The Annals of Probability
1975, Vol. 3, No. 3, 492-502

GENERALIZED DISTRIBUTION FUNCTIONS: THE σ-LOWER FINITE CASE[1,2]

BY GORDON SIMONS

University of North Carolina at Chapel Hill

A mass $m(x) \geqq 0$ is assigned to each point x of a partially ordered countable set X. It is further assumed that $M(x) = \sum_{y \leqq x} m(y) < \infty$ for each $x \in X$. M is called a distribution function. For certain sets X, it is shown that M determines m. For others, M need not determine m uniquely. A theory is presented for σ-lower finite spaces (sets), which are defined in the paper. Such spaces are locally finite. That is, each interval $[x, y] = \{z \in X: x \leqq z \leqq y\}$ has a finite number of points. Möbius functions, which have been defined for locally finite spaces, are used throughout. Distribution functions on a particular σ-lower finite space arise naturally from boundary crossing problems analyzed by Doob and Anderson. The theory is applied to this example and to another.

1. Introduction. Let X be a partially ordered countable set with each point $x \in X$ possessing a nonnegative mass $m(x)$. We assume $M(x) = \sum_{y \leqq x} m(y) < \infty$, $x \in X$, and refer to the function M as the (cumulative) *distribution function*. Sometimes, we shall require the total mass $m(X) = \sum_{x \in X} m(x)$ to be unity.[3] We shall concern ourselves with the following questions:

(i) When does the distribution function determine the individual masses $m(x)$, $x \in X$?

(ii) How are they found when they are determined?

(iii) When is a function $M(x)$, $x \in X$, actually a distribution function?

When X is finite, one has an inversion formula expressed in terms of the *Möbius function* $\mu(x, y)$ of X:

$$m(y) = \sum_{x \leqq y} M(x)\mu(x, y)\,, \qquad y \in X\,. \tag{1}$$

Section 3 of Rota's (1964) fundamental paper on the theory of Möbius functions provides the relevant background. Thus, *the distribution function always determines the individual masses whenever X is finite.*

We shall be concerned with the more difficult situation in which X is infinite.

Received January 14, 1974; revised August 5, 1974.

[1] The general linearly ordered case has been considered in another paper (Simons 1974).

[2] This research was supported by the Air Force Office of Scientific Research under Grant AFOSR 72-2386.

[3] Of course, the requirement $m(X) = 1$ is quite natural in a probabilistic context. However, we shall refrain from making this an assumption *at the outset* since its inclusion would tend to complicate the theory presented below.

AMS 1970 *subject classifications.* Primary 60E05, 06A10; Secondary 60G17.

Key words and phrases. Distribution function, Möbius function, partial ordering, boundary crossing.

The full range of possibilities is much more than we can cope with at this early stage, and we shall be content with a modest beginning. We shall confine all of our attention to *locally finite* spaces since it is for such spaces that a Möbius function is defined. That is, for each pair of points $x, y \in X$, we shall insist that the interval $[x, y] = \{z : x \leqq z \leqq y\}$ be finite.

A rather uninteresting extension of the finite theory discussed in the second paragraph can be made when the number of summands in (1) is finite for each $y \in X$. Formula (1) still applies. We refer to such an X as *lower finite*. A lower finite space is necessarily locally finite.

An interesting example for which X is locally finite but not lower finite arises (apparently unnoticed) in a much cited paper by Doob (1949): Let X consist of a maximal point (0, 1) and pairs of points $(n, 1), (n, 2)$ for $n \geqq 1$. $x = (n, j) < x' = (n', j')$ if and only if $n > n'$, $n' \geqq 0$. See Fig. 1. Let $\{W(t), t \geqq 0\}$ be a standard Wiener process (mean zero and variance t) and let U and L be two lines with U (the upper) having positive slope and intercept, and L (the lower) having negative slope and intercept. Let $M(0, 1) = 1$ and, for $n \geqq 1$, let $M(n, 1)$ and $M(n, 2)$ denote the probability that there exist n time $0 < t_1 < \cdots < t_n$ with $(t_1, W(t_1)), \cdots, (t_n, W(t_n))$ alternately in U and L beginning with U and L, respectively. (Anything may happen before t_1, between times, and after t_n.) Then $\{M(x), x \in X\}$ is a distribution function corresponding to individual masses such as $m(0, 1) = P(W$ never touches U or $L)$, $m(1, 1) = P(W$ touches U but never touches $L)$ and $m(2, 2) = P(W$ touches L before U then touches U then never touches L again).

