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CONVERGENCE RATES FOR A CLASS OF LARGE DEVIATION PROBABILITIES

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For a sequence $\{X_n: 1 \leqq n < \infty\}$ of independent, identically distributed random variables with moment-generating functions, a 1952 theorem of Chernoff asserts that $n^{-1} \log P(S_n \geqq \lambda n) \to \log \rho$, where S_n is the nth partial sum of the X_k's, $\lambda > 0$, and ρ is a constant depending on λ and the distribution of X_1. A 1969 theorem of Sievers, as strengthened by Plachky in 1971, established the convergence of $n^{-1} \log P(W_n \geqq z_n)$ to a constant, where the W_n's have moment-generating functions and belong to a class of random variables more general than partial sums, and the z_n's are numbers such that $n^{-1}z_n \to \lambda > 0$. In a format related to that of Sievers, Behadur in 1971 analyzed the behavior of $n^{-1} \log P(W_n \geqq z_n)$ in situations when it may not converge to a constant. The goal of the present article is to extend the theorems of Chernoff, Sievers, and Bahadur in the direction of obtaining convergence rates (to 0) of the large deviation probabilities $P(W_n \geqq z_n)$ where the z_n's are numbers such that $n^{-\frac{1}{2}}z_n \to \infty$. The method of proof is based on the proof of Chernoff's theorem given, in passing, in a 1960 paper of Bahadur and Ranga Rao.

0. Introduction. If $\{X_n: 1 \leqq n < \infty\}$ is a sequence of independent, identically distributed (i.i.d.) random variables with $E(X_1) = 0$ and Var $(X_1) = 1$, the most elementary form of the central limit theorem asserts that $P(S_n \geqq \lambda n^{\frac{1}{2}}) \to 1 - \Phi(\lambda)$ as $n \to \infty$, where $S_n = \sum_{k=1}^{n} X_k$ and $\Phi(x)$ is the standard normal distribution function. It follows that $P(S_n \geqq z_n)$ tends to 0 as $n \to \infty$ whenever $n^{-\frac{1}{2}}z_n \to \infty$. Chernoff (1952) considered the case $z_n = \lambda n$, for $\lambda > 0$, and showed, for random variables with moment-generating functions (mgf's) $\phi(t) = E(\exp(tX_1)) < \infty$ for some nondegenerate interval of t's, that $n^{-1} \log p(S_n \geqq \lambda n) \to \log \rho$, where ρ is a constant depending on λ and ϕ. The only condition on X_1 for Chernoff's theorem is that the function $Q(t) = \phi'(t)/\phi(t)$ take on the value λ for some t. Chernoff's result has been shown, in the monograph of Bahadur (1971), to hold for random variables not having an mgf if the definition of ρ is expanded slightly. Sievers (1969) considered large deviation probabilities $P(W_n \geqq z_n)$ for $\{W_n: 1 \leqq n < \infty\}$, a sequence of random variables with mgf's $\{\phi_n(t): 1 \leqq n < \infty\}$, and $\{z_n: 1 \leqq 1 < \infty\}$ a sequence of numbers such that $n^{-1}z_n \to \lambda > 0$. Sievers' theorem includes Chernoff's theorem as a special case. As strengthened by Plachky (1971), Sievers' theorem holds under relatively simple conditions on the functions $\log \phi_n(t)$ and their first three derivatives. With conditions no more restrictive than those of Plachky, we extend Sievers' theorem to all sequences $\{z_n: 1 \leqq n < \infty\}$ such that $n^{-\frac{1}{2}}z_n \to \infty$. The main

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theorem of this article, Theorem (2.1), asserts that the difference between $(z_n Q_n^{-1}(z_n))^{-1} \log P(W_n \geqq z_n)$ and a known sequence converges to 0 as $n \to \infty$, where $Q_n(t) = \phi'(t)/\phi(t)$, while Sievers' theorem established the convergence to 0 of the difference between $n^{-1} \log P(W_n \geqq z_n)$ and a known constant in the case $n^{-1}z_n \to \lambda > 0$.

Bahadur (1971) considered a sequence of random variables Y_n with mgf's ψ_n, satisfying some "standard conditions," and investigated the behavior of $n^{-1} \log P(Y_n \geqq 0)$ when this sequence of numbers may not converge to a constant. In particular, if we define $\rho_n = \inf\{\psi_n(t) : 0 \leqq t < \infty\}$, then Bahadur's conditions imply that

$$\lim_{n\to\infty} n^{-1}\{\log P(Y_n \geqq 0) - \log \rho_n\} = 0 .$$

The main theorem of the present article generalizes this statement somewhat, by implying for Y_n's of certain kind the existence of a sequence of numbers $r_n \to \infty$ such that

$$\lim_{n\to\infty} r_n^{-1}\{\log P(Y_n \geqq 0) - \log \rho_n\} = 0 .$$

Here r_n, the "true" rate of convergence, is often not asymptotic to n. Our random variables W_n are related to Bahadur's Y_n by the equation $Y_n = W_n - z_n$, where the z_n's are positive numbers such that $n^{-\frac{1}{2}}z_n \to \infty$. The rate of convergence r_n can be expressed in Bahadur's notation as $r_n = z_n \psi_n^{-1}(\rho_n)$.

