Proposition 2.4, A is a D-set. But, by Lemma 5.3,

$$\begin{aligned}\sup_{x \in A} E_x \tau_A &\leqq \frac{1}{\pi(A)} (1 + \sup_{x,y \in A} |\textstyle\sum_{k=1}^m (P^k(y, a) - P^k(x, A))| \\ &\leqq \frac{1}{\pi(A)} (1 + \sup_{x,y \in A} \alpha(x, y)) .\end{aligned}$$

Then by Proposition 3.3, A is strongly uniform. □

COROLLARY 5.1. *If $\mathscr{A}$ is countably generated and if the set of regular states $\mathscr{R}$ is properly essential, then $\pi(\mathscr{R}) = 1$ and $\alpha(x, y) < \infty$ for every $x, y \in \mathscr{R}$. Moreover, $\alpha(x, y) = \infty$ for every pair (x, y) with $x \in \mathscr{R}$ and $y \notin \mathscr{R}$.*

PROOF. By Theorem 3.1 and Proposition 3.4 $\pi(\mathscr{R}) = 1$ and $\mathscr{R} = \bigcup A_n$ with the A_n strongly uniform and $A_n \uparrow \mathscr{R}$. Then Theorem 5.1 implies that α is finite on $\mathscr{R}$.

If $x \in \mathscr{R}$ and $y \notin \mathscr{R}$, then for A strongly uniform and properly essential, $E_y \tau_A = \infty$ by Proposition 3.4, and, by Lemma 5.3,

$$\infty = \pi(A) E_y \tau_A \leqq 1 + \sup_{z \in A} \alpha(z, y) \leqq 1 + \sup_{z \in A} \alpha(z, x) + \alpha(x, y) .$$

But $A \cup \{x\}$ is strongly uniform, so $\sup_{z \in A} \alpha(z, x) < \infty$, and necessarily $\alpha(x, y) = \infty$. □

LEMMA 5.4. *Let A be strongly uniform and $E_\pi \tau_A < \infty$. Then*

$$\beta(x) \leqq 4(E_x \tau_A + E_\pi \tau_A + b_A)$$

where b_A is the constant in Lemma 5.2.

PROOF. By the triangle inequality and Lemmas 5.1 and 5.2,

$$\begin{aligned}&|\textstyle\sum_{k=1}^n P^k(x, B) - \sum_{k=1}^n \pi_A P^k(B)| \\ &\qquad \leqq |\textstyle\sum_{k=1}^n P^k(x, B) - \int \sum_{k=1}^n P^k(y, B) P^{\tau_A}(x, dy)| \\ &\qquad\qquad + |\textstyle\int (\sum_{k=1}^n P^k(y, B) - \sum_{k=1}^n \pi_A P^k(B)) P^{\tau_A}(x, dy)| \\ &\qquad \leqq 2E_x \tau_A + b_A .\end{aligned}$$

Integrate this inequality with respect to π and use Jensen's inequality to get

$$|\sum_{k=1}^{n} \pi(B) - \sum_{k=1}^{n} \pi_A P^k(B)| \leqq 2E_\pi \tau_A + b_A .$$

These two inequalities yield

$$|\sum_{k=1}^{n} (P^k(x, B) - \pi(B))| \leqq 2(E_x \tau_A + E_\pi \tau_A + b_A)$$

and the assertion follows immediately from this. □

THEOREM 5.2. *Let A be properly essential. Then $\sup_{x \in A} \beta(x) < \infty$ if, and only if, A is strongly uniform and $E_\pi \tau_A < \infty$ (equivalently, $E_{\pi_A} \tau_A{}^2 < \infty$).*