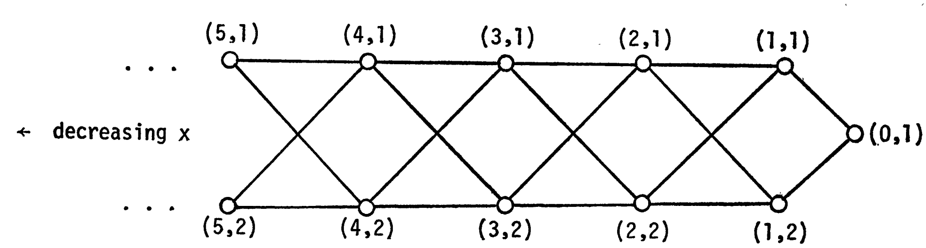

FIG. 1.

Doob describes how to compute $M(x)$, $x \in X$. His interest is in expressing, in terms of these, a probability such as $m(0, 1)$. In particular, he obtains a formula which, in our notation, becomes

(2) $$m(0, 1) = 1 - \textstyle\sum_{n=1}^{\infty} (-1)^{n-1}\{M(n, 1) + M(n, 2)\} .$$

Anderson (1960) obtains a formula for $P(W$ touches U before $L)$. (W does not need to touch L.) This can be expressed in terms of the individual masses as $\sum_{n=1}^{\infty} m(n, 1)$. Anderson evaluates the probability as $M(1, 1) - M(2, 2) + M(3, 1) - M(4, 2) + \cdots$.

For the X of Fig. 1, we find that the distribution function always determines the individual masses. Besides (2), we have the related equations:

$$m(n, j) = M(n, j) - \sum_{k=n+1}^{\infty} (-1)^{k-1}\{M(k, 1) + M(k, 2)\} \qquad n \geqq 1, j = 1, 2 . \tag{3}$$

These equations can be checked by direct substitution.

Before we present some theory, we shall show that a locally finite space can have distribution functions which fail to determine the individual masses. Our example is only slightly more complicated than that of Fig. 1. See Fig. 2. X consists of a maximal point $(0, 1)$ and triplets $(n, 1)$, $(n, 2)$, $(n, 3)$ for $n \geqq 1$. $x = (n, j) < x' = (n', j')$ if and only if $n > n'$, $n' \geqq 0$.

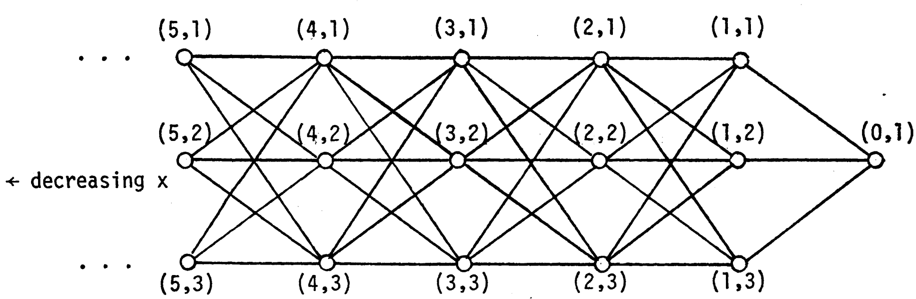

FIG. 2.

EXAMPLE 1. $M(0, 1) = 1$ and $M(n, 1) = M(n, 2) = M(n, 3) = 2^{-n}$, $n \geqq 1$. There are many possible values for the individual masses but all of the possibilities can be expressed as convex combinations of two extremal solutions:

SOLUTION 1a. $m(0, 1) = \frac{1}{2}$; $m(n, 1) = m(n, 2) = m(n, 3) = 2^{-n-1}$ for $n = 2, 4, 6, \cdots$; all other $m(x) = 0$.

SOLUTION 1b. $m(n, 1) = m(n, 2) = m(n, 3) = 2^{-n-1}$ for $n = 1, 3, 5, \cdots$; all other $m(x) = 0$.

EXAMPLE 2. $M(0, 1) = 1$ and $M(n, 1) = M(n, 2) = M(n, 3) = 3^{-n}$, $n \geqq 1$. There is only one solution: $m(0, 1) = \frac{2}{5}$ and $m(n, 1) = m(n, 2) = m(n, 3) = \frac{2}{5} \cdot 3^{-n}$, $n \geqq 1$.

