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RADON–NIKODYM DERIVATIVES WITH RESPECT TO MEASURES INDUCED BY DISCONTINUOUS INDEPENDENT-INCREMENT PROCESSES[1]

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We obtain representation formulas for the Radon–Nikodym derivatives of measures absolutely continuous with respect to measures induced by processes with stationary independent increments. The proofs of these formulas, which have applications in signal detection and estimation problems, call heavily upon recent results in martingale theory, especially a general formula of Doléans-Dade for the logarithm of a strictly positive martingale in terms of a function measuring its jumps.

1. Introduction. Following earlier partial results of Cameron and Martin and of Maruyama, Cameron and Graves [3] presented a by now, well-known formula for the Radon–Nikodym derivative (RND) of a translated Wiener measure w.r.t. the Wiener measure. More specifically, if P_0 is the measure induced by the Wiener process $x(t) = w(t)$ and P_1 is induced by the process

$$x(t) = w(t) + \int_0^t f(u)\, du \tag{1.1}$$

where $f(u)$ is a real-valued L_2-function, then they have shown in Corollary 1, ([3] page 169) that the RND is given by

$$\frac{dP_1}{dP_0} = \exp\{\int_0^1 f\, dx - \tfrac{1}{2}\int_0^1 f^2\, dt\}\,. \tag{1.2}$$

The RND plays the role of the likelihood-ratio statistic in the theory of hypothesis testing for stochastic processes. In this context the above formula has been rediscovered several times and has been widely used in statistical communication engineering as the proper statistic for choosing between the hypotheses (described in the form used by engineers):

$$\begin{aligned} \mathrm{H}_1: \quad \dot{y} &= m + \dot{w} \\ &= \text{signal plus white noise} \\ \mathrm{H}_0: \quad \dot{y} &= \dot{w} \\ &= \text{white noise} \end{aligned} \tag{1.3}$$

where the dots denote formal differentiation with respect to time.

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Now, in many applications, the signal $m(\cdot)$ is itself random and is often—as in feedback communications and feedback control—a function of past $\dot{y}(\cdot)$. Some specific formulae have been obtained for various classes of random signals $m(\cdot)$—most notably for Gaussian $m(\cdot)$. However recently Kailath [13], [14] has obtained a general representation formula for such Radon–Nikodym derivatives which has been quite useful in applications [17] and [26] for reasons described presently. The representation formula is the following: consider the problem of testing between the hypotheses:

$$\text{(1.4)} \qquad \begin{aligned} &H_0: \quad X_t = w_t \quad \text{is a Wiener process} \\ &H_1: \quad X_t = \textstyle\int_0^t z_\tau \, d\tau + w_t \end{aligned}$$

where z_t is a "random signal" and such that w_t is a martingale over some fields $\mathscr{B}_t \supset \mathscr{F}_t = \sigma\{X_s, s \leqq t\}$. The usual assumptions, when z_t is completely independent or past dependent of the noise, can be shown to be special cases of the above. If the probability measures P_0, P_1 respectively induced by $\{X_s\}$ on the space of continuous functions with the natural Borel σ-algebra, are mutually absolutely continuous, then the Radon–Nikodym derivative L_t between their restrictions up to time t is given by:

$$\text{(1.5)} \qquad L_t = \exp\{\textstyle\int_0^t \hat{z}_s \, dX_s - \frac{1}{2} \int_0^t \hat{z}_s^2 \, ds\} \qquad \text{where}$$

$$\hat{z}_s = E\{z_s \,|\, \mathscr{F}_s\} \qquad \text{and} \qquad \mathscr{F}_t = \sigma\{X_s, s \leqq t\}\,.$$

The point of this formula is that it specifies a uniform structure for a large class of signal processes in which the only unknown is the function $\hat{z}(\cdot)$ of (1.5). While explicit formulas for $\hat{z}(\cdot)$ are hard to find, its interpretation as a least-squares estimate permits various approximations to be made to the actual $\hat{z}(\cdot)$. This empirical procedure has been found to be useful in various applications (see e.g., [17], [26]).

Kailath discussed some generalizations of (1.5) in [15], relaxing the assumption of mutual absolute continuity and replacing $w(\cdot)$ by a continuous semimartingale of $w(\cdot)$. In this paper we shall discuss a different class of problems in which the processes involved are discontinuous. More specifically, we shall prove the following result:

Let X_t be a purely discontinuous stochastic process that is right continuous and possesses left-hand limits and let

(1.6a) $\quad \Gamma =$ a Borel subset, bounded away from zero, of the state space.

We shall define the so-called underlying counting process as

$$\text{(1.6b)} \qquad \begin{aligned} p(t, \Gamma) &\triangleq \textstyle\sum_{s \leq t} I_{\{X_s - X_{s-} \in \Gamma\}} \\ &= \text{the number of jumps of } X_s \text{ up to the time } t \\ &\qquad \text{of algebraic magnitude in } \Gamma\,. \end{aligned}$$

We also define

$$\text{(1.7)} \qquad \Pi(\Gamma) \triangleq Ep(1, \Gamma)\,.$$

Now consider the following hypotheses:

H_0: X_t is a process with stationary independent increments, or equivalently a process such that

$$(1.8) \qquad q(t, \Gamma) \triangleq p(t, \Gamma) - \Pi(\Gamma) \cdot t \quad \text{is a martingale on} \quad \mathscr{F}_t = \sigma\{X_s, s \leqq t\}\,, \quad \text{for all} \quad \Gamma \quad \text{as in (1.6a)}$$

H_1: There exists an increasing family of σ-algebras $\{\mathscr{B}_t\}$ such that $\mathscr{B}_t \supset \mathscr{F}_t$ for all t, and a right-continuous random function $\Pi_1(\Gamma, t, \omega)$ that is

(a) a measure on the state space (with its Borel field) for each t and almost every ω;

(b) right continuous with left-hand limits for each Γ and almost every ω;

