The Annals of Probability
1975, Vol. 3, No. 4, 597-607

ON A PROBLEM OF COX CONCERNING POINT PROCESSES IN R^k OF "CONTROLLED VARIABILITY"

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Suppose that the k-dimensional lattice points are displaced independently of each other with the same probability distribution. Denote by Var $N(A)$ the variance of the number of displaced points contained in the set A. The asymptotic behaviour of Var $N(A)$ is determined for large convex A's.

Suppose that the k-dimensional lattice points are displaced independently of each other with the same probability distribution. Denote by Var $N(A)$ the variance of the number of displaced points contained in the set A. The asymptotic behaviour of Var $N(A)$ is determined for large convex bounded A's in the sense that, when the generic displacement random variable η has finite first moment, we show that

$$(0) \qquad w(cA) = \int_{I^k} \operatorname{Var} N(cA - s)\lambda(ds) \sim Kc^{k-1} = K'S(cA) \quad (c \to \infty)$$

for some constants K and K' depending on η and the shape of A, where I^k is the unit cube in R^k, λ denotes Lebesgue measure in R^k, and $S(A)$ is the surface area of A. We discuss the behaviour of $w(A)$ for large $S(A)$ under certain conditions on A, and show the weaker property Var $N(cA) \leqq K''c^{k-1}$ $(c \to \infty)$ for some constant K'' similar to before. The behaviour of $w(A)$ in the case of η having $E|\eta| = \infty$ but $E|\eta|^{1-\delta} < \infty$ for some $0 < \delta < 1$ is also discussed.

1. Introduction. At the Stochastic Point Processes Conference in August 1971, D. R. Cox posed the following problem under the title *Controlled Variability Processes in Two or More Dimensions*:

> Consider a regular network of points in R^2, e.g., the set of lattice points, or the vertices of regular hexagons. Let each point be displaced by a random vector, displacements of different points being independent and identically distributed. Let A be a convex set, and let $N(A)$ denote the number of displaced points contained in the set A. Is it true that as the diameter of A tends to infinity,
>
> $$(*) \qquad \operatorname{Var} N(A)/(\text{circumference of } A)$$
>
> (i) is bounded above, (ii) is bounded away from 0, (iii) tends to a limit? Generalize to more than 2 dimensions.

Received December 6, 1973; revised November 12, 1974.

[1] The second named author expresses his thanks to the IASPS for supporting his participation at the conference held at IBM Yorktown Heights Laboratories.

AMS 1970 *subject classifications.* Primary 60D05, 60K99.

Key words and phrases. Stochastic point processes, random displacements, geometrical probability.

We shall discuss the k-dimensional setting of this problem, denoting by Z^k the set of (rectangular) lattice points $n = (n_1, \cdots, n_k)$, $n_i \in \{0, \pm 1, \cdots\}$, in R^k. Let $\{\eta_n \mid n \in Z^k\}$ be a family of independently distributed R^k-valued random variables (rv's) with common distribution $Q(\cdot)$, and define

$$N(A) = \text{Card}\,\{n \in Z^k \mid n + \eta_n \in A\}$$

for bounded sets $A \subset R^k$. If we replace the lattice points Z^k by a Poisson-distributed family of points with unit rate parameter, then $\text{Var}\, N(A) = EN(A) =$ volume of A for all bounded measurable A. So the point of Cox's question is whether starting from the deterministically distributed set reduces the order of growth of $\text{Var}\, N(cA)$ $(c \to \infty)$ from c^k to c^{k-1}. In one dimension, when $E|\eta_n| < \infty$, (i) holds for (*) but not necessarily either of (ii) or (iii) (see Lewis and Govier (1964) who coined the term "controlled variability," and Corollary 1 and Example 7 of Daley (1971)).

The question becomes more tractable on replacing $\text{Var}\, N(A)$ by its value

$$w(A) = \int_{I^k} \text{Var}\, N(A - s)\lambda(ds) \tag{1.1}$$

averaged over the unit cube $I^k \subset R^k$ with respect to Lebesgue measure $\lambda(\cdot)$. In Section 6 we show what can be said without such averaging. The results are then not as elegant because displacements of A may cause $N(A)$ to oscillate. There is a brief resumée of related results in Kendall (1948) where the R^2 case (and in Kendall and Rankin (1953) the R^k case) of the present problem is discussed using averaging as at (1.1) in the special case of no displacement ($\eta_n = 0$ a.s.). It should also be noted that by introducing a rv ζ independent of $\{\eta_n\}$ uniformly distributed over I^k, the point process N^* defined by

$$N^*(A) = N(A + \zeta)$$

is stationary, and its variance $V(A) = \text{Var}\, N^*(A)$, like $w(A)$, is translation invariant, the two functions being related by

$$V(A) = w(A) + E[E(N(A + \zeta) \mid \zeta) - \lambda(A)]^2\,.$$

We do not propose discussing V here; the one-dimensional case is the function discussed in the work of Lewis and Govier referenced above.