These examples show that *the issue of uniqueness for the individual masses depends on the actual distribution function as well as on the structure of X*. We shall return to these examples later after we have some theory with which to justify our claims.

While the spaces described in Figs. 1 and 2 are not lower finite, they are what we shall refer to as *σ-lower finite*. That is there exists a sequence of partitions $X = A_k + B_k$ with $A_k > B_k$ (i.e., each point of A_k exceeds each point of B_k), $A_k \nearrow X$ as $k \to \infty$, and each A_k is lower finite (i.e., for each $y \in X$ and each $k \geqq 1$, $\{x \in A_k : x \leqq y\}$ is finite). A σ-lower finite space is necessarily locally finite. We shall give a definitive answer to questions (i), (ii) and (iii) for σ-lower finite spaces in Section 3. Although the σ-lower finite assumption is stronger than one might prefer, it permits a wide range of spaces. Departures from this assumption can be analyzed but the assumption permits us to develop a reasonably uncomplicated theory.

2. Some preliminaries. Let X be locally finite. It will be recalled that the Möbius function can be defined recursively by

$$(4)\qquad \begin{aligned} \mu(x, y) &= 1 && \text{for } x = y\,, \\ &= -\textstyle\sum_{x \leqq z < y} \mu(x, z) && \text{for } x < y\,, \\ &= 0 && \text{for } x \nleqq y\,. \end{aligned}$$

Another useful formula is

$$(5)\qquad \mu(x, y) = -\textstyle\sum_{x < z \leqq y} \mu(z, y) \qquad \text{for } x < y\,.$$

Let A' denote the complement of A for each subset A of X.

Suppose $X = A + B + C$ with $A > C > B$, where A, B or C may be the empty set $\varnothing$. Define whenever the number of summands is finite:

$$(6)\qquad \begin{aligned} \mu(x, A) &= -\textstyle\sum_{x \leqq z < A} \mu(x, z)\,, && x \in A'\,, \\ \mu(B, y) &= -\textstyle\sum_{B < z \leqq y} \mu(z, y)\,, && y \in B'\,, \\ \mu(B, A) &= -1 + \textstyle\sum_{B < x, y < A} \mu(x, y)\,. \end{aligned}$$

It is easily checked that

$$(7)\qquad \mu(B, A) = -1 - \textstyle\sum_{B < x < A} \mu(x, A) = -1 - \sum_{B < y < A} \mu(B, y)\,.$$

It is most helpful to have an intuitive understanding of (6). $\mu(x, A)$, $\mu(B, y)$ and $\mu(B, A)$ are actual Möbius function values corresponding to various clustered versions of X. If one views A as a single point and the points of A' as individual entities, (4) with $y = A$ yields the definition of $\mu(x, A)$. In a similar manner, $\mu(B, y)$ arises from (5). $\mu(B, A)$ arises from viewing A and B as points and the points in between as individual entities.

PROPOSITION 1. *Suppose $X = A + B$ where A and B are not empty and $A > B$. Then A and B have at least one minimal and one maximal point, respectively. Moreover, $\mu(b, a_0) = \mu(b, A)$ and $\mu(b_0, a) = \mu(B, a)$ for each $a \in A$, $b \in B$, minimal point $a_0 \in A$ and maximal point $b_0 \in B$.*

PROOF. Let $a \in A$ and $b \in B$. $[b, a]$ is a finite interval and must contain a minimal point in A and maximal point in B. The equalities above easily follow from definitions. □

PROPOSITION 2. *Suppose $X = A + B$ where $A > B$. Then*

$$(8)\qquad \mu(b, a) = -\mu(b, A)\mu(B, a)$$

(with both factors well defined) for each $a \in A$ and $b \in B$.

PROOF. Since $\mu(b_0, A) = \mu(B, a_0) = -1$ for each minimal point $a_0 \in A$ and maximal point $b_0 \in B$, the desired equality follows from Proposition 1 when a is a minimal point of A or b is a maximal point of B. The remainder of the proof uses induction based on the total number of elements in $[b, a]$. The induction step is:

$$\begin{aligned}\mu(b, a) &= -\textstyle\sum_{b<z\leqq a} \mu(z, a) = -\sum_{b<z<A} \mu(z, a) + \mu(B, a) \\ &= \textstyle\sum_{b<z<A} \mu(z, A)\mu(B, a) + \mu(B, a) \\ &= (\textstyle\sum_{b<z\leqq a_0} \mu(z, a_0))\mu(B, a) \\ &= -\mu(b, a_0)\mu(B, a) = -\mu(b, A)\mu(B, a) . \qquad \square\end{aligned}$$