(c) $\mathscr{B}_t$-measurable for all t and Γ;

and such that

$$(1.9) \qquad p(t, \Gamma) - \int_0^t \Pi_1(\Gamma, s, \omega)\, ds$$

is a martingale on $\mathscr{B}_t$ for all Γ as in (1.6a). (See also Remark 6.5.) Then, if the measures induced by X_t on the space of purely discontinuous functions are mutually absolutely continuous, the Radon–Nikodym derivative L_t between their restrictions up to time t can be written

$$(1.10\text{a}) \qquad L_t = \exp\{\int_0^t \int_{|f|<1} f q(ds, dy) + \int_0^t \int_{|f|\geqq 1} f p(ds, dy) - \int_0^t \int_{|f|\leqq 1} (e^f - f - 1)\Pi(dy)\, ds - \int_0^t \int_{|f|\geqq 1} (e^f - 1)\Pi(dy)\, ds\}$$

where

$$(1.10\text{b}) \qquad f = f(y, t, \omega) = \log \frac{\hat{\Pi}_1(dy, t, \omega)}{\Pi(dy)}$$

and

$$(1.10\text{c}) \qquad \hat{\Pi}_1(\Gamma, t, \omega) = E\{\Pi_1(\Gamma, t, \omega) \,|\, \mathscr{F}_t\}\,.$$

When the measure $\Pi_1(\cdot)$ (under hypothesis H_1) is not random, then $\hat{\Pi}_1 = \Pi_1$ and we have the standard formula, probably first given by Skorokhod ([23] pages 230–234) for the RND of measures induced by two mutually absolutely continuous independent-increment processes (without Gaussian parts). Thus (1.10) is the natural generalization of (1.5) to the discontinuous process case.

Previous results. For counting processes (unity-jumps), such problems were first treated by D. Snyder [24] and then by I. Rubin [21]. P. Frost [9] was the first to consider the general class of independent increment processes and he worked out several interesting examples. However the results of Snyder and Frost suffer from a major limitation—the "signal" must be completely independent of the "noise."

This limitation is inherent in their method of proof which is based on a generalized Bayes Rule for stochastic processes—they first write the RND for the random measure $\Pi_1(t, y, \omega)$ when ω is fixed and therefore Π_1 deterministic and then average over all possible values of ω. However this procedure has no meaning if Π_1 can depend upon the past values of X_t.

The Bayes-rule technique has been used in the Wiener case as well by T. Duncan [8] who formalized and extended the procedure used in some pioneering papers by Stratonovich and Sosulin (see, e.g., [25]). But even in the Gaussian case the method is limited to signal processes which are completely independent of the noise. To avoid this restriction, a heavier use of martingale theory is necessary and a four-step proof—using a series of martingale results—was presented by Kailath in the Wiener case [13], [14]. Recently, Bremaud [2] has used this 4-step proof to treat past-dependent signals for point-process noise (M. H. A. Davis [4] has pointed out that Bremaud's proof is marred by a technical error that we shall discuss at a later point.) In this paper we use the 4-step proof for general SII processes and for convenience we outline it here.

A. *Representation for general martingales.* Using the Doléns-Dade exponential formula for local martingales, we show that any martingale on the probability space induced by X_t under H_0 and in particular the Radon–Nikodym derivative, has the form (1.10a) for some function $f(y, t, \omega)$ that is $\mathscr{F}_t$ adapted for any y.
The remaining steps are concerned with identifying the function f.

B. *A Girsanov-type theorem.* The function f is shown to be such that $p(t, \Gamma) - \int_0^t \int e^f \Pi(dy)\, ds$ is a local martingale on $(H_1, \mathscr{F}_t)$.

C. *Innovations.* From (1.8) we obtain an innovations-type result, namely that $p(t, \Gamma) - \int_0^t \hat{\Pi}_1(\Gamma, s, \omega)\, ds$ is a local martingale on $(H_1, \mathscr{F}_t)$.

D. *Uniqueness.* Since any continuous local martingale must be of unbounded variation, B and C above give

$$\int_0^t \hat{\Pi}_1(\Gamma, s, \omega) - \int_0^t \int_\Gamma e^f \Pi(dy)\, ds = 0 \tag{1.11}$$

from which (1.10b), (1.10c) are obtained.

We turn now to the detailed proof. Some further general remarks will be made in the concluding Section 6.

2. General notations. Let Ω be the space of functions $\omega(\cdot)$ from $\mathbb{R}^+ \to \mathbb{R}$ such that $\omega(0) = 0$ and $\omega(\cdot)$ is right-continuous with left-hand limits. We denote by $X_t(\omega)$, $\omega \in \Omega$, the coordinate function on Ω, defined by

$$X_t(\omega) = \omega(t)\,. \tag{2.1}$$

Define

$$\mathscr{F}_t = \sigma\{X_s(\omega), s \leqq t\} \qquad \text{and} \qquad \mathscr{F} = \bigvee_{t \in \mathbb{R}^+} \mathscr{F}_t\,. \tag{2.2}$$

Let Π be a measure on $\mathbb{R}$ with $\Pi(\{0\}) = 0$ and such that

$$\int_{\mathbb{R}} \frac{y^2}{1 + y^2}\, \Pi(dy) < \infty\,. \tag{2.3}$$

Then (see, for example, Hida [12]), there exists a probability space on which X_t is a process with stationary independent increments (SII), no Wiener component, and canonical measure Π. We write P for the measure induced by $X_t(\omega)$ on $(\Omega, \mathscr{F})$.

Let $U_\varepsilon = \{y; |y| > \varepsilon\}$. For Γ a Borel set in U_ε, we define $p(t, \Gamma)$ as the number of jumps in Γ of $\{X_s, s \leqq t\}$, namely

$$p(t, \Gamma) = \sum_{s \leqq t} I_{\{\Delta X_s \in \Gamma\}}\,, \qquad \text{some } \Gamma \in U_\varepsilon \tag{2.4}$$

where I_A is the indicator function of the set A and

$$X_{s-} = \lim_{\tau \uparrow s} X_\tau \qquad \Delta X_s = X_s - X_{s-}\,. \tag{2.5}$$

Then $p(t, \Gamma)$ is Poisson with rate $\Pi(\Gamma)$ under $(\Omega, \mathscr{F}, P)$. Also

$$q(t, \Gamma) = p(t, \Gamma) - \Pi(\Gamma) \cdot t \tag{2.6}$$

is a square integrable $(\mathfrak{F}_t, P)$ martingale.