Denote by

$$\alpha = \eta' - \eta'' \tag{1.2}$$

the difference of two independent rv's η' and η'' with common distribution Q, and for nonzero $u \in R^k$, let $m(u) \equiv m_A(u)$ denote the $(k - 1)$-dimensional volume of the projection A^u of A onto the hyperplane H^u orthogonal to u. Then the character of our results appears most clearly in the following:

THEOREM 1. *When $E|\alpha| < \infty$, and A is any convex bounded closed set $\subset R^k$,*

$$\lim_{c \to \infty} w(cA)/c^{k-1} = E(|\alpha| m_A(\alpha))\,. \tag{1.3}$$

To discuss $\text{Var}\, N(A)$ and its averaged form $w(A)$ we use the simple equations of Lemma 1 in which $\nu(A)$ denotes the number of lattice points contained in the set A, η' and η'' are as in (1.2), and

$$g_A(u) = \lambda(A \cap (A^c - u)) . \tag{1.4}$$

LEMMA 1.

$$\operatorname{Var} N(A) = E\nu((A - \eta') \cap (A^c - \eta'')) , \tag{1.5}$$

$$w(A) = Eg_A(\alpha) = E\lambda((A - \eta') \cap (A^c - \eta'')) . \tag{1.6}$$

PROOF. Let $\xi_n = \chi_A(n + \eta_n)$, where χ_A is the characteristic function of A, and observe that for different n the rv's ξ_n are independent Bernouilli rv's with $E\xi_n = Q(A - n)$. So

$$\begin{aligned}\operatorname{Var} N(A) &= \textstyle\sum_{n \in Z^k} Q(A - n)Q(A^c - n) = E \sum_{n \in Z^k} \chi_{A-n}(\eta')\chi_{A^c-n}(\eta'') \\ &= E\nu((A - \eta') \cap (A^c - \eta'')) ,\end{aligned}$$

proving (1.5). (1.6) follows because

$$\textstyle\int_{I^k} \nu(B - s)\lambda(ds) = \lambda(B) .$$

In the next section we collect together some simple results needed later, and in Section 3 present a proof of (1.3). The work of Section 4 indicates how the shape of A may affect $w(A)/S(A)$, and leads on to Section 5 where $E|\alpha| = \infty$ can lead to orders of growth for $w(cA)$ intermediate between c^{k-1} and c^k.

2. Elementary results. We assume throughout that $k \geqq 2$, and that A is a convex bounded closed set in R^k with positive k-dimensional volume. Write

$$D \equiv D_A = A - A = \{x - y \mid x, y \in A\} .$$

The following geometrical facts are simply demonstrated and stated without proof.

(F1) *D_A is convex, central-symmetric with respect to the origin* 0. *If A is central-symmetric with respect to* 0, $D = 2A = \{2x \mid x \in A\}$.

For fixed $u \in R^k$, $u \neq 0$, and real c, define

$$r(u) \equiv r_A(u) = \sup_{cu \in D} |cu| , \tag{2.1}$$

$$\rho \equiv \rho_A = \inf_{u \neq 0} r_A(u) , \tag{2.2}$$

and let e_u denote the unit vector in the direction of u ($u \neq 0$).

(F2)
$$\begin{aligned}r_A(u) &= r_A(e_u) = \sup \{c \mid A \cap (A - ce_u) \neq \emptyset\} , \\ \rho_A &= \inf \{|u| \mid A \cap (A - u) = \emptyset\} .\end{aligned}$$

(F3) *ρ_A is the inner radius of D_A, and also the width of A (i.e., the minimum of the orthogonal projections of A onto all possible lines in R^k). The width of D_A is $2\rho_A$.*

Denote by (u, v) the scalar product of elements $u, v \in R^k$.

(F4) *There exists a unit vector* $e_a \in R^k$ *such that* $r_A(e_a) = \rho_A$, *and for any other unit vector* e_u, $(e_u, e_a)r_A(e_u) \leqq r_A(e_a)$.

We note that

$$(2.3) \qquad \rho_A' \equiv \sup_{u \neq 0} r_A(u)$$

is the diameter of A, $2\rho_A'$ *is the diameter of* D_A.