PROPOSITION 3. *Suppose $X = A + B$ where $A > B$ and $B \neq \emptyset$. Let x_0 be a minimal point of A. Further, let x and y be distinct points satisfying $x \neq x_0$ and $y > x_0$. Then*

$$\textstyle\sum_{B<z\leqq y, z\neq x_0} \mu(x, z) = 0 .$$

PROOF. Using (4):

$$\begin{aligned}0 &= \textstyle\sum_{x\leqq z\leqq y} \mu(x, z) = \sum_{x\leqq z\leqq y, z\in B} \mu(x, z) + \sum_{x\leqq z\leqq y, z\in A} \mu(x, z) \\ &= \textstyle\sum_{x\leqq z<x_0} \mu(x, z) + \sum_{B<z\leqq y} \mu(x, z) \\ &= -\mu(x, x_0) + \textstyle\sum_{B<z\leqq y} \mu(x, z) = \sum_{B<z\leqq y, z\neq x_0} \mu(x, z) . \qquad \square\end{aligned}$$

We describe now, in terms of the notation used in the definition, the three possible types of σ-lower finite spaces:

Type I: X is lower finite.
Type II: X is not lower finite and $\mu(B_{k+1}, A_k) = 0$ for infinitely many k.
Type III: X is not lower finite and $\mu(B_{k+1}, A_k) = 0$ for only finitely many k.

PROPOSITION 4. *The distinction between types* II *and* III *is independent of the particular sequence of partitions $X = A_k + B_k$, $k \geqq 1$.*

PROOF. By viewing the points of B_{k+2} and the points of A_k as single points, it follows from Proposition 2 that $\mu(B_{k+2}, A_k) = -\mu(B_{k+2}, A_{k+1})\mu(B_{k+1}, A_k)$. In turn,

$$\mu(B_{k+l}, A_k) = \pm\textstyle\prod_{j=0}^{l-1} \mu(B_{k+j+1}, A_{k+j}) , \quad l \geqq 2 . \tag{9}$$

It follows that the type, II or III, is unaltered by adding or deleting partitions from a given sequence. $\square$

3. Some theory. We shall successively examine σ-lower finite spaces of types I, II and III.

We have already commented about type I spaces (lower finite spaces) in the introduction. They are so easily analyzed because each set $\{x: x \leqq y\}$, $y \in X$, is finite and the restriction of a distribution function (on X) to $\{x: x \leqq y\}$ is a distribution function on that finite space. Therefore, questions (i), (ii) and (iii) (appearing in the introduction) are easily answered with the use of (1).

Suppose X is a σ-lower finite space of type II. X has the following important property:

PROPOSITION 5. *For a type* II *σ-lower finite space X, each set $\{x: \mu(x, y) \neq 0\}$, $y \in X$, is finite.*

PROOF. In view of (9), we may assume, without sacrificing generality, that $\mu(B_{k+1}, A_k) = 0$ for $k \geqq 1$. Suppose $y \in X$. Then, necessarily, $y \in A_k$ for some k. Since X is not lower finite, A_{k+1} must be a proper subset of X. Consequently, there exists a point $x_0 \in B_{k+1}$. Since $[x_0, y]$ is a finite set, it suffices to show that $\mu(x, y) = 0$ for each $x \notin [x_0, y]$, We only need to consider $x \in B_{k+1}$. For such an x, we have, with successive applications of Proposition 2, $\mu(x, y) = -\mu(x, A_k)\mu(B_k, y) = \mu(x, A_{k+1})\mu(B_{k+1}, A_k)\mu(B_k, y) = 0$. □

The following theorem tells us that (1) holds for type II spaces:

THEOREM 1. *For any locally finite space,* (1) *holds for a given $y \in X$ whenever the number of nonzero summands in* (1) *is finite.*

PROOF. Under the assumption,

$$\sum_{x \leqq y} \sum_{z \leqq x} |m(z)\mu(x, y)| \leqq M(y) \sum_{\{x: x \leqq y, M(x) \neq 0\}} |\mu(x, y)| < \infty .$$