We also recall some more difinitions. Following [27], a process X_t is said to be *purely discontinuous* if $X_t = \lim_{\varepsilon \downarrow 0} [\sum_{s \leqq t} \Delta X_s I_{\{|\Delta X_s| > \varepsilon\}}]$. Observe that since X_t is assumed to be SII with no Wiener component, it is purely discontinuous with possibly an additional deterministic term linear in t ([10] Chapter 6). A process that jumps only at the discontinuity points of X_s is called *quasi-left-continuous.*

A martingale of the form $Y_t - A_t$ where Y_t is purely discontinuous and A_t is continuous and of bounded variation is called a *compensated sum of jumps martingale*. It has been shown by Meyer ([20] page 101) that every martingale can be decomposed into a continuous martingale plus a compensated sum of jumps martingale. Under the above assumptions, $\mathscr{F}_t$ can support only martingales that are compensated sums of jumps.

Finally we shall need to define certain classes of functions F^Q_{loc} and F^P_{loc}. We follow here closely Kunita and Watanabe [2].

$$\begin{aligned} \Phi &= \{\text{Borel } [0, \infty) \times \mathscr{F} \text{ measurable processes } \phi_t, \text{ such that } \phi_T(\omega) \in \mathscr{F}_T, \ \forall \text{ stopping time } T\}\,; \\ \Phi_{\text{rc}} &= \{\text{bounded right continuous processes with left-hand limits}\}\,; \\ L &= \Phi \cap \bar{\Phi}_{\text{rc}} \text{ where } \bar{\Phi}_{\text{rc}} \text{ is the completion of } \Phi_{\text{rc}} \text{ with respect to the seminorm} \\ &\qquad \|\phi\|_{(t)} = E(\textstyle\int_0^t \phi_s^2\, ds)^{\frac{1}{2}}\,. \end{aligned} \tag{2.7}$$

Let F^Q be the class of Borel $[0, \infty) \times$ Borel $(-\infty, \infty) \times \mathscr{F}$ measurable functions such that

$$h(t, y, \omega) \in L \qquad \text{for any fixed } y \tag{2.8}$$

and

$$\|h\|_t = E(\textstyle\int_0^t \int_{\mathbf{R}} h^2 \Pi(dy)\, ds) < \infty \qquad \text{all } t\,. \tag{2.9}$$

Also let F^{P+} be the class of *positive* Borel $[0, \infty) \times$ Borel $(-\infty, \infty) \times \mathscr{F}$ measurable functions $g(s, y, \omega)$ such that $g \in \Phi$ for all y and

$$E \int_0^t \int_{\mathbf{R}} g(s, y, \omega)\Pi(dy)\, ds < \infty\,. \tag{2.10}$$

We define

$$F^P = F^{P+} - F^{P+}\,. \tag{2.11}$$

The class F^Q_{loc} is defined as F^Q with (2.9) replaced by

$$E\left(\int_0^{t\wedge T_n} \int_{\mathbf{R}} h^2\Pi(dy)\, ds\right)^{\frac{1}{2}} < \infty \qquad \text{all } t \text{ and all } n \tag{2.12}$$

for some increasing sequence of stopping times $T_n \uparrow \infty$ a.s. The class F^P_{loc} is defined similarly.

We also define in the appendix the following stochastic integrals

$$Q_h(t) = \int_0^t \int_{\mathbf{R}} h(s, y, \omega)q(ds, dy) \qquad \text{for } h \in F^Q_{\text{loc}} \tag{2.13}$$

$$P_g(t) = \int_0^t \int_{\mathbf{R}} g(s, y, \omega)p(ds, dy)\,. \qquad \text{for } g \in F^P_{\text{loc}}\,. \tag{2.14}$$

3. Representation of martingales of discontinuous SII processes.

THEOREM 1. *Let* L_t *be a* compensated sum of jumps martingale *on* $(\Omega, \mathscr{F}, P)$ *with* $L_t > 0$, $L_{t-} > 0$ *and* $L_0 = 1$. *Then there exists a Borel* $[0, \infty) \times$ *Borel* $(-\infty, \infty) \times \mathscr{F}$ *measurable function* $f(t, y, \omega)$ *such that, for all fixed* y, $f(T, y, \omega)$ *is* $\mathscr{F}_T$ *measurable for all stopping times* T *and with*

$$g = fI_{\{|f|\geq 1\}}\,; \qquad h = fI_{\{|f|<1\}}\,, \tag{3.1}$$

we have

$$h \in F^Q_{\text{loc}} \tag{3.2}$$

$$g \in F^P_{\text{loc}} \tag{3.3}$$

$$\int_0^t \int_{\mathbf{R}} (e^{g(s,y,\omega)} - 1)\Pi(dy)\, ds < \infty \qquad \textit{all } t \tag{3.4}$$

and L_t *is given by*

$$L_t = \exp\{P_g(t) + Q_h(t) - \int_0^t \int_{\mathbf{R}} (e^g - 1)\Pi(dy)\, ds - \int_0^t \int_{\mathbf{R}} (e^h - 1 - h)\Pi(dy)\, ds\}\,. \tag{3.5}$$

