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DEDUCTION OF WOLD REPRESENTATION OF STATIONARY PROCESSES FROM CRAMÉR REPRESENTATION OF SECOND-ORDER PROCESSES[1]

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The Wold representation of wide-sense stationary processes with continuous parameter is obtained from a general Cramér representation of second-order processes, continuous in quadratic mean.

1. Introduction. The concepts and notation will be the same as in [1]. For a wide-sense stationary, purely nondeterministic stochastic process $\{x(t), -\infty < t < \infty\}$, there is the well known Wold representation [2]

$$x(t) = \int_{-\infty}^{t} g(t-u)\, dz(u)\,, \qquad t \in (-\infty, \infty) \tag{1}$$

where $\{z(t), -\infty < t < \infty\}$ is a process with orthogonal increments such that $E|dz(t)|^2 = \|dz(t)\|^2 = dt$ and $H(x; t) = H(z, t)$, $t \in (-\infty, \infty)$. The function $g(t)$, $t \geqq 0$, is a nonrandom function such that

$$\int_0^\infty |g(t)|^2\, dt < \infty\,.$$

For any purely nondeterministic second-order process $\{x(t), -\infty < t < \infty\}$, continuous in quadratic mean, there is the Cramér representation [1]

$$x(t) = \sum_{n=1}^{N} \int_{-\infty}^{t} g_n(t, u)\, dz_n(u)\,, \qquad t \in (-\infty, \infty) \tag{2}$$

where $\{z_n(t), -\infty < t < \infty\}$, $n = \overline{1, N}$ (N may be infinite), are mutually orthogonal processes with orthogonal increments for which

$$H(x; t) = \sum_{n=1}^{N} \oplus H(z_n; t)\,, \qquad t \in (-\infty, \infty)\,. \tag{3}$$

The measures $dF_n(t) = \|dz_n(t)\|^2$, $n = \overline{1, N}$, are ordered by absolute continuity

$$dF_1 > dF_2 > \cdots > dF_N\,. \tag{4}$$

The nonrandom functions $g_n(t, u)$, $-\infty < u < t$, $n = \overline{1, N}$, satisfy the condition

$$\sum_{n=1}^{N} \int_{-\infty}^{t} |g_n(t, u)|^2\, dF_n(u) < \infty\,,$$

for each $t \in (-\infty, \infty)$.

Let R_n be the class of measures equivalent to the measure dF_n with respect to the relation $>$. According to [1], the correlation function $r(s, t) = Ex(s)\overline{x(t)} = (x(s), x(t))$, $-\infty < s, t < \infty$, of the process $\{x(t)\}$ uniquely determines its spectral type, i.e., the sequence $R_1 > R_2 > \cdots > R_N$, where $dF_n \in R_n$, $n = \overline{1, N}$.

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The subject of this note is to obtain, starting from the general representation (2), the representation (1) for stationary process, i.e., for a process whose correlation function $r(s, t)$ depends only on the difference of the arguments $s - t$.

2. Deduction. The condition $r(s, t) = r(s - t)$ is equivalent to the existence of a one-parameter group of unitary operators $\{U_t, -\infty < t < \infty\}$ in $H(x)$ such that

$$x(t) = U_t x(0) , \qquad t \in (-\infty, \infty) . \tag{5}$$

Let a, b $(a < b)$ be two arbitrary numbers and let $H(x; a, b) = H(x; b) \ominus H(x; a)$. We have

$$H(x; a, b) = \textstyle\sum_{n=1}^{N} \oplus H(z_n; a, b) .$$

Consider the cyclic space $H(z_n; a, b)$. It is generated by the "arc" $\{z_n(t), a \leqq t \leqq b\}$, in the sense that any element y of $H(z_n; a, b)$ is of the form

$$y = \textstyle\int_a^b f(u)\, dz_n(u) , \qquad f(u) \in L_2(dF_n) .$$

Consider now the cyclic space $H(z_n; a + t, b + t)$ where t is an arbitrary number. According to the definition of U_t, we have

$$U_t H(z_n; a, b) = H(z_n; a + t, b + t) .$$

Since we know that isomorphic cyclic spaces have the same spectral type ([3] Chapter X), we deduce that the measures

$$dF_n(u) = ||dz_n(u)||^2 , \qquad a \leqq u \leqq b$$

and

$$d_u F_n(u + t) = ||d_u z_n(u + t)||^2 , \qquad a \leqq u \leqq b$$

corresponding, respectively, to the generating "arcs" $\{z_n(u), a \leqq u \leqq b\}$ and $\{z_n(u + t), a \leqq u \leqq b\}$, are equivalent.