PROOF. The "if" assertion follows directly from Lemma 5.4, since, for A strongly uniform and properly essential, $\sup_{x \in A} E_x \tau_A < \infty$. For the converse, let β be bounded on A. Since

$$\gamma(n, x) \leqq \frac{1}{n} \beta(x) ,$$

A is uniform by Theorem 2.1, hence A is a D-set by Proposition 2.4. By Lemma 5.3 with $\mu = \pi$,

$$\begin{aligned} E_\pi \tau_A &\leqq \frac{1}{\pi(A)} \{1 + \sup_{x \in A, n} |\sum_{k=1}^{n} (P^k(x, A) - \pi(A))|\} \\ &\leqq \frac{1}{\pi(A)} \{1 + \sup_{x \in A} \beta(x)\} < \infty . \end{aligned}$$

Using Lemma 5.3 again with $\mu = \delta_x$, where $x \in A$,

$$E_x \tau_A \leqq \frac{1}{\pi(A)} \{1 + \sup_{y \in A} \alpha(y, x)\} \leqq \frac{1}{\pi(A)} \{1 + 2 \sup_{y \in A} \beta(y)\} .$$

It follows that $\sup_{x \in A} E_x \tau_A < \infty$, and by Proposition 3.3, A is strongly uniform. □

COROLLARY 5.2. *If $E_\pi \tau_A < \infty$ for some strongly uniform A, then $\pi(\mathscr{R}) = 1$ and $\beta(x) < \infty$, if and only if, $x \in \mathscr{R}$.*

PROOF. Since $E_\pi \tau_A < \infty$, necessarily A is properly essential. Then, by Proposition 3.4 and Theorem 5.2, $\pi(\mathscr{R}) = 1$ and $\beta(x) < \infty$ for $x \in \mathscr{R}$. If $x \notin \mathscr{R}$, then $E_x \tau_A = \infty$, and by Lemma 5.3

$$\begin{aligned} \infty = \pi(A) E_x(\tau_A) &\leqq 1 + \sup_{y \in A, n} |\sum_{k=1}^{n} (P^k(y, A) - P^k(x, A))| \\ &\leqq 1 + \sup_{y \in A} \beta(y) + \beta(x) . \end{aligned}$$

But then $\beta(x) = \infty$ since $\sup_{y \in A} \beta(y) < \infty$ by Theorem 2. □

Orey (1971) introduces the quantity

$$D^n(\mu; A; \nu, B) = \pi(B) \sum_{k=1}^{n} \mu P^k(A) - \pi(A) \sum_{k=1}^{n} \nu P^k(B)$$

and studies the boundedness of D^n as $n \to \infty$ and its convergence (as a sharpening of ratio limit theorems) for π σ-finite. For π a probability the following result shows that D is closely related to β.

PROPOSITION 5.1. (a) *For all* x, y,

$$\tfrac{1}{2}\max(\beta(x), \beta(y)) \leqq \sup_{A,B\in\mathscr{A},n} |D^n(\delta_x, A; \delta_y, B)| \leqq \tfrac{1}{2}(\beta(x) + \beta(y)) .$$

(b) *For all initial distributions* μ, ν,

$$\sup_{A,B\in\mathscr{A},n} |D^n(\mu, A; \nu, B)| \leqq \tfrac{1}{2}\int \beta(x)(\mu(dx) + \nu(dx)) .$$

PROOF. First note that

$$D^n(\delta_x, A; \delta_y, B) = \pi(B)\sum_{k=1}^n (P^k(x, A) - \pi(A)) - \pi(A)\sum_{k=1}^n (P^k(y, B) - \pi(B)) .$$

The right-hand inequality in part (a) follows directly from this. For the left-hand inequality, first let $B = \mathscr{X}$ to get

$$\sup_{A\in\mathscr{A},n} D^n(\delta_x, A; \delta_y, \mathscr{X}) = \sup_{A\in\mathscr{A},n} \sum_{k=1}^n (P^k(x, A) - \pi(A)) = \tfrac{1}{2}\beta(x) .$$

Similarly, taking $A = \mathscr{X}$ gives a lower bound of $\beta(y)/2$.