PROOF. We shall specialize a general result of Doléans-Dade ([6] Theorem 2) to the case of discontinuous SII processes. We may define

$$f_s' = \log\left(1 + \frac{\Delta L_s}{L_{s-}}\right), \qquad \text{where } \Delta L_s = L_s - L_{s-} \tag{3.6}$$

because $L_t > 0$ and $L_{t-} > 0$ imply $\Delta L_t / L_{t-} > -1$. Let

$$h_s' = f_s' I_{\{|f_s'|<1\}}\,; \tag{3.7}$$

$$g_s' = f_s' I_{\{|f_s'|\geq 1\}}\,. \tag{3.8}$$

Now we shall use the notations [6]

$$\hat{\phi} = e^{\phi} - 1 , \qquad \hat{\hat{\phi}} = e^{\phi} - 1 - \phi ,$$
$$S_t^{\phi} = \sum_{s \leqq t} \phi_s \qquad \text{if } \{s \mid \phi_s \neq 0\} \text{ is countable,}$$
$$\Lambda^1_{\mathrm{loc}} = \{\phi \mid S_t^{|\phi|} \text{ is locally integrable}\} ,$$
$$\Lambda^2_{\mathrm{loc}} = \{\phi \mid \phi^2 \in \Lambda^1_{\mathrm{loc}}\} .$$

If $\phi \in \Lambda^1_{\mathrm{loc}}$, let $\tilde{S}_t^{\phi}$ be the unique, predictable, locally integrable process which is a difference of two increasing processes and such that

$$S_t^{\phi} - \tilde{S}_t^{\phi} = Q_t^{\phi} \tag{3.9}$$

is a local martingale. If ϕ is only in Λ^2_{loc}, such an $\tilde{S}_t^{\phi}$ may not exist, but Doléans-Dade ([6] page 190) has shown that we can still find a unique L_2 local martingale, again denoted Q_t^{ϕ}, such that

$$\Delta Q_t^{\phi} = \phi_t \qquad \text{a.s.} \quad \forall t . \tag{3.10}$$

Then Doléans-Dade has shown that

$$g' \in \Lambda^1_{\mathrm{loc}} , \qquad \hat{g}' \in \Lambda^1_{\mathrm{loc}} , \qquad \hat{h}' \in \Lambda^2_{\mathrm{loc}} , \qquad \hat{\hat{h}}' \in \Lambda^1_{\mathrm{loc}} \tag{3.11 a}$$

and

$$L_t = \exp\{(S_t^{g'} - \tilde{S}_t^{\hat{g}'}) + (Q_t^{\hat{h}'} - S_t^{\hat{\hat{h}}'})\} \tag{3.11 b}$$

(see also Remark 6.4). Now we shall identify these quantities for our case. First note that $S_t^{g'}$ has jumps only at discontinuity points of the martingale L_t of an SII process, so that the jump points of $S_t^{g'}$ will be jump points of the sample paths ([1] page 66, [18] page 16 and [5] Theorem VI. 42). Therefore, $S_t^{g'}$ is a quasi-left-continuous functional and moreover, it is purely discontinuous. Then it is shown in the Appendix (cf. (A.12)) that there exists a function $g(t, y, \omega) \in F^P_{\mathrm{loc}}$ such that

$$S_t^{g'} = \sum_{|\Delta X_s| > 0, s \leqq t} g(s-, \Delta X_s, \omega) = \int_0^t \int_{\mathbf{R}} g(s, y, \omega) p(ds, dy) . \tag{3.12}$$

We also have by definition of $S_t^{g'}$ that

$$g(s-, \Delta X_s, \omega) = g_s'(\omega) \qquad \text{where } g_s'(\omega) \neq 0 , \text{ and} \tag{3.13 a}$$

$$|g(s, y, \omega)| \geqq 1 . \tag{3.13 b}$$

Also

$$S_t^{\hat{g}'} = \sum_{s \leqq t} (e^{g_s'} - 1) = \int_0^t \int_{\mathbf{R}} (e^{g(s,y,\omega)} - 1) p(ds, dy) \tag{3.14}$$

is locally integrable because $\hat{g}' \in \Lambda^1_{\mathrm{loc}}$. Hence

$$\int_0^t \int_{\mathbf{R}} (e^g - 1) p(ds, dy) - \int_0^t \int_{\mathbf{R}} (e^g - 1) \Pi(ds)\, dy \tag{3.15}$$

is a local martingale and we can identify

$$\tilde{S}^{\hat{g}'} = \int_0^t \int_{\mathbf{R}} (e^g - 1) \Pi(ds)\, dy . \tag{3.16}$$

Now, since $\hat{\hat{h}}' \in \Lambda^1_{\mathrm{loc}}$ and is bounded by $(e+2)$,

$$\sum |\hat{\hat{h}}'|^2 \leqq (e+2) \sum |\hat{\hat{h}}'| , \tag{3.17}$$

so that $\hat{\hat{h}}'$ belongs to Λ^2_{loc} as well. Hence

$$h' = \hat{h}' - \hat{\hat{h}}' \in \Lambda^2_{\mathrm{loc}} \tag{3.18}$$

and we can write

$$Q_t^{\hat{h}'} - S^{\hat{\hat{h}}'} = Q_t^{h'} + Q_t^{\hat{\hat{h}}'} - S_t^{\hat{\hat{h}}'} = Q_t^{h'} - \tilde{S}_t^{\hat{\hat{h}}'} . \tag{3.19}$$

Moreover, $Q_t^{h'}$ is an L^2-local martingale, so there exists (cf. (A.6) of the Appendix), a function $h(t, y, \omega)$ in F^Q_{loc} such that

$$Q_t^{h'} = \int_0^t \int_R h(s, y, \omega) q(ds, dy) . \tag{3.20}$$

Also, as $Q^{h'}$ has jumps of absolute value less than 1, $|h| \leqq 1$,

$$h(t_-, \Delta X_t, \omega) = h_t'(\omega) \qquad \text{where } h_t'(\omega) \neq 0 \tag{3.21}$$

and

$$\tilde{S}^{\hat{\hat{h}}'} = \int_0^t \int_R (e^{h(s,y,\omega)} - h(s, y, \omega) - 1)\Pi(dy)\, ds . \tag{3.22}$$

Substituting (3.12), (3.16), (3.20), (3.22) into (3.11) gives the stated formula (3.5). To complete the proof we observe that P_g and Q_h have no common jumps; therefore $g \cdot h = 0$ and we can define $f(s, y, \omega) = g(s, y, \omega)$ when $h(s, y, \omega) = 0$ and $f(s, y, \omega) = h(s, y, \omega)$ when $g(s, y, \omega) = 0$. Then f has the required properties and

$$f(s_-, \Delta X_s, \omega) = f_s'(\omega) \qquad \text{where } f_s'(\omega) \neq 0 . \tag{3.23}$$

Finally,

$$\int_0^t \int_R (e^g - 1)\Pi(dy)\, ds < \infty \tag{3.24}$$

follows from [16] Lemma 6.1.