Denote by $V_i(A)$ the sum of the $\binom{k}{i}$ projections of A onto the various i-dimensional coordinate subspaces obtained by setting $k - i$ of the coordinates equal to zero. Then Davenport (1951) showed

(F5) *For any convex set* A,

$$(2.4) \qquad |\lambda(A) - \nu(A)| \leqq 1 + \textstyle\sum_{i=1}^{k-1} V_i(A) \equiv T(A)\,.$$

3. Proof of Theorem 1. In view of Lemma 1, we examine $g(u)$ defined at (1.4) whence it follows that

$$(3.1) \qquad g(u) \leqq \lambda(A)$$

with equality for all $|u| \geqq r(u)$. To obtain a lower bound on $g(u)$, denote the projection of $A \cap (A - u) \equiv A_1$ onto H^u by $A_1{}^u$, and let $m^1(u) \equiv m_A{}^1(u)$ be its $(k - 1)$-dimensional volume. Then geometrical considerations lead us easily to the two inequalities

$$(3.2) \qquad |u|m^1(u) \leqq g(u) \leqq |u|m(u)\,.$$

We proceed to show that for u such that $|u| \leqq r(u)$,

$$(3.3) \qquad m^1(u) \geqq m(u)(1 - |u|/r(u))^{k-1}\,.$$

To see this, take some fixed $x \in A \cap (A - r(u)e_u)$, the latter being nonempty since A is closed, and denote by x^u its projection onto H^u. Define

$$(3.4) \qquad A_x{}^u = x^u + (1 - |u|/r(u))(A^u - x^u) = (|u|/r(u))x^u + (1 - |u|/r(u))A^u\,.$$

Take $y^u \in A_x{}^u$ as the projection of

$$y = (|u|/r(u))x + (1 - |u|/r(u))z$$

where $z \in A$, observing that $y \in A$ by convexity. Now $x + r(u)e_u \in A$, and y^u is also the projection of

$$y + u = (|u|/r(u))(x + r(u)e_u) + (1 - |u|/r(u))z\,,$$

so $y + u \in A$, and hence, $y \in A \cap (A - u)$ and $y^u \in A_1{}^u$. Thus $A_x{}^u \subseteqq A_1{}^u$, which used in conjunction with (3.4) and the first of the relations

$$(3.5) \qquad m_{cA}(u) = c^{k-1}m_A(u)\,, \qquad r_{cA}(u) = cr_A(u) \quad (c > 0)\,.$$

proves (3.3). Substituting (3.3) into (3.2) and using (3.1) leads to

$$(3.6) \qquad E[|\alpha_D|m_A(\alpha)(1 - |\alpha|/r_A(\alpha))^{k-1}] \leqq w(A) \leqq E[|\alpha_D|m_A(\alpha)] + \lambda(A)P(R^k\backslash D)$$

where for any Borel set $L \subseteqq R^k$, $\alpha_L = \alpha$ if $\alpha \in L$, $= 0$ otherwise, and $P(L) = pr\{\alpha \in L\}$. If $E|\alpha| < \infty$, then $cP(R^k\backslash cD) \to 0$ for $c \to \infty$, so using (3.5) in (3.6) we find that then

$$\begin{aligned} E[|\alpha|m_A(\alpha)] &\geqq \limsup_{c\to\infty} w(cA)/c^{k-1} \\ &\geqq \liminf_{c\to\infty} E[|\alpha_{cD}|m_A(\alpha)(1-|\alpha|/cr_A(\alpha))^{k-1}] \\ &= E[|\alpha|m_A(\alpha)]\,. \end{aligned}$$

This completes our proof of Theorem 1.

Recalling that $m(u)r(u) = m(e_u)r(e_u)$, substitution of (3.3) into (3.2) and maximizing over $|u|/r(u)$ shows that $k^{-1}(1-k^{-1})^{k-1}r(u)m(u) \leqq g(u)$, so combining with (3.1) and the fact that $4(1-k^{-1})^k \geqq 1$ for $k \geqq 2$ leads to the elementary geometrical