The inequalities in part (b) follow directly from part (a) by using Jensen's inequality, first for μ replacing δ_x, then for ν replacing δ_y. □

LEMMA 5.5. *Let* $E_\pi\tau_A < \infty$. *Then, for any starting distribution* μ,

$$\pi(A)|E_\mu\tau_A - E_\pi\tau_A| \leqq ||\mu - \pi|| \sup_{x\in A}\beta(x) + \liminf_n \left|\sum_{k=1}^n (\mu P^k(A) - \pi(A))\right| .$$

PROOF. Observe that

$$\begin{aligned}\sum_{k=1}^n (\mu P^k(A) - \pi(A)) &= (E_\mu - E_\pi)\{\textstyle\sum_{k=1}^n \chi_{[\tau_A=k]} \sum_{j=k}^n \chi_{[X_j\in A]}\} \\ &= (E_\mu - E_\pi)\{\textstyle\sum_{k=1}^n \chi_{[\tau_A=k]} \sum_{j=k}^n (\chi_{[X_j\in A]} - \pi(A))\} \\ &\qquad + \pi(A)(E_\mu - E_\pi)\{\textstyle\sum_{k=1}^n (n - k + 1)\chi_{[\tau_A=k]}\} \\ &= I + II ,\end{aligned}$$

say. Now, using the strong Markov property,

$$\begin{aligned}|I| &= |(E_\mu - E_\pi)\{\textstyle\sum_{k=1}^n \chi_{[\tau_A=k]} \sum_{j=0}^{n-k} (P^j(X_k, A) - \pi(A))\}| \\ &\leqq ||\mu - \pi|| \sup_{x\in A}\beta(x) ,\end{aligned}$$

and

$$\begin{aligned}II &= \pi(A)\sum_{k=1}^n k(P_\pi[\tau_A = k] - P_\mu[\tau_A = k]) + (n+1)(P_\pi[\tau_A > k] - P_\mu[\tau_A > k]) \\ &\to \pi(A)(E_\pi\tau_A - E_\mu\tau_A) .\end{aligned}$$

The lemma follows directly from this. □

THEOREM 5.3. *If the chain is aperiodic and if* $E_\pi\tau_A < \infty$ *for some strongly uniform* A, *then*

$$\sum_{n=1}^\infty (P^n f(x) - \int f(y)\pi(dy))$$

converges uniformly for $f \in M$ *with* $||f|| \leqq 1$ *for each regular* x, *and the series converges in* $L_1(\mathscr{X}, \mathscr{A}, \pi)$ *uniformly in* $||f|| \leqq 1$.

PROOF. Let A be the set in the hypothesis and b_A be the quantity in Lemma 5.2. Let $\hat\beta(x) = 4(E_x\tau_A + E_\pi\tau_A + b_A)$, so $\beta \leqq \hat\beta$ by Lemma 5.4. For any $c > 0$, let $\hat\beta_c = \min(c, \hat\beta)$.

Next let $x \in \mathscr{R}$. Then $|\sum_{k=1}^{n} (P^k(x, A) - \pi(A))| \leqq \beta(x) < \infty$ for all n, hence there exists a subsequence n_m such that

$$\lim_{m\to\infty} \textstyle\sum_{k=1}^{n_m} (P^k(x, A) - \pi(A))$$

exists as a finite limit. Let $\mu_m = P^{n_m}(x, \cdot)$. Since the chain is aperiodic, $||\mu_m - \pi|| \to 0$. Also

$$\begin{aligned}\liminf_k |\textstyle\sum_{j=1}^{k} (\mu_m P^j(A) - \pi(A))| &= \liminf_k |\textstyle\sum_{j=1}^{k} (P^{n_m+j}(x, A) - \pi(A))| \\ &\leqq \liminf_l |\textstyle\sum_{j=n_m+1}^{n_l} (P^j(x, A) - \pi(A))| \to 0\end{aligned}$$

as $m \to \infty$. Applying Lemma 5.5, $E_{\mu_m}\tau_A \to E_\pi \tau_A$. But then $P^{n_m}\hat\beta(x) \to \int \hat\beta(y)\pi(dy)$. Since $\hat\beta_c \in M$, it is also true that $P^{n_m}\hat\beta_c \to \int \hat\beta_c(y)\pi(dy)$, and it follows that

$$\int_{[\hat\beta(y)>c]} \hat\beta(y)P^{n_m}(x, dy) \to \int_{[\hat\beta(y)>c]} \hat\beta(y)\pi(dy)$$

as $m \to \infty$ for each c.