Feller (1969), interested primarily in the law of the iterated logarithm, derived theorems of the form $r_n^{-1} \log P(S_n \geqq z_n) \to -1$, where the S_n are row sums of a triangular array of random variables with mgf's. As can be expected from the generality of the triangular array setup (containing the possibility of a lack of normal convergence), Feller's conditions are quite complicated, and the relation between z_n and r_n, even in simple cases, is not always clear. Our theorem (in which normal convergence is assumed) implies directly that, for $n^{-\frac{1}{2}}z_n \to \infty$ and $n^{-1}z_n \to 0$ as $n \to \infty$, $r_n = (2n)^{-1}z_n^2$ when S_n is the nth partial sum of a sequence of i.i.d. random variables, and that $r_n = z_n Q_n^{-1}(z_n)$, or a known multiple thereof, in a triangular array situation satisfying all the conditions on Q_n.

In Section 1, we develop the mathematical preliminaries and lemmas needed for the rest of the paper. The main theorem is stated and proved in Section 2. We then use the main theorem to derive, in Section 3, large deviation theorems for weighted sums of the form $\sum_{k=1}^n a_{nk} X_k$. In particular, we obtain extensions of a recent theorem of [3] which established the rate of convergence to 0 of the probability $P(\sum_{k=1}^n a_{nk} X_k \geqq \lambda \sum_{k=1}^n a_{nk})$.

1. Preliminaries. The development follows the main lines of the proof of Chernoff's theorem given on pages 1015–1018 of Bahadur and Ranga Rao (1960), as an aside to their derivation of a much sharper large deviation theorem. We have a sequence $\{W_n : 1 \leqq n < \infty\}$ of random variables with $E(W_n) = 0$ and $\operatorname{Var}(W_n) = n$ for all positive integers n, and whose mgf's $\phi_n(t)$ all exist in an interval $|t| < B$, where $B \leqq \infty$ is a positive constant.

For each n, we define a random variable $Y_n = n^{-\frac{1}{2}}(W_n - z_n)$, which has mgf $\psi_n(t) = \exp(-n^{-\frac{1}{2}}z_n t)\phi_n(n^{-\frac{1}{2}}t)$ and distribution function (df) $G_n(y)$. For each h, $|h| < B$, we construct an "associated" random variable $\bar{Y}_n$ through the definition of its df $\bar{G}_n(y)$ by

$$d\bar{G}_n(y) = [e^{hy}/\psi_n(h)]\, dG_n(y) .$$

The mgf $\bar{\psi}_n(t)$ of $\bar{Y}_n$ is

$$\bar{\psi}_n(t) = \psi_n(t+h)/\psi_n(h) = \exp(-n^{-\frac{1}{2}}z_n t)[\phi_n(n^{-\frac{1}{2}}(t+h))/\phi_n(n^{-\frac{1}{2}}h)] .$$

The proof of the following lemma is identical with the proof of Lemma 2 on page 1017 of Bahadur and Ranga Rao (1960):

(1.1) LEMMA. *For each h, $|h| < B$,*

$$P(W_n \geqq z_n) = \exp(-n^{-\frac{1}{2}}z_n h)\phi_n(n^{-\frac{1}{2}}h)h \int_0^\infty e^{-hy}\,[\bar{G}_n(y) - \bar{G}_n(0)]\, dy .$$

By successive differentiation of the mgf $\bar{\psi}_n(t)$ of $\bar{Y}_n$, we obtain:

(1.2) LEMMA. *If $Q_n(t) = \phi_n'(t)/\phi_n(t)$, then*

$$E(\bar{Y}_n) = n^{-\frac{1}{2}}Q_n(n^{-\frac{1}{2}}h) - n^{-\frac{1}{2}}z_n$$

and

$$\mathrm{Var}\,(\bar{Y}_n) = n^{-1}Q_n'(n^{-\frac{1}{2}}h) .$$

We now put our first condition on W_n: for all sufficiently large n, we require that z_n lie in the range of Q_n. (We will formalize all the conditions in the statement of the main theorem.) When $n^{-1}z_n \to 0$, this condition is actually weaker than condition (ii), case $k = 1$, of Plachky (1971), for the latter condition requires that $Q_n(t) \approx nc_1(t)$ for $|t| < B$. Because $Q_n(0) = 0$ and $Q_n'(t) > 0$ always, being the variance of a nondegenerate "associated" random variable, the Sievers–Plachky condition really requires that Q_n take any value which is less than a constant multiple of n. The next lemma follows immediately from Lemma (1.2).

(1.3) LEMMA. *If z_n lies in the range of Q_n, there exists a unique solution $h = h_n$ to the equation $E(\bar{Y}_n) = 0$, and $h_n = n^{\frac{1}{2}}Q_n^{-1}(z_n)$.*

We next impose the condition that there exist numbers σ_1^2 and σ_2^2 such that $0 < \sigma_1^2 \leqq n^{-1}Q_n'(Q_n^{-1}(z_n)) \leqq \sigma_2^2 < \infty$ for all sufficiently large n. This condition is comparable to Plachky's conditions (ii), case $k = 2$, and (iii), which require that $n^{-1}Q_n'(t) \to c_2(t) > 0$ for each t such that $|t| < B$.