4. Mutually absolutely continuous probability measures on $(\Omega, \mathscr{F})$. We have defined P to be the probability measure for which $\{X_t, \mathscr{F}_t\}$ is a process with stationary independent increments on $(\Omega, \mathscr{F})$.

If P_1 is another probability measure on $(\Omega, \mathscr{F})$ such that $P_1 \ll P$ and $P \ll P_1$, then, if $E^{\mathscr{F}_t}$ denotes the conditional expectation w.r.t. the field $\mathscr{F}_t$ under the measure P,

$$L_t = E^{\mathscr{F}_t} \frac{dP_1}{dP} \tag{4.1}$$

is an $(\mathscr{F}_t, P)$ martingale. Moreover $EL_t = 1$ and $L_t < \infty$ and $L_t > 0$ a.s. By a theorem of Meyer ([19] page 99), $L_{t-} > 0$ a.s. all t and therefore Theorem 1 may be applied to L_t. In other words L_t can be written as in (3.5).

THEOREM 2 (Girsanov Type). *Let $f(t, y, \omega)$ be the function of Theorem 1 with*

$L_t = E^{\mathscr{F}_t}(dP_1/dP)$. *Then for any* $\Gamma \in U_\varepsilon$, *the process*

$$q_1(t, \Gamma) = p(t, \Gamma) - \int_0^t \int_\Gamma e^{f(s,y,\omega)}\Pi(dy)\, ds \tag{4.2}$$

is a $(P_1, \mathscr{F}_t)$ *local martingale.*

PROOF. Recall that $q(t, \Gamma)$ is a square integrable $(P, \mathscr{F}_t)$ martingale. Then

$$q_1(t, \Gamma) = q(t, \Gamma) - \int_0^t \int_\Gamma (e^f - 1)\Pi(dy)\, ds\,. \tag{4.3}$$

Now, $q(t, \Gamma)$ is a $(P_1, \mathscr{F}_t)$ local martingale if and only if $q_1(t, \Gamma)L_t$ is a $(P, \mathscr{F}_t)$ local martingale, because for $s \leqq t$ and $A \in \mathscr{F}_s$ we have

$$\begin{aligned} E_1 I_A q_1(t, \Gamma) &= E I_A \frac{dP_1}{dP} q_1(t, \Gamma) = E I_A q_1(t, \Gamma) E^{\mathscr{F}_t} \frac{dP_1}{dP} \\ &= E I_A q_1(t, \Gamma) L_t \end{aligned} \tag{4.4}$$

and in particular

$$E_1 I_A q_1(s, \Gamma) = E I_A q_1(s, \Gamma) L_s \tag{4.5}$$

so that

$$E_1^{\mathscr{F}_s} q_1(t, \Gamma) = q_1(s, \Gamma) \tag{4.6a}$$

if and only if

$$E^{\mathscr{F}_s} q_1(t, \Gamma) L_t = q_1(s, \Gamma) L_s\,. \tag{4.6b}$$

But the change of variables formula ([7] Theorem 8) with $F(x, y) = xy$ gives

$$L_t q_1(t, \Gamma) = \int_0^t L_{s-}\, q_1(ds, \Gamma) + \int_0^t q_1(s-, \Gamma)\, dL_s + \sum_{s \leqq t} \Delta L_s \cdot \Delta q_1(s, \Gamma) \tag{4.7}$$

and from (3.6), $\Delta \Gamma_s = (e^{f_s'} - 1)L_{s-}$, hence

$$\begin{aligned} \sum_{s \leqq t} \Delta L_s \cdot \Delta q_1(s, \Gamma) &= \sum_{s \leqq t} L_{s-}(e^{f_s'} - 1)\Delta q_1(s, \Gamma) \\ &= \sum_{s \leqq t} L_{s-}(e^{f_s'} - 1)\Delta p(s, \Gamma) \end{aligned} \tag{4.8}$$

since $\int\int e^f \Pi(dy)\, ds$ is a continuous function. But, since $\Gamma \in U_\varepsilon$, the last expression is just

$$\int_0^t \int_\Gamma L_s(e^f - 1)p(ds, dy) = \sum_{s \leqq t, |\Delta X_s| > 0} L_{s-}(e^{f(s-, \Delta X_s, \omega)} - 1) \tag{4.9}$$

so, using (4.3), expression (4.7) becomes

$$\begin{aligned} L_t q_1(t, \Gamma) = \int_0^t L_{s-}\, q(ds, \Gamma) &- \int_0^t L_s \int_\Gamma (e^f - 1)\Pi(dy)\, ds \\ &+ \int_0^t q_1(s-, \Gamma)\, dL_s + \int_0^t \int_\Gamma L_s(e^f - 1)p(ds, dy)\,. \end{aligned} \tag{4.10}$$

The first and the third term are $(P, \mathscr{F}_t)$ local martingales; another local martingale is obtained by combining the second and the fourth term. This completes the proof.

5. "Signals" in discontinuous-type "noise". Our objective now is to identify the function $f(t, y, \omega)$ obtained in Theorem 1 for the following case:

Let $\mathscr{B}_t$ be an increasing family of σ-fields such that $\mathscr{B}_t \supset \mathscr{F}_t$ for all t.