LEMMA 2. *For any* $u, u' \in R^k$ *different from* 0 *and* $k \geqq 2$,

$$(4k-4)^{-1}r(u)m(u) \leqq \lambda(A) \leqq r(u')m(u')\,. \tag{3.7}$$

We conclude this section with an expression for $E|\alpha|m(\alpha)$ as an integral over the surface F_A of A. Denote by μ_A the surface measure on F_A, so that $\mu(F_A) = S(A)$. Through every $x \in F_A$ there is at least one supporting hyperplane of A. We denote its (outward) normal by n_x, noting that it is uniquely defined except on a μ-null set. Without loss we exclude x from this set, and define R_x to be the line $\{x + cn_x \mid -\infty < c < \infty\}$, L_x the half-line $\{x - cn_x \mid c \geqq 0\}$, and Q_x the projection of the measure Q onto R_x (thus, $Q_x(B) = Q\{\eta \mid \eta, n_x) \in B\}$ for one-dimensional Borel sets B). Write $w_x(A) = w(L_x)$ for displacements having distribution Q_x in the one-dimensional space R_x. From the proof of Lemma 1,

$$w_x(A) = \int_{-\infty}^{\infty} Q_x(L_x + s)Q_x(L_x^c + s)\,ds = \int_{-\infty}^{\infty} Q\{(\eta, n_x) \leqq s\}Q\{(\eta, n_x) > s\}\,ds\,.$$

LEMMA 3. $E|\alpha|m(\alpha) = \int_{F_A} w_x(A)\mu(dx)$.

PROOF. For a unit vector u we have $m_A(u) = \int_{F_A} (u, n_x)^+\mu(dx)$, so

$$E|\alpha|m(\alpha) = \int_{F_A} E(\alpha, n_x)^+\mu(dx)\,.$$

$$E(\alpha, n_x)^+ = E(\eta' - \eta'', n_x)^+ = Eg_{L_x}((\eta', n_x) - (\eta'', n_x)) = w_x(A)$$

by Lemma 1, and that completes the proof.

4. Bounds for $w(A)/S(A)$.

For any Borel set $L \neq R^k$ define

$$\delta_L = \min\,(1/k, \inf\{|u|/r_A(u) \mid u \in L^c\})\,.$$

Then since for any fixed $u \neq 0$, $g(xe_u)$ is monotonic nondecreasing in $x \geqq 0$, it follows by referring to (3.3), (3.7) and (3.2) that

$$g(u) \geqq \delta_L(1-\delta_L)^{k-1}\lambda(A) \quad (u \in L^c)\,. \tag{4.1}$$

Suppose from now on that $L \subset D$. Then as a generalization of (3.6) there follow the inequalities

$$\begin{aligned} E[|\alpha_L|m_A(\alpha)(1-|\alpha|/r_A(\alpha))^{k-1}] + \delta_L(1-\delta_L)^{k-1}\lambda(A)P(L^c) \\ \leqq w(A) \leqq E[|\alpha_L|m_A(\alpha)] + \lambda(A)P(L^c)\,. \end{aligned} \tag{4.2}$$

Now it is clear that since $\lambda(A) > 0$,

$$m(u) \leqq S(A)/2 \quad (u \neq 0)\,, \tag{4.3}$$

so taking $u' = e_u$ in (3.7) and recalling (F3) yields

$$\lambda(A) \leqq \rho_A S(A)/2\,. \tag{4.4}$$

To obtain a lower bound on $\lambda(A)/S(A)$, recall that $S(A) = S_k Em_A(\sigma)$ where the rv σ is uniformly distributed on the unit sphere in R^k and S_k is the ratio of the surface area of a unit sphere in R^k to the volume of the unit sphere in R^{k-1}. This identity is known as Cauchy's surface area formula (cf. Eggleston (1958)) and is a special case of our Lemma 3. So there must exist some u' for which

$$S(A) \leqq S_k m_A(u')\,, \tag{4.5}$$

and from (3.7) we then have

$$4(k-1)\lambda(A) \geqq r_A(u')m_A(u') \geqq \rho_A S(A)/S_k\,. \tag{4.6}$$

Combining (4.2), (4.4), and (4.6) gives

THEOREM 2. *For any Borel set* $L \subset D$,

$$\begin{aligned} &E[|\alpha_L|\{m_A(\alpha)/S(A)\}\{1-|\alpha|/r_A(\alpha)\}^{k-1}] \\ &\quad + \{\delta_L(1-\delta_L)^{k-1}\rho_A/4(k-1)S_k\}P(L^c) \\ &\qquad \leqq w(A)/S(A) \leqq E[|\alpha_L|m_A(\alpha)/S(A)] + (\rho_A/2)P(L^c)\,. \end{aligned} \tag{4.7}$$

Observe that using (F4) and (3.7) with (4.6) yields

LEMMA 4. $m_A(e_a) \geqq S(A)/4(k-1)S_k$.

THEOREM 3. *Suppose* $E|\alpha| < \infty$. *Then*

$$|w(A) - E|\alpha|m_A(\alpha)|/S(A) < f(k, Q, \rho_A) \tag{4.8}$$

where for fixed k *and* Q, $f(k, Q, r) \to 0$ $(r \to \infty)$.