Our third condition, akin to Plachky's condition (iv) that $n^{-1}Q_n''(t)$ be locally bounded on the interval $|t| < B$, requires that $n^{-1}Q_n''(t_n + Q_n^{-1}(z_n))$ be a bounded sequence for all sequences $\{t_n : 1 \leqq n < \infty\}$ such that $t_n \to 0$ as $n \to \infty$. We obtain:

(1.4) LEMMA. *If the above three conditions hold and $\bar{\sigma}_n^2 = \mathrm{Var}\,(\bar{Y}_n)$, then $\lim_{n\to\infty} \bar{G}_n(y\bar{\sigma}) = \Phi(y)$.*

PROOF. We apply the continuity theorem for mgf's of Curtiss (1942), after showing that $\bar{\phi}_n(\bar{\sigma}_n^{-1}t)$, the mgf of $\bar{\sigma}_n^{-1}\bar{Y}_n$, converges to $\exp(t^2/2)$ for $|t| \leqq B_0 < B$. We have

$$\log \bar{\phi}_n(\bar{\sigma}_n^{-1}t) = -n^{-\frac{1}{2}}z_n\bar{\sigma}_n^{-1}t + \log \phi_n(n^{-\frac{1}{2}}\bar{\sigma}_n^{-1}t + Q_n^{-1}(z)) - \log \phi_n(Q_n^{-1}(z_n)) .$$

Set $B_n(u) = \max_{1 \leq n < \infty} |n^{-1}Q_n''(n^{-\frac{1}{2}}\bar{\sigma}_n^{-1}u + Q_n^{-1}(z_n))| < \infty$ for $|u| < |t|$, and take $B_0 < B$, where B comes from the third condition stated above. By expanding $f_n(t) = \log \phi_n(n^{-\frac{1}{2}}\bar{\sigma}_n^{-1}t + Q_n^{-1}(z_n))$ in a Taylor series with remainder about $t = 0$, we obtain for $|t| \leqq B_0$ that

$$f_n(t) = \log \phi(Q_n^{-1}(z_n)) + n^{-\frac{1}{2}}\bar{\sigma}_n^{-1}tQ_n(Q_n^{-1}(z_n)) + \tfrac{1}{2}n^{-1}\bar{\sigma}_n^{-2}t^2Q_n'(Q_n^{-1}(z_n)) + R_n$$

where $|R_n| \leqq 6^{-1}n^{-\frac{3}{2}}\bar{\sigma}_n^{-3}t^3B_n(u_n)$ for some $|u_n| \leqq |t|$. By Lemma (1.2), $\bar{\sigma}_n^2 = n^{-1}Q_n'(Q_n^{-1}(z_n)) \geqq \sigma_1^2$ for sufficiently large n, so $\log \bar{\phi}_n(\bar{\sigma}_n^{-1}t) \to t^2/2$ as $n \to \infty$.

2. The main theorem. We can now state and prove the following extension of the theorem of Sievers (1969):

(2.1) THEOREM. *If $\{W_n: 1 \leqq n < \infty\}$ is a sequence of random variables with mgf's $\{\phi_n(t): 1 \leqq n < \infty\}$ such that, for $Q_n(t) = \phi_n'(t)/\phi(t)$ and $\{z_n: 1 \leqq n < \infty\}$ a sequence of real positive numbers with $n^{-\frac{1}{2}}z_n \to \infty$ as $n \to \infty$,*

(i) *$E(W_n) = 0$, $\mathrm{Var}(W_n) = n$, and $\phi_n(t) < \infty$ for $|t| < B$ for all n, where $0 < B \leqq \infty$;*

(ii) *z_n lies in the range of Q_n for all sufficiently large n, and $z_nQ_n^{-1}(z_n) \to \infty$ as $n \to \infty$;*

(iii) *there exist numbers σ_1^2 and σ_2^2 such that $0 < \sigma_1^2 \leqq n^{-1}Q_n'(Q_n^{-1}(z_n)) \leqq \sigma_2^2 < \infty$ for all sufficiently large n; and*

(iv) *for some B_0, $0 < B_0 < B$, and all sequences $\{t_n: 1 \leqq n < \infty\}$ such that $|t_n| \leqq B_0$ and $t_n \to 0$ as $n \to \infty$, the sequence of numbers $n^{-1}Q_n''(t_n + Q_n^{-1}(z))$ is bounded, then*

$$\lim_{n\to\infty} (z_nQ_n^{-1}(z_n))^{-1}\{\log P(W_n \geqq z_n) - \log \phi_n(Q_n^{-1}(z_n))\} = -1 .$$

PROOF. Taking $h = t_n = n^{\frac{1}{2}}Q_n^{-1}(z_n)$, we have from Lemma (1.1) that $P(W_n \geqq z_n) = \exp(-z_nQ_n^{-1}(z_n))\phi_n(Q_n^{-1}(z_n))\ I_n$, where $I_n = t_n \int_0^\infty \exp(-t_ny) \times [\bar{G}_n(y) - \bar{G}_n(0)]\,dy$. For the proof of the theorem, it suffices to show that $\lim_{n\to\infty} (z_nQ_n^{-1}(z_n))^{-1} \log I_n = 0$. Because $0 \leqq \bar{G}_n(y) - \bar{G}_n(0) \leqq 1$ for all n and y, we know that $I_n \leqq 1$, so that $\limsup_{n\to\infty} (z_nQ_n^{-1}(z_n))^{-1} \log I_n \leqq 0$. On the other hand, by Lemmas (1.2) and (1.4) and condition (iii), we know that, for $y > 0$, $\liminf_{n\to\infty} \bar{G}_n(y) \geqq \Phi(\sigma_2^{-1}y)$. Therefore, for $\varepsilon > 0$, we have $I_n \geqq t_n \int_\varepsilon^\infty \exp(-t_ny)[\bar{G}_n(y) - \bar{G}_n(0)]\,dy \geqq [\bar{G}_n(\varepsilon) - \bar{G}_n(0)] \exp(-t_n\varepsilon)$ which implies that $\liminf_{n\to\infty} (z_nQ_n^{-1}(z_n))^{-1} \log I_n \geqq \lim_{n\to\infty} n^{\frac{1}{2}}z_n^{-1}\varepsilon = 0$, because $\liminf_{n\to\infty} [\bar{G}_n(\varepsilon) - \bar{G}_n(0)] \geqq \Phi(\sigma_2^{-1}\varepsilon) - \Phi(0)$, and $z_nQ_n^{-1}(z_n) \to \infty$ and $n^{-\frac{1}{2}}z_n \to \infty$ as $n \to \infty$. That completes the proof.