Assume moreover that there exists a function $\Pi_1(\Gamma, t, \omega)$ with the following properties:

(a) $\Pi_1(\cdot, s, \omega)$ is a.e. a measure on $\mathbb{R}$ with $\Pi_1(\{0\}, s, \omega) = 0$.
(b) $\Pi_1(\Gamma, t, \omega)$ is $\mathscr{B}_t$ measurable for any $\Gamma \in U_\varepsilon$ and all t.
(c) $\Pi_1(\Gamma, \cdot, \omega)$ is right continuous for each $\Gamma \in U_\varepsilon$.
(d) $E_1 \int_0^t \Pi_1(\Gamma, s, \omega)\, ds < \infty$ for all $\Gamma \in U_\varepsilon$ and all t;

and such that

$$p(t, \Gamma) - \int_0^t \Pi_1(\Gamma, s, \omega)\, ds \tag{5.1}$$

is a $(P_1, \mathscr{B}_t)$ martingale (see also Remark 6.e). We say then that X_t has a random nonanticipative canonical measure $\Pi_1(\Gamma, t, \omega)$ under $(P_1, \mathscr{B}_t)$.

THEOREM 3 (Innovations). *Set*

$$\hat{\Pi}_1(\Gamma, t, \omega) = E_1^{\mathscr{F}_t}\Pi_1(\Gamma, t, \omega)\,. \tag{5.2}$$

Under the above assumptions,

$$\hat{q}(t, \Gamma) = p(t, \Gamma) - \int_0^t \hat{\Pi}_1(\Gamma, s, \omega)\, ds \tag{5.3}$$

is a $(P_1, \mathscr{F}_t)$ martingale, for any Borel set $\Gamma \in U_\varepsilon$.

PROOF. Clearly $\hat{q}(t, \Gamma)$ is $\mathscr{F}_t$-measurable for all t. Also, for $t \geqq s$

$$\begin{aligned} E_1^{\mathscr{F}_s}&[p(t, \Gamma) - p(s, \Gamma)] \\ &= E_1^{\mathscr{F}_s}E_1^{\mathscr{B}_s}[p(t, \Gamma) - p(s, \Gamma)] \\ &= E_1^{\mathscr{F}_s} \int_s^t \Pi_1(\Gamma, \tau, \omega)\, d\tau = \int_s^t [E_1^{\mathscr{F}_s}\Pi_1(\Gamma, \tau, \omega)]\, d\tau \\ &= \int_s^t \{E_1^{\mathscr{F}_s}[E_1^{\mathscr{F}_\tau}\Pi_1(\Gamma, \tau, \omega)]\}\, dt \\ &= \int_s^t [E_1^{\mathscr{F}_s}\hat{\Pi}_1(\Gamma, \tau, \omega)]\, d\tau = E_1^{\mathscr{F}_s} \int_s^t \hat{\Pi}_1(\Gamma, \tau, \omega)\, d\tau\,. \end{aligned} \tag{5.4}$$

We have applied here (5.1), (5.2) and Fubini's Theorem.

The following lemma provides the last step toward the identification of $f(t, y, \omega)$.

LEMMA 4 (Uniqueness). *Let $L_t = E^{\mathscr{F}_t}(dP_1/dP)$ and P_1, P be mutually absolutely continuous. Let $\Pi_1(\Gamma, t, \omega)$ be as in Theorem* 3. *Then for any $\varepsilon > 0$, $\hat{\Pi}_1(\cdot, t, \omega) \ll \Pi(\cdot)$ on U_ε* a.s. *all t and the function $f(t, y, \omega)$ found in Theorem* 1 *is given by*

$$f(t, y, \omega) = \log \frac{\hat{\Pi}_1(dy, t, \omega)}{\Pi(dy)}\,. \tag{5.5}$$

PROOF. We have

$$\begin{aligned} p(t, \Gamma) &= q_1(t, \Gamma) + \int_0^t \int_\Gamma e^{f(t,y,\omega)}\Pi(dy)\, d\tau \\ &= \hat{q}(t, \Gamma) + \int_0^t \hat{\Pi}_1(\Gamma, \tau, \omega)\, d\tau \end{aligned}$$

so that

$$\int_0^t \left[\int_\Gamma e^{f(\tau,y,\omega)}\Pi(dy) - \hat{\Pi}_1(\Gamma, \tau, \omega)\right] d\tau = q_1(t, \Gamma) - \hat{q}(t, \Gamma) \tag{5.6}$$

and the latter is a local martingale. Clearly it is continuous, and, since

$\int_\Gamma e^f \Pi(dy)$ and $\hat{\Pi}(\Gamma, t, \omega)$ are positive, it is a difference of two increasing processes. Moreover $q_0 - \hat{q}_0 = 0$, so that by Corollary to Theorem 1.3 [16], $q_1(t, \Gamma) - \hat{q}(t, \Gamma) = 0$, or

$$\int_0^t [\int_\Gamma e^{f(\tau, y, \omega)} \Pi(dy)\, d\tau - \hat{\Pi}_1(\Gamma, \tau, \omega)]\, d\tau = 0\,. \tag{5.7}$$

We have therefore

$$\int_\Gamma e^{f(t, y, \omega)} \Pi(dy) = \hat{\Pi}_1(\Gamma, t, \omega) \qquad \text{for all } \Gamma \in U_\varepsilon \text{ and all } t \tag{5.8}$$

which implies the absolute continuity and (5.5).

Putting these results together, we have the following result stated in Section 1.

Let the probability measures P and P_1 be mutually absolutely continuous and let the coordinate process X_t be such that

under P: X_t has stationary independent increments

under P_1: X_t is such that for the underlying counting processess $p(t, \Gamma)$ defined in Section 2, the process

$$q_1(t, \Gamma) = p(t, \Gamma) - \int_0^t \Pi_1(t, \Gamma, \omega)\, d\tau \tag{5.9}$$

is a $\mathscr{B}_t$ martingale; $\mathscr{B}_t$ are any increasing fields including the fields $\mathscr{F}_t = \sigma\{X_s, s \leqq t\}$ and $\Pi_1(t, \Gamma, \omega)$ is a function with the properties (a) and (d) above.