PROOF. Using first Lemma 1 and (3.2), and then (4.3), gives for any A

$$w(A)/S(A) \leqq E|\alpha|m_A(\alpha)/S(A) \leqq E|\alpha|/2 < \infty\,, \tag{4.9}$$

so to prove (4.8) we need only to show that $(E|\alpha|m_A(\alpha) - w(A))/S(A)$ can be made small. Consider the first term in (4.7): since $(1-x)^{k-1} \geqq 1-(k-1)x$ for $x > 0$ and $k \geqq 2$, and the second term in (4.7) is nonnegative, we have

$$\begin{aligned} (E|\alpha_L|m_A(\alpha) - w(A))/S(A) &\leqq (k-1)E[\{|\alpha_L|^2/r_A(\alpha)\}\{m_A(\alpha)/S(A)\}] \\ &\leqq E|\alpha_L|^2/2\rho_A\,. \end{aligned} \tag{4.10}$$

Write $\alpha_r = \alpha_{S(r)}$ for the sphere $S(r)$ of diameter r and center at 0. Then since $|\alpha_r| \leqq r$ and

$$|\alpha_r|^2/r = |\alpha_r - \alpha_{r^{\frac12}}|^2/r + |\alpha_{r^{\frac12}}|^2/r \leqq |\alpha - \alpha_{r^{\frac12}}| + |\alpha|/r^{\frac12}\,,$$

setting $L = S(\rho) = S(\rho_A)$ in the right hand side of (4.10) gives

$$2(E[|\alpha_\rho|m_A(\alpha)] - w(A))/S(A) \leqq E|\alpha - \alpha_{\rho^{\frac12}}| + E|\alpha|/\rho^{\frac12}\,. \tag{4.11}$$

Thus the right-hand side of (4.11) $\to 0$ as $\rho \to \infty$. Bounded convergence proves that $E||\alpha - \alpha_\rho|m_A(\alpha)|/S(A) \to 0$ $(\rho \to \infty)$, and coupling this with the convergence to zero from (4.11) completes our proof.

THEOREM 4. *Suppose that* $0 < E|\alpha| < \infty$. *Then for any finite constants* $a_1 > 0$, $a_2 > 1$, *and any* A *such that*

$$\rho_A \geqq a_1 , \qquad \rho_A'/\rho_A \leqq a_2 , \tag{4.12}$$

there are finite positive constants b_1, b_2 *such that*

$$b_1 \leqq w(A)/S(A) \leqq b_2 . \tag{4.13}$$

PROOF. The inequality at (4.9) shows that any $b_2 \geqq E|\alpha|/2$ satisfies (4.13) without any restrictions on ρ_A and ρ_A'. For the other inequality, take any $u \neq 0$ and let u' be as at (4.5). Then by (3.7),

$$\frac{m(u)}{S(A)} \geqq \frac{m(u)}{S_k m(u')} \geqq \frac{r(u')}{S_k 4(k-1)r(u)} \geqq \frac{\rho_A/\rho_A'}{4(k-1)S_k} \equiv d_A , \quad \text{say}.$$

For any positive $d \leqq d_A$, reference to (4.7) shows that

$$\begin{aligned} w(A)/S(A) \geqq\; & dE[|\alpha_L|(1 - |\alpha|/r(\alpha))^{k-1}] \\ & + \delta_L(1-\delta_L)^{k-1}\rho_A P(L^c)/4(k-1)S_k . \end{aligned} \tag{4.13$'$}$$

Taking $L = (1/2k)D$ so that $\delta_L(1-\delta_L)^{k-1} \geqq 1/4k$ and $|\alpha_L|(1 - |\alpha|/r(\alpha))^{k-1} \geqq |\alpha_L|/2$,

$$2w(A)/S(A) \geqq dE|\alpha_L| + a_3 P(L^c) \tag{4.14}$$

where $a_3 = a_1/16k(k-1)S_k$. Since $0 < E|\alpha|$, there exists positive d_1 such that $\pi_1 \equiv P\{|\alpha| > d_1\} > 0$, and thus

$$\begin{aligned} 2w(A)/S(A) &\geqq dd_1 P\{|\alpha| > d_1, \alpha \in (1/2k)D\} + a_3 P\{|\alpha| > d_1, \alpha \notin (1/2k)D\} \\ &\geqq \pi_1 \min(dd_1, a_3) > 0 . \end{aligned}$$

Equation (4.14) shows immediately the result

COROLLARY 4.1. *If* $E|\alpha| = \infty$, *then* $w(A_n)/S(A_n) \to \infty$ *as* $n \to \infty$ *for any sequence* $\{A_n\}$ *satisfying* (4.12) *and having* $\rho_{A_n} \to \infty$.