Thus, Fubini's theorem applies. Then,

$$\begin{aligned} \sum_{x \leqq y} M(x)\mu(x, y) &= \sum_{x \leqq y} \sum_{z \leqq x} m(z)\mu(x, y) \\ &= \sum_{z \leqq y} m(z) \sum_{z \leqq x \leqq y} \mu(x, y) = m(y) . \end{aligned}$$ □

While (1) can be used directly to answer question (iii) for type II spaces, it is easier to use the following theorem:

THEOREM 2. *A function M on a type* II *σ-lower finite space X is a distribution function if and only if*

(a) *the sum in* (1) *is nonnegative for each $y \in X$, and*
(b) *the set $\{x: x \leqq y, |M(x)| \geqq \varepsilon\}$ is finite for each $\varepsilon > 0$ and $y \in X$.*

PROOF. Assume (a) and (b), and fix y. Find an A_k containing y and an $x_k \leqq y$ which is a minimal point of A_k. Define m in terms of M by (1). Except for a finite number of $x \in X$, $\mu(x, z) = 0$ for all z in any given finite subset of X. Thus, we may interchange the order of summation below, and we have with the aid of Proposition 3:

$$\begin{aligned} \sum_{B_k < z \leqq y, z \neq x_k} m(z) &= \sum_{x \leqq y} M(x) \sum_{B_k < z \leqq y, z \neq x_k} \mu(x, z) \\ &= M(y)\mu(y, y) - M(x_k)\mu(x_k, x_k) = M(y) - M(x_k) . \end{aligned}$$

Letting $k \to \infty$ leads to $\sum_{z \leqq y} m(z) = M(y)$. Thus, M is a distribution function. Conversely, if M is a distribution function for the individual masses $m(x)$, $x \in X$, then (a) follows from Theorem 1 and (b) follows from the local finiteness of X. □

Suppose X is a σ-lower finite space of type III. Depending on the structure of X, there may be two distinct sets of individual masses with the same distribution function. This contrasts with type I and type II spaces where the answer to question (i) is "Always."

Proposition 4 permits us to assume that $A_1 \neq \varnothing$ and $\mu(B_k, A_1) \neq 0$ for each $k \geqq 1$. For each $x \in X$, choose an arbitrary A_k containing x and define $\beta(x) = -\mu(B_k, x)/\mu(B_k, A_1)$.

PROPOSITION 6. *β is well defined in the sense that the value of $\beta(x)$ does not depend on how one chooses k. $\beta(x) = \mu(B_1, x)$ for $x \in A_1$. β is not identically zero.*

PROOF. For $x \in A_k$, $\mu(B_{k+1}, x) = -\mu(B_{k+1}, A_k)\mu(B_k, x)$ (cf., (8)), and $\mu(B_{k+1}, A_1) = -\mu(B_{k+1}, A_k)\mu(B_k, A_1)$ (cf., (8)). Thus $\beta(x)$ is well defined. If $x \in A_1$, $\mu(B_1, A_1) = -1$ and $\beta(x) = \mu(B_1, x)$. Finally, if x_k is a minimal point of A_k, then $\mu(B_k, x_k) = -1$ and $\beta(x_k) = \mu(B_k, A_1)^{-1} \neq 0$. □

Let β^+ and β^- denote the positive and negative parts of β, respectively.

THEOREM 3. *For each $y \in X$,*

$$\sum_{x \leqq y} \beta^+(x) = \sum_{x \leqq y} \beta^-(x) . \tag{10}$$

Thus, if $\sum_{x \leqq y} |\beta(x)| < \infty$ for each $y \in X$, $\{\beta^+(x)\}$ and $\{\beta^-(x)\}$ represent two distinct sets of individual masses with the same distribution function.

PROOF. Fix y and choose a minimal point x_k of A_k satisfying $x_k \leqq y$, for each A_k containing y. Then $\sum_{B_k < z \leqq y, z \neq x_k} \mu(B_k, z) = 0$ (cf., Proposition 3), and hence,

$$\sum_{B_k < z \leqq y, z \neq x_k} \beta(z) = 0 . \tag{11}$$

Then (10) follows by letting $k \to \infty$. □

A space X will be called a *determining space* if every distribution function on X corresponds to a unique set of individual masses.

THEOREM 4. *A type* III *σ-lower finite space X is a determining space if and only if $\sum_{x \leqq y_0} |\beta(x)| = \infty$ for some $y_0 \in X$.*

We shall defer the proof of the "if" part until later. The "only if" part is immediate from Theorem 3.