Then $\Pi(\Gamma)$, the canonical measure of X_t under P, is absolutely continuous w.r.t. $\hat{\Pi}_1(t, \Gamma, \omega)$ on U_ε for any $\varepsilon > 0$ and the RND $L_t = E^{\mathscr{F}_t}(dP_1/dP)$ is given by (3.5), with

$$f(t, y, \omega) = \log \frac{\hat{\Pi}_1(t, dy, \omega)}{\Pi(dy)} \tag{5.10}$$

6. Remarks.

REMARK 6.1. Our treatment assumes mutually absolute continuity between P and P_1. We are not aware presently of the most general conditions under which this holds. Grigelionis [11] has shown that if

$$f(t, y, \omega) = \log \frac{\hat{\Pi}_1(t, dy, \omega)}{\Pi(dy)} \tag{6.1}$$

is bounded by a constant, then the two measures P, P_1 are indeed mutually absolutely continuous. In this case the RND can be written in the form:

$$L_t = E^{\mathscr{F}_t} \frac{dP_1}{dP} = \exp\{Q_f(t) - \int_0^t \int_R (e^f - 1 - f)\Pi\,(dy)\, ds\}\,. \tag{6.2}$$

REMARK 6.2. The special case when X_t is a unity-jump process (a Poisson process under P) was treated in Bremaud [2] using Kunita and Watanabe's [16] theory of L^2-local martingales. However, this treatment is not directly applicable to discontinuous processes. Explicitly, for $T_N = \inf\{t \mid |L_t| \geqq N\}$, the process $L_{t \wedge T_N}$ is not uniformly bounded in general, since we have no information about the value of $L_{t\wedge T_N}$ for $t \geqq T_N$. Therefore, we do not have L^2-local martingales. To avoid this difficulty, we consider here separately "big" and "small" jumps. Observe that $Q_t^{h'}$ as defined in (3.10) has jumps of absolute value not greater than 1 because of (3.7) and therefore $Q_{t\wedge T_N}^{h'}$ is uniformly bounded by $(N+1)$ for suitable T_N, and $Q_t^{h'}$ is an L^2-local martingale.

REMARK 6.3. To avoid confusion, we clarify here some of Doléans-Dade's results [6]: Lemma 2 is incorrect as it stands, because the last inequality on page 192 does not hold under the assumption of the lemma. However, it is easily corrected by changing the assumption to

$$\frac{1}{v} \leqq 1 + \frac{\Delta\alpha_t}{\alpha_{t-}} \leqq v \qquad \text{for some } 1 < v < \infty . \tag{6.3}$$

Now, Lemma 1 will continue to hold if we assume

$$u \leqq 1 + \frac{\Delta\alpha_t}{\alpha_{t-}} \qquad \text{or} \qquad 0 < 1 + \frac{\Delta\alpha_t}{\alpha_{t-}} \leqq \frac{1}{u} \tag{6.4}$$

and Theorem 2 will then be correct if we take

$$\begin{aligned} g &= fI_{\{|f|\geqq 1\}} \\ h &= fI_{\{|f|<1\}} . \end{aligned} \tag{6.5}$$

(Observe the absolute values in the indicator functions.)

REMARK 6.4. If the process X_t is under H_0 a general SII process (i.e., having both a continuous and a discontinuous part) then (1.4) changes to (see [6] Theorem 2)

$$L_t = \exp\{(\beta_t^c - \tfrac{1}{2}\langle\beta^c, \beta^c\rangle_t) + (S_t^{g'} - \tilde{S}_t^{\hat{g}'}) + (Q_t^{\hat{h}'} - S_t^{\hat{\hat{h}}'})\} \tag{6.6}$$

where β_t is a continuous local martingale. The hypotheses change to:

H_0: X_t has SII, i.e., $X_t = X_t^d + w_t$ where w_t is Wiener

H_1: $X_t = X_t^d + X_t^c$ where

$p(t, \Gamma) - \int_0^t \Pi_1(s, \Gamma, \omega)\, ds$ is an $(\mathscr{F}_t, P)$ martingale

$X_t^c - \int_0^t z_1(s, \omega)\, ds$ is an $(\mathscr{F}_t, P)$ Wiener process .

Then Kailath [14] has shown that

$$\beta_t^c = \int_0^t \hat{z}_1(s)\, d\nu_s \tag{6.7}$$

where

$$\nu_t = X_t^c - \int_0^t \hat{z}_1(s, \omega)\, ds \tag{6.8}$$

is a Wiener process so that L_t will be given by (3.5) multiplied by

$$\exp\{\int_0^t \hat{z}_1(s)\, d\nu_s - \tfrac{1}{2}\int_0^t \hat{z}_1^2(s)\, ds\} . \tag{6.9}$$

REMARK 6.5. Observe that as shown in [22], for an *arbitrary* jump process X_t with $E_1 p(t, \Gamma) < \infty$ and arbitrary family $\{\mathscr{B}_t\}$ of σ-fields with $\mathscr{B}_t \supset \mathscr{F}_t$ all t, there exists by Meyer's decomposition a $\mathscr{B}_t$-predictable increasing process $A(\cdot, \Gamma)$ s.t.

$$p(t, \Gamma) - A(t, \Gamma) \tag{6.10}$$

is a $(P_1, \mathscr{B}_t)$-martingale. Conditions under which $A(t, \Gamma)$ is absolutely continuous are not known to us. This is not restrictive though, since under mutual absolute continuity, even if $A(t, \Gamma)$ is not absolutely continuous, its $\mathscr{F}_t$-predictable dual projection $A^{(3)}(t, \Gamma)$ ([5] page 110) must be so by (5.7). The $\mathscr{F}_t$-predictable dual projection is just the $\mathscr{F}_t$-predictable increasing process that makes $p(t, \Gamma) - A^{(3)}(t, \Gamma)$ as $(P_1, \mathscr{F}_t)$-martingale, and clearly if

$$A(t, \Gamma) = \int_0^t \Pi_1(s, \Gamma)\, ds \qquad \text{then}$$

$$A^{(3)}(t, \Gamma) = \int_0^t \hat{\Pi}(s, \Gamma)\, ds .$$

APPENDIX

Stochastic integrals. We follow here closely Kunita and Watanabe [16]. In Section 2 we have defined the processes

$$p(t, \Gamma) = \sum_{s \leq t} I_{\{X_s - X_{s-} \in \Gamma\}} \qquad \Gamma \in U_\varepsilon \tag{A.1}$$

$$q(t, \Gamma) = p(t, \Gamma) - \Pi(\Gamma) \cdot t \tag{A.2}$$

and the class of functions F^P, F^Q, F^P_{loc}, F^Q_{loc}. For $h \in F^Q$ of the form

$$h(t, y, \omega) = \phi(t, \omega) I_{\{y \in \Gamma\}} ; \qquad \Gamma \in U_\varepsilon ;\ \phi \in L , \tag{A.3}$$

we shall define a stochastic integral

$$Q_h = \int_0^t \int_{\mathbf{R}} h(s, y, \omega) q(ds, dy) = \int_0^t \phi(s, \omega) q(ds, \Gamma) . \tag{A.4}$$