NOTE. In the language of Feller (1969), the result of the theorem would be expressed as

$$P(W_n \geqq z_n) = \exp(-r_n + o(r_n))$$

as $n \to \infty$, where $r_n = \{1 - \log \phi_n(Q_n^{-1}(z)\}z_n Q_n^{-1}(z_n)$.

Theorem (2.1) is related to Theorem 2.2 of Bahadur (1971) in the following way. Bahadur started with random variables Y_n whose mgf's $\psi_n(t)$ assume minimum values $\rho_n < 1$ on their domains, say at points $t = h_n$, i.e. $\rho_n = \psi_n(h_n) < 1$. Then he showed that $\lim_{n\to\infty} n^{-1}\{\log P(Y_n \geqq 0) - \log \rho_n\} = 0$. If we define $W_n = Y_n + z_n$, then $\phi_n(t) = e^{z_n t}\psi_n(t)$ and $Q_n(t) = \phi_n'(t)/\phi_n(t) = z_n + \{\psi_n'(t)/\psi_n(t)\}$. Therefore $Q_n(h_n) = z_n$ if and only if $\psi_n'(h_n) = 0$, i.e. if and only if $\rho_n = \psi_n(Q_n^{-1}(z_n))$. In this case, $\log \phi_n(Q_n^{-1}(z_n)) = z_n Q_n^{-1}(z_n) + \log \rho_n$, and so $\log P(Y_n \geqq 0) - \log \rho_n = \log P(W_n \geqq z_n) - \log \phi_n(Q_n^{-1}(z)) + z_n Q_n^{-1}(z_n)$. From our Theorem (2.1), it then follows that

$$\lim_{n\to\infty} (z_n Q_n^{-1}(z_n))^{-1}\{\log P(Y_n \geqq 0) - \log \rho_n\} = 0 .$$

The value of this complement to Bahadur's statement is that it covers cases when $z_n Q_n^{-1}(z_n)$ may not be asymptotic to n. Examples of such cases are discussed in Section 3 of this article.

When the sequence $\{z_n : 1 \leqq n < \infty\}$ is such that $n^{-1}z_n \to \lambda > 0$ as $n \to \infty$, we obtain the theorem of Sievers in the form given by Plachky (1971):

(2.2) COROLLARY. (Sievers–Plachky). *If* $\{W_n : 1 \leqq n < \infty\}$ *is a sequence of random variables with* mgf*'s* $\{\phi_n(t) : 1 \leqq n < \infty\}$ *such that, for* $Q_n(t) = \phi_n'(t)/\phi_n(t)$ *and* $\{z_n : 1 \leqq n < \infty\}$ *a sequence of real numbers with* $n^{-1}z_n \to \lambda > 0$ *as* $n \to \infty$,

(i*) $E(W_n) = 0$, $\mathrm{Var}\,(W_n) = n$, *and* $\phi_n(t) < \infty$ *for* $|t| < B$ *for all* n, *where* $0 < B \leqq \infty$;

(ii*) *for* $0 \leqq t < B$, $\lim_{n\to\infty} n^{-1} \log \phi_n(t) = c_0(t) < \infty$;

(iii*) *for* $0 \leqq t < B$, $\lim_{n\to\infty} n^{-1}Q_n(t) = c_1(t) < \infty$, *and there exists an* h, $0 < h < B$, *with* $c_1(h) = \lambda$;

(iv*) *for* $0 \leqq t < B$, $\lim_{n\to\infty} n^{-1}Q_n'(t) = c_2(t)$, *where* $0 < c_2(t) < \infty$; *and*

(v*) *for* $0 < t < B$, *the sequence* $n^{-1}Q_n''(t)$ *is locally bounded, then*

$$\lim_{n\to\infty} n^{-1} \log P(W_n \geqq z_n) = c_0(h) - \lambda h .$$

3. Weighted sums. The results of this section are all concerned with a sequence $\{X_n : 1 \leqq n < \infty\}$ of i.i.d. random variables with $E(X_1) = 0$, $\mathrm{Var}\,(X_1) = 1$, and mgf $\phi(t) = E(\exp(tX_1)) < \infty$ for $|t| < B$, where $0 < B \leqq \infty$. We have a double array $\{a_{nk} : 1 \leqq k \leqq n, 1 \leqq n < \infty\}$ of nonnegative real numbers such that $\sum_{k=1}^n a_{nk}^2 = 1$, and we impose the following condition on these weights so that the sum $\sum_{k=1}^n a_{nk} X_k$ is not dominated by a relatively few terms:

Condition W. The weights are normalized so that $\sum_{k=1}^n a_{nk}^2 = 1$, and there exist numbers α and θ, $0 < \alpha \leqq 1$, $0 < \theta \leqq 1$, such that, for every sufficiently large n, at least αn of the a_{nk}'s exceed or equal $\theta\sigma_n$, where $\sigma_n = \max\{a_{nk} : 1 \leqq k \leqq n\}$.