Define

$$\begin{aligned} \nu(x, y) &= \mu(x, y) && \text{for } x \in A_1 , \\ &= \mu(x, y) + \mu(x, A_1)\beta(y) && \text{for } x \in B_1 . \end{aligned}$$

$\nu(x, y) = 0$ whenever $x \in B_k$ and $y \in A_k$ for some $k \geqq 1$ (cf., (8)). Likewise, $\nu(x, y) = 0$ whenever $x \in A_1$ and $x \nleqq y$ (cf., (4)). Thus, the sum $\sum_{x \in X} M(x)\nu(x, y)$ has at most a finite number of nonzero summands for each $y \in X$.

PROPOSITION 7. *Let M be a distribution function corresponding to the individual masses $\{m(x)\}$ and let $\alpha = m(B_1)$. M and α together determine m. In particular,*

$$m(y) = \sum_{x \in X} M(x)\nu(x, y) + \alpha\beta(y) , \qquad y \in X . \tag{12}$$

PROOF. $\alpha < \infty$ since there exists a point $y_0 \in A_1$ and $\alpha = m(B_1) \leqq M(y_0) < \infty$. Fix $k \geqq 1$ and view B_k as a point. Then (cf., (1)),

$$m(y) = \sum_{x \in A_k} M(x)\mu(x, y) + m(B_k)\mu(B_k, y) , \qquad y \in A_k . \tag{13}$$

Next, view both $D = \{z \in A_1 : z \leqq y_0\}$ and B_k as points. Then (cf., (1)),

$$M(y_0) - \alpha = m(D) = M(y_0) \cdot 1 + \textstyle\sum_{x \in B_1 - B_k} M(x)\mu(x, A_1) + m(B_k)\mu(B_k, A_1) .$$

Consequently,

$$\alpha + \textstyle\sum_{x \in B_1 - B_k} M(x)\mu(x, A_1) + m(B_k)\mu(B_k, A_1) = 0 . \tag{14}$$

Combining (13) and (14), we obtain for $y \in A_k$,

$$\begin{aligned} m(y) &= \textstyle\sum_{x \in A_k} M(x)\mu(x, y) + (\sum_{x \in B_1 - B_k} M(x)\mu(x, A_1) + \alpha)\beta(y) \\ &= \textstyle\sum_{x \in X} M(x)\nu(x, y) + \alpha\beta(y) . \end{aligned}$$ □

PROOF OF THEOREM 4 ("if" part). Let $\{m_1(x)\}$ and $\{m_2(x)\}$ be distinct sets of individual masses which give rise to the distribution function M. From (12), we have $\Delta m(x) = \Delta\alpha\beta(x)$, where $\Delta m = m_2 - m_1$ and $\Delta\alpha = \Delta m(B_1)$. Since $\Delta m(x) \neq 0$ for some x, $\Delta\alpha \neq 0$. Thus for any $y \in X$,

$$\textstyle\sum_{x \leqq y} |\beta(x)| = |\Delta\alpha^{-1}| \sum_{x \leqq y} |\Delta m(x)| \leqq |2\Delta\alpha^{-1}| M(y) < \infty .$$ □

THEOREM 5. *A function M on a type* III *σ-lower finite space is a distribution function if and only if*

(a) *there exists an α which makes the right-hand side of* (12) *nonnegative for each $y \in X$, and*

(b) *the set $\{x : x \leqq y, |M(x)| \geqq \varepsilon\}$ is finite for each $\varepsilon > 0$ and $y \in X$.*

Each such α corresponds to a unique set of individual masses $\{m(x)\}$ defined by (12), *and for this set, $\alpha = m(B_1)$.*

PROOF. The proof of the "if" part parallels that for Theorem 2. But here one defines m by (12), using the α predicated in assumption (a), instead of (1). The complications brought in by β (both directly and indirectly through $\nu(x, y)$) are taken care of by using (11). One obtains $M(y) = \sum_{x \leqq y} m(x)$, $y \in X$. If α were not equal to $m(B_1)$, Proposition 7 would be contradicted. For the converse, (a) is a consequence of Proposition 7 and (b) is due to the local finiteness of X. □

Let M be a distribution function. It is easy to see that the possible value for $\alpha = m(B_1)$ is an interval $[\alpha_{\min}, \alpha_{\max}]$, perhaps a point.