We recall that the last integral is a stochastic integral as defined by Kunita and Watanabe ([16] Theorem 2.1) and is further clarified in a remark of Meyer ([20] page 80). Namely one defines the last integral as a Stieltjes integral with the integrand ϕ replaced by a predictable version of ϕ. In our case, this amounts to working with the Stieltjes integral

$$\int_0^t \phi(s_-, \omega) q(ds, \Gamma) . \tag{A.5}$$

Now an arbitrary process $h \in F^Q$ can be approximated by linear combinations $h^{(n)}$ of processes of the form (A.3), s.t. $\|h - h^{(n)}\|_t \to 0$. Then $Q_{h^{(n)}}$ is a Cauchy sequence and we define $Q_h = \lim Q_{h^{(n)}}$, and Q_h is a square integrable compensated sum of jumps martingale. Conversely any square integrable compensated sum of jumps martingale on an SII process can be written as

$$\int_0^t \int_{\mathbf{R}} h(s, y, \omega) q(ds, dy) \tag{A.6}$$

for some $h \in F^Q$. This follows directly from [16] Proposition 5.2, since we can easily verify that for a process with SII, the pair $(\Pi(dy - x), t)$ forms a Levy system and that the basic process $q_\varepsilon(t, E)$ of Kunita and Watanabe can be written as

$$q_\varepsilon(t, E) = \int_0^t \int_{|y| > \varepsilon} I_{\{y \in E - X_{s-}\}}\, q(ds, dy) . \tag{A.7}$$

The above definitions can also be easily extended to locally square integrable martingales with F_{loc}^{Q} as defined in Section 2.

For $g \in F_{\text{loc}}^{P}$ we define

$$P_g(t) = \int_0^t \int_{\mathbf{R}} g(s, y, \omega) p(dy, ds) = \sum_{|\Delta X_s|>0, s \leq t} g(s-, \Delta X_s, \omega) . \tag{A.8}$$

Then $P_g(t)$ is a purely discontinuous quasi-left-continuous process with

$$EP_g(t \wedge T_n) = E \int_0^{t \wedge T_n} \int_{\mathbf{R}} g(s, y, \omega) \Pi(dy)\, ds \tag{A.9}$$

and hence

$$E|P_g(t \wedge T_n)| \leqq E \int_0^{t \wedge T_n} \int_{\mathbf{R}} |g| \Pi(dy)\, ds < \infty . \tag{A.10}$$

Conversely, any purely discontinuous quasi-left-continuous process φ_t with the property

$$E|\varphi_{t \wedge T_n}| < \infty \tag{A.11}$$

can be written as

$$\varphi_t = P_g(t) \tag{A.12}$$

for some $g \in F_{\text{loc}}^{P}$. To see this, observe that we can write

$$\varphi_t = \sum_{\Delta \varphi_s > 0} \Delta \varphi_s + \sum_{\Delta \varphi_s < 0} \Delta \varphi_s \tag{A.13}$$

and show (A.12) for each component separately, so that we may assume that φ_t has *only positive jumps.*

Now define

$$\varphi_t^{N,m} = \sum_{s \leq t, |\Delta X_s| > 1/m} (\Delta \varphi_s \wedge N) . \tag{A.14}$$

Then $\varphi_t^{N,m}$ purely discontinuous and $\varphi_t^{N,m} \uparrow \varphi_t$ as $N, m \uparrow \infty$, so that we can show (A.12) only for $\varphi_t^{N,m}$. Next observe that $E\varphi_{t \wedge T_n}^{N,m} < \infty$ and hence $\varphi_{t \wedge T_n}^{N,m}$ is a positive submartingale of class (DL) (see [9] page 101) that is also quasi-left-continuous, so that there exists a continuous increasing process $A_t^{N,m}$, such that

$$w_t^{N,m} = \varphi_t^{N,m} - A_t^{N,m} \tag{A.15}$$

is a local martingale, and since $\varphi_t^{N,m}$ has bounded jumps and $A_t^{N,m}$ is continuous, $w_t^{N,m}$ is also locally square integrable. Therefore $w_t^{N,m}$ can be written as

$$w_t^{N,m} = \int_0^t \int_{\mathbf{R}} h(s, y, \omega) q(ds, dy) \tag{A.16}$$

for some $L \in F_{\text{loc}}^{Q}$ and now it is clear that

$$\varphi_t^{N,m} = \sum_{s \leq t} h(s-, \Delta X_s, \omega) . \tag{A.17}$$

NOTE 1. Observe that we have changed somewhat the definitions of Kunita and Watanabe [16] to deal with the *size* of the jumps rather than the position after the jump; this is more natural in the SII case.

NOTE 2. For φ_t as in (A.11) that is also an additive functional, Watanabe [27] has shown that

$$\varphi_t = P_g(t) \tag{A.18}$$

where

$$g(s-, \Delta X_s, \omega) = g(X_{s-}, X_s) = \bar{g}(X_{s-}, \Delta X_s) \tag{A.19}$$

namely the functional g is only a function of the value X_{s-} of the sample path immediately before the jump (and of the value of the jump). In general g is dependent on all the past $\{X_\tau(\omega), \tau < s\}$.

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