So we have for each t,

$$d_u F_n(u + t) = \varphi_n(t, u)\, dF_n(u) \qquad (\varphi_n(0, u) = 1) ,$$

where $\varphi_n(t, u) > 0$ a.e. (dF_n) is the Radon-Nikodym derivative.

Setting

$$d_u \hat{Z}_n(u + t) = \varphi_n^{-\frac{1}{2}}(t, u) d_u z_n(u + t)$$
$$(d_u \hat{Z}_n(u + t) = 0 \text{ if } \varphi_n(t, u) = 0)$$

we have

$$||d_u \hat{Z}_n(u + t)||^2 = ||d\hat{Z}_n(u)||^2 .$$

The only solution of the last equation is

$$||d\hat{Z}_n(u)||^2 = k_n du$$

where k_n is a positive number.

Let

$$\tilde{Z}_n(t) = k_n^{-\frac{1}{2}} \hat{Z}_n(t)$$

and

$$\tilde{G}_n(t, u) = \varphi_n^{\frac{1}{2}}(t, u) k_n^{\frac{1}{2}} g_n(t, u) .$$

Then (2) becomes

$$x(t) = \sum_{n=1}^{N} \int_{-\infty}^{t} \tilde{G}_n(t, u)\, d\tilde{Z}_n(u)\,, \qquad t \in (-\infty, \infty) \tag{6}$$

where

$$H(x; t) = \sum_{n=1}^{N} \oplus H(\tilde{Z}_n; t)\,, \qquad t \in (-\infty, \infty)$$

and

$$\|d\tilde{Z}_n(u)\|^2 = du\,, \qquad n = \overline{1, N}\,.$$

So we proved that the measures dF_n, $n = \overline{1, N}$, are equivalent to the ordinary Lebesgue measure du.

We have

$$\begin{aligned} x(t) &= U_t x(0) = U_t \sum_{n=1}^{N} \int_{-\infty}^{0} \tilde{G}_n(0, u)\, d\tilde{Z}_n(u) \\ &= \sum_{n=1}^{N} \int_{-\infty}^{0} \tilde{G}_n(0, u) d_u z_n(t + u) = \sum_{n=1}^{N} \int_{-\infty}^{t} \tilde{G}_n(0, v - t)\, d\tilde{Z}_n(v)\,. \end{aligned}$$

If we set $\tilde{G}_n(0, u - t) = g_n(t - u)$ in (6), we get

$$x(t) = \sum_{n=1}^{N} \int_{-\infty}^{t} g_n(t - u)\, d\tilde{Z}_n(u)\,, \qquad t \in (-\infty, \infty)\,. \tag{7}$$

It remains to be shown that in (7) $N = 1$. Let us suppose that $N \geqq 2$. Consider the nonzero element

$$z = \int_0^\infty \overline{g_2(u)}\, d\tilde{Z}_1(u) - \int_0^\infty \overline{g_1(u)}\, d\tilde{Z}_2(u)\,;$$

it belongs to the subspace $H(\tilde{Z}_1) \oplus H(\tilde{Z}_2)$ of the space $H(x)$. A sqort calculation yields

$$\begin{aligned} (x(t), z) &= 0\,, && t \leqq 0 \\ &= \int_0^t g_1(t - u) g_2(u)\, du - \int_0^t g_2(t - u) g_1(u)\, du = 0\,, && t > 0 \end{aligned}$$

for all $t \in (-\infty, \infty)$. That means z is orthogonal to $H(x)$, which contradicts $0 \neq z \in H(x)$. This contradiction shows that in (7) $N = 1$, which completes the deduction.

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REFERENCES

[1] CRAMÉR, H. (1964). Stochastic processes as curves in Hilbert space. *Theory. Probability Appl.* **9** 193–204.

[2] HANNER, O. (1949). Deterministic and non-deterministic stationary stochastic processes. *Ark. Mat.* **1** 151–177.

[3] PLESNER, A. I. (1969). *Spectral Theory of Linear Operators.* Moscow. Ungar, New York.

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