PROPOSITION 8.

$$\alpha_{\min} = \sup\{-\textstyle\sum_{x \in X} M(x)\nu(x, y)/\beta(y) : \beta(y) > 0\} . \tag{15}$$

$$\alpha_{\max} = \inf\{-\textstyle\sum_{x \in X} M(x)\nu(x, y)/\beta(y) : \beta(y) < 0\} . \tag{16}$$

If $\sum_{x \leqq z} |\beta(x)| = \infty$ for some z (and necessarily for all z), then

$$\alpha_{\min} = \alpha_{\max} = \lim_{k \to \infty} \{\textstyle\sum_{B_k < y < A_1} |\sum_{x \in X} M(x)\nu(x, y)| / \sum_{B_k < y < A_1} |\beta(y)|\} . \tag{17}$$

If $m(B_k)|\mu(B_k, A_1)| \to 0$ as $k \to \infty$, then

$$\alpha_{\min} = \alpha_{\max} = -\lim_{k \to \infty} \textstyle\sum_{B_k < x < A_1} M(x)\mu(x, A_1) . \tag{18}$$

PROOF. Formulas (15) and (16) follow easily from Theorem 5. Suppose $\sum_{x \leq z} |\beta(z)| = \infty$. We may suppose $z \in A_1$. Then from (12), we have for each k:

$$|\alpha \sum_{B_k < y < A_1} |\beta(y)| - \sum_{B_k < y < A_1} |\sum_{x \in X} M(x)\nu(x, y)|| \leqq \sum_{B_k < y < A_1} m(y) \leqq M(z) < \infty .$$

The first sum $\to \infty$ as $k \to \infty$, and (17) follows. Formula (18) follows directly from (14). □

The condition for (18) is difficult to validate directly since one is not likely to know the value of $m(B_k)$ without knowing α. However, $m(B_k) \leqq M(x)$, $x \in A_k$, $k \geqq 1$. So, an indirect validation is possible.

We now turn our attention toward the evaluation of the total mass $m(X)$. A requirement such as $m(X) = 1$ sometimes determines a unique choice for the individual masses when a distribution function, by itself, does not. The most direct formula, of course, is $m(X) = \sum_{x \in X} m(x)$. However, there are some advantages in securing formulas which primarily involve the values of the distribution function.

Let $X^* = X + \{x^*\}$ be an augmented space satisfying $x^* > X$. If $m(x^*) = 0$, then $M(x^*) = m(X)$. If X^* is locally finite, then $\mu^*(x, x^*) = \mu(x, \phi)$, $x \in X$, where μ^* denotes the extension of μ to X^*.[4]

PROPOSITION 9. *Suppose X^* is a σ-lower finite space. Then X is a σ-lower finite space of the same type as X^*. If X^* is of type* I *or type* II,

$$m(X) = -\sum_{x \in X} M(x)\mu(x, \phi) , \tag{19}$$

a sum with only a finite number of nonzero summands.

If X^* is of type III, *then A_1 is a finite set and*

$$m(X) = -\sum_{x \in A_1} M(x)\mu(x, \phi) - \alpha\mu(B_1, \phi) , \tag{20}$$

where $\alpha = m(B_1)$.

PROOF. Briefly, (20) follows from (cf., (1)):

$$0 = m(x^*) = m(X)\mu^*(x^*, x^*) + \sum_{x \in A_1} M(x)\mu(x, \phi) + m(B_1)\mu(B_1, \phi) .$$

Formula (19) is shown similarly. The details are left to the reader. □

Proposition 9 is still usable when X^* is not σ-lower finite: Express X as $\{x_n, n \geqq 1\}$. Then define $X_n = \{z \in X: z \leqq x_j$ for some $j = 1, \cdots, n\}$ and $X_n^* = X_n + \{x^*\}$, $n \geqq 1$. If X is σ-lower finite, then so is each X_n^* (and of the same type as X). Furthermore, $m(X_n) \nearrow m(X)$ as $n \to \infty$.

THEOREM 6. *Suppose X is a σ-lower finite space. If X is of type* I *or* II, *then*

$$m(X) = \lim_{n \to \infty} \sum_{x \in X} \{M(x) \sum_{y \in X_n} \mu(x, y)\} . \tag{21}$$

[4] We shall superscript an extended function with an asterisk only when it aids clarity. Here, $\mu(x, \phi)$ has a different meaning from $\mu^*(x, \phi)$. The latter is zero for all $x \in X$.