The following examples show that the condition $\rho_A' \leqq a_2\rho_A$ (or some similar constraint on the "elongatedness" of A) is needed for (4.13) to hold.

EXAMPLE 1. Take $A_n \in R^2$ to be the rectangle with sides of lengths 1 and n, parallel to the axes, and let $\eta = (1, 0)$ with probability .5, $= (-1, 0)$ with probability .5. Then $|\alpha| = 0$ or 2 with probability .5 each, $g_{A_n}(\alpha) = 0$ or 2 with probability .5 each, and $w(A_n) = 1$ (all n), whereas $S(A_n) = 2(n+1)$.

EXAMPLE 2. Take $A_n \in R^2$ to be the rectangle with sides of length n^2 and n, parallel to the axes, and let $\eta = (Y, 0)$ where Y is a Cauchy distributed random variable with density $2/\pi(1+4y^2)$. Then α has a standard Cauchy distribution, $g_{A_n}(\alpha) = n\min(n^2, |\alpha|)$, and $\pi w(A_n) = 2n^3 \arctan(1/n^2) + n\log(1+n^4)$, so that

$w(A_n)/S(A_n) \to 0$ and $\rho_{A_n} \to \infty$, but, (cf. (4.12)) $\rho'_{A_n}/\rho_{A_n} \to \infty$. If alternatively A_n is a rectangle $n \times n^2$ rather than $n^2 \times n$, then $w(A_n)/S(A_n) \to \infty$.

It may be possible to replace the second condition at (4.12) by others. For example, the conclusion (4.13) is true if $P\{(\alpha, e_u) \neq 0\} > 0$ for every unit vector u. Alternatively, instead of conditions on α we may average $w(A)$ so as to make it not merely translation invariant but also rotation invariant. Define

$$\hat{w}(A) = Ew(\Gamma A) \tag{4.15}$$

where Γ is uniformly distributed over the group $\mathscr{U}_k$ of orthogonal transformations in R^k.

THEOREM 5. *If* $0 < E|\alpha| < \infty$, *given any finite constant* $a_1 > 0$ *and* A *such that* $\rho_A \geqq a_1$, *there exist constants* b_1 *and* b_2 *such that*

$$0 < b_1 \leqq \hat{w}(A)/S(A) \leqq b_2 < \infty\,. \tag{4.16}$$

OUTLINE OF PROOF. The existence of b_2 follows as in Theorem 4.

Consider e_a as at (F4). By Lemma 2, (3.7) and (F4),

$$\frac{m(u)}{S(A)} \geqq \frac{m(u)}{4(k-1)S_k m(e_a)} \geqq \frac{r(e_a)}{16(k-1)^2 S_k r(u)} \geqq 2d_2(e_a, e_u)^+$$

where $1/d_2 = 32(k-1)^2 S_k$ and $(e_a, e_u)^+ = \max(0, (e_a, e_u))$. Proceeding much as in Theorem 4, with $L = (1/2k)D$,

$$w(A)/S(A) \geqq d_2 E[|\alpha_L|(e_a, e_\alpha)^+] + a_3 P(L^c)\,. \tag{4.17}$$

Since for any $\Gamma \in \mathscr{U}_k$, $\rho_A = \rho_{\Gamma A} = r_{\Gamma A}(\Gamma e_a) = r_{\Gamma A}(e_a)$, $|\alpha_{\Gamma L}|(e_{\Gamma a}, e_\alpha)^+ = |(\Gamma^{-1}\alpha)_L|(e_a, e_{\Gamma^{-1}\alpha})^+$, and $P((\Gamma L)^c) = P\{\Gamma^{-1}\alpha \in L^c\}$, the expression for $Ew(\Gamma A)$ obtained from (4.17) can be bounded below.

5. Behaviour of $w(A)/S(A)$ **when** $E|\alpha| = \infty$. We consider in this section the behaviour of $w(A_n)$ for sequences $\{A_n\}$ of sets for which $\rho_{A_n} \to \infty$. In the last section, we have seen (Theorem 4 and Example 2) that both $w(A_n) = o(S(A_n))$ and $S(A_n) = O(w(A_n))$ are possible. What is always true, without any assumptions on either α or $\{A_n\}$, is that

$$w(A_n)/\lambda(A_n) \to 0 \qquad \text{for} \quad \rho_{A_n} \to \infty\,. \tag{5.1}$$

To see this, refer to (4.2) and use (3.7), so that

$$w(A)/\lambda(A) \leqq \delta_L(1-\delta_L)^{k-1}P(L^c) + E|\alpha_L|/4(k-1)\rho_A\,.$$

Take $A = A_n$, $L = S(\varepsilon_n \rho_{A_n})$, where $\varepsilon_n \to 0$ but $\varepsilon_n \rho_{A_n} \to \infty$, and (5.1) follows.