The theorem in [3] constructs a sequence $\{r_n : 1 \leqq n < \infty\}$ of positive numbers such that

$$\lim_{n\to\infty} r_n^{-1} \log P(\sum_{k=1}^n a_{nk} X_k \geqq \lambda \sum_{k=1}^n a_{nk}) = -1$$

for a range of positive numbers λ. In this section, we use the theorem of Section 2 in order to extend the theorem of [3] to probabilities of the form $P(\sum_{k=1}^n a_{nk} X_k \geqq \lambda(\sum_{k=1}^n a_{nk})^q)$ where $\frac{1}{2} < q < \infty$.

As in the earlier sections of this paper, we deal with a sequence of positive numbers $\{z_n : 1 \leqq n < \infty\}$ such that $n^{-\frac{1}{2}} z_n \to \infty$. To simplify the notation, we set

$$P(n) = P(\sum_{k=1}^n a_{nk} X_k \geqq n^{-\frac{1}{2}} z_n) .$$

We obtain the first result with no conditions on the mgf of X_1 beyond existence in a nondegenerate interval.

(3.1) THEOREM. *If $n^{-1} z_n \to 0$ as $n \to \infty$, then*

$$\lim_{n\to\infty} 2nz_n^{-2} \log P(n) = -1 .$$

PROOF. In Theorem (2.1), we take $W_n = n^{\frac{1}{2}} \sum_{k=1}^n a_{nk} X_k$, and so $\phi_n(t) = \prod_{k=1}^n \phi(n^{\frac{1}{2}} a_{nk} t)$ and $Q_n(t) = n^{\frac{1}{2}} \sum_{k=1}^n a_{nk} Q(n^{\frac{1}{2}} a_{nk} t)$. Under *Condition* W, we have that $1 \leqq n^{\frac{1}{2}} \sigma_n \leqq (\alpha\theta^2)^{-\frac{1}{2}}$. The range of Q_n contains z_n because $n^{-1} z_n \to 0$ and $n^{-\frac{1}{2}} \sum_{k=1}^n a_{nk} Q(n^{\frac{1}{2}} a_{nk} t) \geqq n^{-\frac{1}{2}} \alpha n \theta \sigma_n Q(n^{\frac{1}{2}} \theta \sigma_n t) \geqq \alpha\theta Q(\theta t)$. Furthermore, if $h_n = Q_n^{-1}(z_n)$, then $n^{-1} z_n \geqq \alpha\theta Q(\theta h_n)$ so that $h_n \leqq \theta^{-1} Q^{-1}((\alpha\theta)^{-1} n^{-1} z_n)$. It follows that $h_n \to 0$ as $n \to \infty$. In a Taylor series expansion, $Q(t) = Q(0) + Q'(0)t + \frac{1}{2}Q''(\delta)t^2 = t + \frac{1}{2}Q''(\delta)t^2$, where $0 < \delta < t$. Therefore

$$\begin{aligned} n^{-1} z_n &= n^{-1} Q_n(h_n) = n^{-\frac{1}{2}} \sum_{k=1}^n a_{nk} Q(n^{\frac{1}{2}} a_{nk} h_n) \\ &= n^{-\frac{1}{2}} \sum_{k=1}^n a_{nk} \{n^{\frac{1}{2}} a_{nk} h_n + \tfrac{1}{2} Q''(\delta_{nk}) n a_{nk}^2 h_n^2\} \\ &= h_n + \tfrac{1}{2} n^{\frac{1}{2}} h_n^2 \sum_{k=1}^n a_{nk}^3 Q''(\delta_{nk}) , \end{aligned}$$

where $0 < \delta_{nk} < n^{\frac{1}{2}} a_{nk} h_n \leqq n^{\frac{1}{2}} \sigma_n h_n \to 0$ as $n \to \infty$. Therefore

$$\begin{aligned} |n^{-1} z_n h_n^{-1} - 1| &= \tfrac{1}{2} n^{\frac{1}{2}} h_n |\sum_{k=1}^n a_{nk}^3 Q''(\delta_{nk})| \\ &\leqq \tfrac{1}{2} n^{\frac{1}{2}} h_n^2 |Q''(n^{\frac{1}{2}} \sigma_n h_n)| \sigma_n \sum_{k=1}^n a_{nk}^2 \\ &\leqq \tfrac{1}{2} n^{\frac{1}{2}} \sigma_n h_n |Q''(n^{\frac{1}{2}} \sigma_n h_n)| \to 0 \end{aligned}$$

as $n \to \infty$ because $n^{\frac{1}{2}} \sigma_n \leqq (\alpha\theta^2)^{-\frac{1}{2}}$, $|Q''(0)| = |E(X_1^3)| < \infty$, and $h_n \to 0$ as $n \to \infty$. From this, it follows that $h_n \sim n^{-1} z_n$, where the symbol "$\sim$" indicates that the ratio of the two sides tends to one as $n \to \infty$. Conditions (iii) and (iv) of Theorem (2.1) follow without difficulty using the bounds on σ_n and the fact that $\sum_{k=1}^n a_{nk}^2 = 1$. It remains to investigate the behavior of $(z_n h_n)^{-1} \log \phi_n(h_n)$. A Taylor expansion yields