If X is of type III,

(22) $$m(X) = \alpha + \lim_{n\to\infty} \sum_{x \in A_1} \{(M(x) - \alpha) \sum_{y \in X_n} \mu(x, y)\},$$

where $\alpha = m(B_1)$. *Each sum has a finite number of nonzero summands.*

PROOF. The details easily follow from (6), (19), (20) and the foregoing discussion.

4. Applications. We shall confine our applications to the two spaces illustrated in Figs. 1 and 2. Both are σ-lower finite spaces of type III. The basic facts about these spaces are summarized in Table 1. For Fig. 1, $\alpha_{\min} = \alpha_{\max} = \sum_{n=1}^{\infty} (-1)^{n-1}\{M(n, 1) + M(n, 2)\}$ (cf., (18)). This is consistent with (2). For Fig. 2, define $S_0 = M(0, 1)$ and, for $n \geqq 1$, $S_n = \sum_{k=1}^{n} (-2)^{k-1}\{M(k, 1) + M(k, 2) + M(k, 3)\} + (-2)^n \min (M(n, 1), M(n, 2), M(n, 3))$. Then, $\alpha_{\min} = \sup \{S_1, S_3, S_5, \cdots\}$ and $\alpha_{\max} = \inf \{S_0, S_2, S_4, \cdots\}$ (cf., (15) and (16)).

Finally, we turn our attention to Examples 1 and 2. We see in Example 1 that the first and second solutions correspond to $m(x) = \frac{1}{2}\beta^-(x)$ and $m(x) = \frac{1}{2}\beta^+(x)$ $(x \in X)$, respectively. For the first solution $\alpha = \alpha_{\min} = \frac{1}{2}$, and for the second $\alpha = \alpha_{\max} = 1$. For Example 2, (18) applies, and we obtain $\alpha = \frac{3}{5}$. With this, the values of $m(x)$ follow immediately from (12).

TABLE 1

X	Fig. 1	Fig. 2
A_n	$\{x : x > (n, 1)\}$	$\{x : x > (n, 1)\}$
$\mu((n, j), (n', j'))$	$(-1)^{n-n'}$ for $n' < n$ 1 for $n = n'$, $j = j'$ 0 otherwise	$-(-2)^{n-n'-1}$ for $n' < n$ 1 for $n = n'$, $j = j'$ otherwise
$\mu(B_n, (n', j'))$	$(-1)^{n-n'}$ for $n' < n$	$-(-2)^{n-n'-1}$ for $n' < n$
$\mu((n, j), A_{n'})$	$(-1)^{n-n'+1}$ for $n' \leqq n$	$-(-2)^{n-n'}$ for $n' \leqq n$
$\mu(B_n, A_{n'})$	$(-1)^{n-n'+1}$ for $n' \leqq n$	$-(-2)^{n-n'}$ for $n' \leqq n$
$\beta(n, j)$	$(-1)^{n+1}$	$-(-2)^{-n}$
$\sum_{x \leqq (0,1)} \lvert\beta(x)\rvert$	∞	4
$\sum_{x \leqq (n,j)} \beta^+(x)$	∞	2^{1-n}
$\sum_{x \leqq (n,j)} \beta^-(x)$	∞	2^{1-n}
$\nu((n, j), (n', j'))$	1 for $n = n' = 0$, $j = j' = 1$ $(-1)^{n-n'-1}$ for $n' > n \geqq 1$ -1 for $n = n' \geqq 1$, $j \neq j'$ 0 otherwise	1 for $n = n' = 0$, $j = j' = 1$ $(-2)^{n-n'-1}$ for $n' > n \geqq 1$ $+\frac{1}{2}$ for $n = n' \geqq 1$, $j = j'$ $-\frac{1}{2}$ for $n = n' \geqq 1$, $j \neq j'$ 0 otherwise

REFERENCES

[1] ANDERSON, T. W. (1960). A modification of the sequential probability ratio test to reduce the sample size. *Ann. Math. Statist.* **31** 165–197.

[2] DOOB, J. L. (1949). Heuristic approach to the Kolmogorov–Smirnov theorems. *Ann. Math. Statist.* **20** 393–403.

[3] ROTA, GIAN-CARLO (1964). On the foundations of combinatorial theory I. Theory of Möbius functions. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **2** 340–368.

[4] SIMONS, GORDON (1974). Generalized distribution functions: the linearly ordered case with applications to nonparametric statistics. *Ann. Probability* **2** 501–508.

DEPARTMENT OF STATISTICS
UNIVERSITY OF NORTH CAROLINA
CHAPEL HILL, NORTH CAROLINA 27514