THEOREM 6. *Let* $\{A_n\}$ *be a sequence of sets for which* $\rho_n \equiv \rho_{A_n} \to \infty$ *and* $\sup_n \rho'_{A_n}/\rho_n < \infty$. *Then for* $\delta \in (0, 1)$,

$$M_\delta \equiv E|\alpha|^\delta < \infty \tag{5.2}$$

implies

$$\sup_n w(A_n)/\rho_n^{1-\delta}S(A_n) < \infty . \tag{5.3}$$

Conversely, when (5.3) *holds,* $M_\gamma < \infty$ *for any* $0 \leqq \gamma < \delta$.

COROLLARY 6.1.

$$\sup\{\delta \leqq 1 \mid M_\delta < \infty\} = \sup\{\delta \leqq 1 \mid \sup_n w(A_n)/\rho_n^{1-\delta}S(A_n) < \infty\} .$$

PROOF. The corollary is little more than a restatement of the theorem, which will be proved from the observations that, as has been shown in the proof of Theorem 4, there exist positive constants d and a such that

$$dE|\alpha_L| + a\rho_A P(L^c) \leqq 2w(A)/S(A) \qquad \text{for} \quad L = (1/2k)D,\ A \in \{A_n\} , \tag{5.4}$$

and that for general L and A,

$$2w(A)/S(A) \leqq E|\alpha_L| + \rho_A P(L^c) . \tag{5.5}$$

Writing $\alpha_r = \alpha$ if $|\alpha| < r$, $= 0$ otherwise, observe that

$$E|\alpha_r| \leqq r^{1-\delta}E|\alpha_r|^\delta \leqq r^{1-\delta}M_\delta , \tag{5.6 a}$$

$$rP\{|\alpha| > r\} \leqq r^{1-\delta}E|\alpha - \alpha_r|^\delta \leqq r^{1-\delta}M_\delta . \tag{5.6 b}$$

Applying (5.6) to (5.5) with L a sphere of radius ρ_n and $A = A_n$ shows that (5.2) implies (5.3). For the partial converse, using the boundedness condition ρ'_{A_n}/ρ_n in (5.4) shows that when (5.3) holds, $\rho_n{}^\delta P\{|\alpha| > \rho_n\}$ is uniformly bounded in n, hence $x^{\gamma-1}P\{|\alpha| > x\}$ is integrable on $(0, \infty)$ for $0 < \gamma < \delta$, so $M_\gamma < \infty$ as asserted.

6. Bounds for Var $N(A)$. Our work to date has answered a modified version of Cox's problem to which we now return. Theorem 7 is an analogue of Theorem 4, and gives affirmative answers to (i) and (under certain conditions) to (ii). Example 3 shows that it is not possible to give any affirmative answer to (iii) in general.

THEOREM 7. (a) *If* $E|\alpha| < \infty$, *then* Var $N(A)/S(A)$ *is bounded above.*
(b) *If* $\rho_A'/\rho_A \leqq a_2$ *then for some positive constants* $c_1 = c_1(a_2, k)$ *and* B_k,

$$\operatorname{Var} N(A)/S(A) \geqq c_1[E|\alpha_{\rho/2k}| + \rho P\{|\alpha| > \rho/2k\} - B_k \tag{6.1}$$

where $\rho = \rho_A$ *and* $\alpha_r = \alpha$ *if* $|\alpha| < r$, $= 0$ *otherwise.*

PROOF. Lemma 1 with (F5) yields

$$\begin{aligned} &|w(A) - \operatorname{Var} N(A)| \\ &\quad = |E\lambda((A - \eta') \cap (A^c - \eta'')) - E\nu((A - \eta') \cap (A^c - \eta''))| \\ &\quad \leqq |E\lambda(A - \eta') - E\nu(A - \eta')| \\ &\qquad + |E\lambda((A - \eta') \cap (A - \eta'')) - E\nu((A - \eta') \cap (A - \eta'')) \\ &\quad \leqq 2T(A) . \end{aligned} \tag{6.2}$$