$$\begin{aligned} \log \phi(t) &= \log \phi(0) + Q(0)t + \tfrac{1}{2}Q'(0)t^2 + (\tfrac{1}{6})Q''(\delta)t^3 \\ &= \tfrac{1}{2}t^2 + (\tfrac{1}{6})Q''(\delta)t^3 \end{aligned}$$

for $0 < \delta < t$. Therefore

$$\log \phi_n(h_n) = \sum_{k=1}^n \log \phi(n^{\frac{1}{2}} a_{nk} h_n)$$
$$= \tfrac{1}{2} n h_n^2 \sum_{k=1}^n a_{nk}^2 + (\tfrac{1}{6}) n^{\frac{3}{2}} h_n^3 \sum_{k=1}^n a_{nk}^3 Q''(\delta_{nk}) ,$$

where $0 < \delta_{nk} < n^{\frac{1}{2}} a_{nk} h_n \leqq n^{\frac{1}{2}} \sigma_n h_n \to 0$ as $n \to \infty$. Therefore

$$(z_n h_n)^{-1} \log \phi_n(h_n) = \tfrac{1}{2}(n^{-1} z_n h_n^{-1})^{-1} + (\tfrac{1}{6}) n^{\frac{3}{2}} z_n^{-1} h_n^2 \sum_{k=1}^n a_{nk}^3 Q''(\delta_{nk}) .$$

It follows from this that $(z_n h_n)^{-1} \log \phi_n(h_n) \to \frac{1}{2}$ as $n \to \infty$ because the remainder term tends to 0 as before. The proof of the theorem is completed by observing that $(z_n h_n)^{-1} \sim z_n^{-1}(n^{-1} z_n)^{-1} = n z_n^{-2}$.

It is to be noted that the above theorem includes the case of sums of i.i.d. random variables, upon taking $a_{nk} = n^{-\frac{1}{2}}$ for each n and k. The next corollary deals with a sequence of z_n's of particular interest. For notational case, we write

$$A \leqq \overline{\underline{\lim}}_{n\to\infty} C_n \leqq B$$

to indicate that $A \leqq \liminf_{n\to\infty} C_n \leqq \limsup_{n\to\infty} C_n \leqq B$. We use this notation throughout the remainder of the paper.

(3.2) COROLLARY. *If $A_n = \sum_{k=1}^n a_{nk}$, $\frac{1}{2} < q < 1$, and $\lambda > 0$, then, as $n \to \infty$,*

$$P(\sum_{k=1}^n a_{nk} X_k \geqq \lambda A_n^q) = \exp(-r_n + o(r_n)) ,$$

where $r_n = \frac{1}{2}\lambda^2 A_n^{2q} = \frac{1}{2}\lambda^2 n^q \gamma_n$ for a bounded sequence of positive numbers γ_n such that $0 < (\alpha\theta)^{2q} \leqq \overline{\underline{\lim}}_{n\to\infty} \gamma_n \leqq (\alpha\theta^2)^{-q} < \infty$.

PROOF. From Theorem (3.1), we know that $r_n = (2n)^{-1} z_n^2$ in general. Here $z_n = \lambda n^{\frac{1}{2}} A_n^q$ so that $r_n = \frac{1}{2}\lambda^2 A_n^{2q}$. The result follows from the string of inequalities $\alpha\theta n^{\frac{1}{2}} \leqq \alpha n \theta \sigma_n \leqq A_n \leqq n\sigma_n \leqq (\alpha\theta)^{-\frac{1}{2}} n^{\frac{1}{2}}$.

If we set $a_{nk} = n^{-\frac{1}{2}}$ in the above corollary, so that $\alpha = \theta = 1$, we obtain an extension of the original Chernoff theorem asserting that, for $\frac{1}{2} < q < 1$ and $\lambda > 0$,

$$\lim_{n\to\infty} n^{-q} \log P(S_n \geqq \lambda n^{(1+q)/2}) = -\tfrac{1}{2}\lambda^2 .$$

The next theorem is an improvement of the theorem in [3] that follows from the main theorem.

(3.3) THEOREM. *If $n^{-\frac{1}{2}} A_n^{-1} z_n \to \lambda > 0$ as $n \to \infty$, where $A_n = \sum_{k=1}^n a_{nk}$, $\lambda(\alpha\theta)^{-1}$ lies in the range of Q, and $M_\lambda = \theta^{-1} Q^{-1}(\lambda(\alpha\theta)^{-1})$ lies in the domain of ϕ, then*

$$-\infty < M_\lambda(c_1 - 1) \leqq \overline{\underline{\lim}}_{n\to\infty} z_n^{-1} \log P(n) \leqq m_\lambda(c_2 - 1) < 0 ,$$

where $m_\lambda = (\alpha\theta^2)^{\frac{1}{2}} Q^{-1}(\lambda)$, $c_1 = \alpha(\lambda M_\lambda)^{-1} \log \phi(\theta m_\lambda)$ and $c_2 = 1 - \alpha(\lambda M_\lambda)^{-1} L(\theta m_\lambda)$ for $L(t) = tQ(t) - \log \phi(t)$.