Let $V^n(x, dy)$ be the absolute variational measure of $P^n(x, dy) - \pi(dy)$. Let $f \in M$ and $\bar f(x) = f(x) - \int f(y)\pi(dy)$, Then, using the invariance of π at the first equality,

$$\begin{aligned}|\textstyle\sum_{j=n+1}^{n+l} P^j\bar f(x)| &= |\textstyle\int \sum_{j=1}^{l} P^j\bar f(x)(P^n(x, dy) - \pi(dy))| \\ &\leqq \textstyle\int |\sum_{j=1}^{l} P^j\bar f(x)| V^n(x, dy) \leqq 2||f|| \int \hat\beta(y)V^n(x, dy) \\ &\leqq 2c||f|| \cdot ||P^n(x, \cdot) - \pi|| \\ &\quad + \textstyle\int_{[\hat\beta(y)>c]} \hat\beta(y)(P^n(x, dy) + \pi(dy))\,.\end{aligned} \tag{5.2}$$

Now, given $\varepsilon > 0$, the integral will be less than ε for $c = c_\varepsilon$ sufficiently large and $n \in \{n_m\}$ sufficiently large.

Choose $n(\varepsilon) \in \{n_m\}$ so the integral is bounded by ε and $||P^{n(\varepsilon)}(x, \cdot) - \pi|| \leqq \varepsilon/2c_\varepsilon$. Then (2) yields

$$\sup_l |\textstyle\sum_{j=n(\varepsilon)+1}^{n(\varepsilon)+l} P^j\bar f(x)| \leqq \varepsilon(||f|| + 1)\,.$$

But then, for $n > n(\varepsilon)$,

$$\begin{aligned}\sup_l |\textstyle\sum_{j=n+1}^{n+l} P^j\bar f(x)| &\leqq |\textstyle\sum_{j=n(\varepsilon)+1}^{n} P^j\bar f(x)| + \sup_l |\textstyle\sum_{j=n(\varepsilon)+1}^{n+l} P^j\bar f(x)| \\ &\leqq 2\varepsilon(||f|| + 1)\,.\end{aligned}$$

Thus the series converges uniformly in $||f|| \leqq 1$ for this particular $x \in \mathscr{R}$, and this is true for each $x \in \mathscr{R}$. Since $\pi(\mathscr{R}) = 1$ by Corollary 5.2, the convergence holds a.e.-π.

Since the partial sums are bounded in modulus by $||f||\hat\beta$ and $\hat\beta$ is π integrable, the series converges in $L_1(\mathscr{X}, \mathscr{A}, \pi)$ for each $f \in M$. If the convergence were not uniform in $||f|| \leqq 1$, then there would be a sequence $f_n \in M$ with $||f_n|| \leqq 1$ for which the uniformity failed. But, letting

$$g_n(x) = \sup_{k,l} |\textstyle\sum_{j=n+1}^{n+l} (P^j f_k(x) - \int f_k(y)\pi(dy))|\,,$$

the g_n are measurable, bounded by $\hat\beta$, and converge to 0 a.e.-π. Hence

$$\sup_{k,l} \textstyle\int |\sum_{j=n+1}^{n+l} P^j f_k(x) - \int f_k(y)\pi(dy)|\pi(dx) \leqq \int g_n(x)\pi(dx) \to 0$$

by the dominated convergence theorem.

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