Denote the largest of the $\binom{k}{i}$ components of $V_i(A)$ by U_i. Observe that each component of $V_i(A)$ $(1 \leqq i \leqq k-2)$ is the projection of a component of $V_{i+1}(A)$, so that by using Lemma 2,

$$V_i(A) \leqq \binom{k}{i} 4iU_{i+1}/\rho_A \leqq \binom{k}{i}\{(4k-4)/\rho_A\}^{k-i-1}U_{k-1}\,. \tag{6.3}$$

We assert that

$$V_{k-1}(A) \leqq k^{\frac{1}{2}}S(A)/2\,; \tag{6.4}$$

to verify this assertion, we use the Holder inequality in

$$\begin{aligned} S(A) &= \textstyle\int_{F_A} \mu_A(dx) = \int_{F_A} [\sum_{i=1}^k \lambda^2(dx_i)]^{\frac{1}{2}} \geqq \int_{F_A} k^{-\frac{1}{2}} \sum_{i=1}^k \lambda(dx_i) \\ &= 2k^{-\frac{1}{2}}V_{k-1}(A)\,. \end{aligned} \tag{6.5}$$

Thus

$$T(A) - 1 \leqq (\textstyle\sum_{i=1}^{k-2} \binom{k}{i}\{(4k-4)/\rho_A\}^{k-i-1} + k^{\frac{1}{2}}/2)S(A)\,, \tag{6.6}$$

and so $T(A)/S(A)$ has an upper bound B_k depending only on the dimensionality k. Combining this observation with (4.9) proves part (a) of the theorem.

To prove (b), we use (4.13)′ with $L = S(\rho_A/2k) \equiv S(\rho/2k)$ and the inequality $\log(1-x) \geqq -x/(1-x)$ $(0 < x < 1)$ to give

$$w(A)/S(A) \geqq d_A e^{-\frac{1}{2}}E|\alpha_{\rho/2k}| + P\{|\alpha| > \rho/2k\}/\{2ka_2e^{\frac{1}{2}} \cdot 4(k-1)S_k\}\,.$$

Now use (6.6) again and (6.1) follows.

Observe that the bound in (6.1) need not be positive. However, setting $2B_k/c_1 = c_2$ in (6.1) yields the

COROLLARY 7.1. *If $\rho_A'/\rho_A \leqq a_2$, then there exists a positive constant $c_2 = c_2(a_2, k)$ such that when $E|\alpha| > c_2$,*

$$\liminf_{\rho_A \to \infty} \operatorname{Var} N(A)/S(A) > 0\,.$$

The following examples illustrate the necessity of our conditions. Example 3, in addition to showing that (iii) need not hold, also shows that if we wish (ii) to be satisfied, then $E|\alpha|$ cannot be very small. Example 4 shows that the constant c_2 in Corollary 7.1 must depend on a_2. Example 5 shows that the requirement that $E|\alpha|$ be large without the requirement that ρ_A also be large is not enough to give a positive lower bound for $\operatorname{Var} N(A)/S(A)$.

EXAMPLE 3. Let $A \subset R^2$ be the square with center at the origin and vertices at $(\pm 1, \pm 1)$, and let $\eta = (\pm 0.1, \pm 0.1)$ with probability 0.25 each. Then $\operatorname{Var} N(cA) = 2n - 0.25$ whenever $|n - c| < 0.1$ for some positive integer n, $= 0$ otherwise.

EXAMPLE 4. Let $A \subset R^2$ be the parallelogram with vertices $(0, \frac{1}{2})$, $(nj, \frac{1}{2})$, $(2nj, n + \frac{1}{2})$, $(nj, n + \frac{1}{2})$, and let $\eta = \pm(\frac{1}{4}j, \frac{1}{4})$ with probability .5 each. For large j, $E|\alpha|$ is large whereas $\operatorname{Var} N(A) = 0$ for every n and j.

EXAMPLE 5. Take any A, and let η be such that $P\{\eta = 0\} = 1 - p < 1$.

Observe that we can simultaneously have $E|\alpha|$ arbitrarily large and p arbitrarily small. It follows from (1.5) that

$$\operatorname{Var} N(A) \leqq p(2-p) \sup_{x \in I^k} \nu(A+x) ,$$

which when coupled with (2.4), (6.6), and (4.4), shows that Var $N(A)/S(A)$ may be made arbitrarily small by choice of p.

Acknowledgments. Our investigation began in 1971 when P.G. showed that if $E|\eta|^2 < \infty$, then the limit in Theorem 1 equals the surface integral in Lemma 3. In 1972 D.S. gave the treatment above using Lemma 1. The authors express their sincere thanks to Mrs. Valerie Isham of Imperial College, London for her helpful criticism and remarks, and to Daryl Daley for his kindness in aiding them to bring the paper to its final form, which also contains a few of his ideas.

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