PROOF. We take W_n, ϕ_n, and Q_n as in the proof of Theorem (3.1). First we see that the range of Q_n contains z_n because $Q_n(t) \geqq n^{\frac{1}{2}} \alpha\theta A_n Q(n^{\frac{1}{2}} \theta \sigma_n t)$ since $n\sigma^u \geqq A_n$, and so $n^{-\frac{1}{2}} A_n^{-1} Q_n(t) \geqq \alpha\theta Q(n^{\frac{1}{2}} \theta \sigma_n t)$, from which the existence of

$h_n = Q_n^{-1}(z_n)$ follows from the conditions on λ. We have then that $z_n = Q_n(h_n) \geqq n^{\frac{1}{2}}\alpha n\theta\sigma_n Q(n^{\frac{1}{2}}\theta\sigma_n h_n)$ so that $n^{-\frac{1}{2}}A_n^{-1}z_n \geqq \alpha\theta Q(n^{\frac{1}{2}}\theta\sigma_n h_n)$, from which the vital bound $n^{\frac{1}{2}}\sigma_n h_n \leqq \theta^{-1}Q^{-1}(\lambda(\alpha\theta)^{-1}) + o(1) = M_\lambda + o(1)$ follows, where $o(1) \to 0$ as $n \to \infty$. Furthermore, since $1 \leqq n^{\frac{1}{2}}\sigma_n \leqq (\alpha\theta^2)^{-\frac{1}{2}}$, we have the inequalities

$$(\alpha\theta^2)^{\frac{1}{2}}Q^{-1}(n^{-\frac{1}{2}}A_n^{-1}z_n) \leqq h_n \leqq \theta^{-1}Q^{-1}((\alpha\theta)^{-1}n^{-\frac{1}{2}}A_n^{-1}z_n) ,$$

which imply that

$$\lambda^{-1}m_\lambda \leqq \varlimsup_{n\to\infty} (n^{-\frac{1}{2}}A_n^{-1}z_n)^{-1}h_n \leqq \lambda^{-1}M_\lambda .$$

Therefore $\liminf_{n\to\infty} z_n Q_n^{-1}(z_n) \geqq \lim_{n\to\infty} m_\lambda z_n = \infty$, as $z_n \sim \lambda n^{\frac{1}{2}}A_n \geqq \lambda n^{\frac{1}{2}}\alpha n\theta\sigma_n \geqq \lambda\alpha\theta n$, completing the verification of (ii) of Theorem (2.1). The verification of conditions (iii) and (iv) of the main theorem is straightforward, and so it remains only to study the quantity $(z_n h_n)^{-1}\log\phi_n(h_n)$. We have

$$\begin{aligned} z_n h_n - \log\phi_n(h_n) &= z_n h_n - \textstyle\sum_{k=1}^n \log\phi(n^{\frac{1}{2}}a_{nk}h_n) \\ &\leqq z_n h_n - \alpha n \log\phi(n^{\frac{1}{2}}\theta\sigma_n h_n) \end{aligned}$$

so that

$$1 - (z_n h_n)^{-1}\log\phi_n(h_n) \leqq 1 - \alpha(n^{-1}z_n h_n)^{-1}\log\phi(n^{\frac{1}{2}}\theta\sigma_n h_n)$$

and

$$\liminf_{n\to\infty} (z_n h_n)^{-1}\log\phi_n(h_n) \geqq \alpha(\lambda M_\lambda)^{-1}\log\phi(\theta m_\lambda) = c_1 > 0 .$$

On the other hand, setting $L(t) = tQ(t) - \log\phi(t)$, we obtain

$$\begin{aligned} z_n h_n - \log\phi_n(h_n) &= h_n Q_n(h_n) - \log\phi_n(h_n) = \textstyle\sum_{k=1}^n L(n^{\frac{1}{2}}a_{nk}h_n) \\ &\geqq \alpha n L(n^{\frac{1}{2}}\theta\sigma_n h_n) \geqq \alpha n L(\theta h_n) , \end{aligned}$$

because $L(t)$ is positive and monotonically increasing for $t > 0$. Therefore

$$\begin{aligned} \limsup_{n\to\infty} (z_n h_n)^{-1}\log\phi_n(h_n) &\leqq 1 - \liminf_{n\to\infty} (z_n h_n)^{-1}\alpha n L(\theta h_n) \\ &\leqq 1 - \alpha(\lambda M_\lambda)^{-1}L(\theta m_\lambda)^{-1} = c_2 < 1 . \end{aligned}$$

The conclusion now follows from Theorem (2.1) and the bounds on h_n.

If, in Theorem (3.3), we set each $a_{nk} = n^{-\frac{1}{2}}$, then the statement reduces to a slight improvement of the original Chernoff theorem, in view of the facts that $\alpha = 1, \theta = 1, A_n = n^{\frac{1}{2}}, m_\lambda(c_2 - 1) = M_\lambda(c_1 - 1) = -\lambda Q^{-1}(\lambda) + \log\phi(Q^{-1}(\lambda)) = \log\rho$, and $n^{-\frac{1}{2}}z_n \sim \lambda n$. Finally, we have the theorem of [3] itself:

(3.4) COROLLARY. *If $\lambda(\alpha\theta)^{-1}$ lies in the range of Q, and $\theta^{-1}Q^{-1}(\lambda(\alpha\theta)^{-1})$ lies in the domain of ϕ, then as $n \to \infty$,*

$$P(\textstyle\sum_{k=1}^n a_{nk}X_k \geqq \lambda A_n) = \exp(-r_n + o(r_n))$$

where $r_n = \frac{1}{2}\lambda^2 A_n^2\delta_n$ for a bounded sequence positive numbers δ_n such that $0 < 2\lambda^{-1}(\alpha\theta^2)^{\frac{1}{2}}m_\lambda(1 - c_2) \leqq \varliminf_{n\to\infty} \delta_n \leqq 2(\lambda\alpha\theta)^{-1}M_\lambda(1 - c_1) < \infty$, in the notation of Theorem (3.3).

We now extend the theorem on weighted sums to the only remaining case, that when $n^{-1}z_n \to \infty$ as $n \to \infty$. We note here that the conditions for this theorem are much more stringent than those of either of the previous results. In addition, the "lim sup" assertion is informative only when $\alpha^2\theta^4 > \frac{1}{2}$.

(3.5) THEOREM. *If* $\phi(t) < \infty$ *for all real* t, *and*

(i′) $\lim_{h\to\infty} Q(h) = \infty$;
(ii′) $\lim_{h\to\infty} Q'(h) = a$, *where* $0 < a < \infty$; *and*
(iii′) $Q''(h)$ *is bounded for* $0 \leqq h < \infty$, *then, for* $n^{-1}z_n \to$ *as* $n \to \infty$,

$$\frac{\alpha^2\theta^4 - 2}{a\alpha\theta^2} \leqq \overline{\lim_{n\to\infty}}\, 2nz_n^{-2} \log P(n) \leqq \frac{1 - 2\alpha^2\theta^4}{a\alpha\theta^2}\,.$$

PROOF. We take W_n, ϕ_n, and Q_n as in the proof of Theorem (3.1). Now $Q_n(t) \geqq \alpha n\theta Q(\theta t)$ so that $n^{-1}Q_n(t) \geqq \alpha\theta Q(\theta t) \geqq n^{-1}z_n$ for all sufficiently large t by condition (i′), and therefore, for all large n we can find an h_n with $Q_n(h_n) = z_n$. Furthermore, $h_n \to \infty$ as $n \to \infty$ because $h_n \geqq (\alpha\theta^2)^{\frac{1}{2}}Q^{-1}((\alpha\theta^2)^{\frac{1}{2}}n^{-1}z_n)$, and therefore $z_n h_n \to \infty$ also. Condition (iii) of Theorem (2.1) can be verified using condition (ii′) of this theorem in view of the fact that we obtain

$$\alpha\theta^2 a \leqq \overline{\lim_{n\to\infty}}\, n^{-1}Q_n'(h_n) \leqq (\alpha\theta^2)^{-\frac{1}{2}}a$$

from the application of *Condition* W. Condition (iv) of the main theorem is satisfied because of condition (iii′), so it remains only to determine the behavior of the quantity $(z_n h_n)^{-1} \log \phi_n(h_n)$. From the bounds

$$(\alpha\theta^2)^{\frac{1}{2}}Q^{-1}((\alpha\theta^2)^{\frac{1}{2}}n^{-1}z_n) \leqq h_n \leqq \theta^{-1}Q^{-1}((\alpha\theta)^{-1}n^{-1}z_n)$$

and an application of L'Hôspital's Rule, we find that

$$(\alpha\theta^2)a \leqq \overline{\lim_{n\to\infty}}\, (z_n h_n)/(nh_n^2) \leqq (\alpha\theta^2)^{-1}a\,.$$

From the latter inequalities and bounds on $\log \phi_n(h_n) = \sum_{k=1}^n \log \phi(n^{\frac{1}{2}}a_{nk}h)$ obtained from *Condition* W, further applications of L'Hôspital's Rule yield that

$$\tfrac{1}{2}(\alpha\theta^2)^2 \leqq \overline{\lim_{n\to\infty}}\, (z_n h_n)^{-1} \log \phi_n(h_n) \leqq \tfrac{1}{2}(\alpha\theta^2)^{-2}\,,$$

for $\lim_{h\to\infty} h^{-2} \log \phi(ch) = \frac{1}{2}ac^2$ for all $c > 0$. The assertion of the theorem then follows the conclusion of Theorem (2.1).

Note that $\alpha^2\theta^4 - 2 \leqq 1 - 2\alpha^2\theta^4$ must hold because of the $\alpha^2\theta^4 \leqq 1$. The following corollary deals with a sequence of z_n's of special interest:

(3.6) COROLLARY. *If* $A_n = \sum_{k=1}^n a_{nk}$, $q > 1$, *and* $\lambda > 0$, *then, under the conditions of Theorem* (3.5), *as* $n \to \infty$,

$$P(\textstyle\sum_{k=1}^n a_{nk}X_k \geqq \lambda A_n^q) = \exp(-r_n + o(r_n))\,,$$

where $r_n = \frac{1}{2}\lambda^2 A_n^{2q}a^{-1}\gamma_n$ *for a bounded sequence of positive numbers* γ_n *such that*

$$(2\alpha^2\theta^4 - 1)/(\alpha\theta^2) \leqq \overline{\lim_{n\to\infty}}\, \gamma_n \leqq (2 - \alpha^2\theta^4)/(\alpha\theta^2)\,.$$

In the above corollary, the upper bound is always finite, while the lower bound exceeds 0 only if $\alpha^2\theta^4 > \frac{1}{2}$. If we set $a_{nk} = n^{-\frac{1}{2}}$ in the above corollary, so that $\alpha = \theta = 1$, we obtain an extension of Chernoff's theorem to the case of "very large deviations." The assertion would be that, for S_n the nth partial sum of i.i.d. random variables satisfying the conditions of Theorem (3.5),

$$\lim_{n\to\infty} n^{-q} \log P(S_n \geqq \lambda n^{(1+q)/2}) = -\tfrac{1}{2}\lambda^2 a^{-1}$$

for every $q > 1$.

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