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GENERALISATION AND APPLICATION OF SOME RESULTS OF IBRAGIMOV ON CONVERGENCE TO NORMALITY

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We modify certain results obtained by I. A. Ibragimov concerning the remainder term in the Central Limit Theorem and the remainder term in the approximation of the distribution function for normed sums of i.i.d. rv's by a portion of the Chebyshev series and we then use these to complete some results appearing in a joint article by C. C. Heyde and the author (*Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **21** 255–268) (as well as in a similar article by F. N. Galstian (*Theor. Probability Appl.* **16** 3, 528–533)).

1. Introduction. Let X_i, $i = 1, 2, 3, \cdots$ be a sequence of independent and indentically distributed random variables with $EX_i = 0$ and $EX_i^2 = 1$. Write F for the distribution function and f for the characteristic function of X_i and put $S_n = \sum_1^n X_i$, $n \geqq 1$. Let $F_n(x) = P(S_n \leqq n^{\frac{1}{2}}x)$ and $\Delta_n = \sup_x |F_n(x) - \Phi(x)|$ where Φ is the distribution function for a standard normal variate. Further, let $\{n_i, i = 1, 2, 3, \cdots\}$ be an infinite subsequence of the positive integers such that $n_{i+1} > n_i$ for all i and $\lim_{i\to\infty} n_{i+1}/n_i = C$ for C a constant satisfying $1 \leqq C < \infty$.

In [5], Ibragimov gives conditions which are both necessary and sufficient for the relation

$$\Delta_n = O(n^{-\frac{1}{2}\delta}), \qquad 0 < \delta \leqq 1 \tag{1}$$

to hold for $n = 1, 2, 3, \cdots$. If in (1), instead of n taking successively the values $1, 2, 3, \cdots$, it takes the values $n_1, n_2, n_3, \cdots$ of the sequence $\{n_i\}$, we show (Theorem 1) that Ibragimov's conditions continue to be both necessary and sufficient for (1) to hold.

This result allows us similarly to generalise two theorems of Ibragimov concerning the remainder term in the approximation of $F_n(x)$ by a portion of the Chebyshev series. The Chebyshev series corresponding to the random variable $n^{-\frac{1}{2}} S_n$ has the form

$$F_n(x) \sim \Phi(x) + \frac{1}{(2\pi)^{\frac{1}{2}}} e^{-\frac{1}{2}x^2} \sum_{j=1}^k Q_j(x) n^{-\frac{1}{2}j} \tag{2}$$

(see Gnedenko and Kolmogorov [2] Section 38) where the $Q_j(x)$ are polynomials of degree $3j - 1$ whose coefficients depend on the first $j + 2$ moments of X_i.

This expansion has the disadvantage that we must know the first $j + 2$

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moments of X_i (and hence the existence of at least $E|X_i|^{j+1}$ and in Cauchy limit EX_i^{j+2}) before we can even write down $Q_j(x)$. We overcome this problem by using a formulation due to Ibragimov [6]: An arbitrary numerical sequence $\beta_1 = 0, \beta_2 = 1, \beta_3, \beta_4, \cdots$ is prescribed and on the basis of this sequence, polynomials $Q_j(x)$ are formed in such a way that their coefficients are expressed in terms of $\beta_1, \beta_2, \cdots, \beta_{j+2}$ in the same way as the coefficients of the classical polynomials $Q_j(x)$ are expressed in terms of the cumulants $\kappa_1, \kappa_2, \cdots, \kappa_{j+2}$ of X_i. For further details see [4] or [6]. As the expansion in (2) can now be written down without presupposing the existence of any moments of higher order than the second, we shall henceforth interpret the $Q_j(x)$'s in this way.

Setting

$$G_{kn}(x) = \Phi(x) + \frac{1}{(2\pi)^{\frac{1}{2}}} e^{-\frac{1}{2}x^2} \sum_{j=1}^{k} Q_j(x) n^{-\frac{1}{2}j}$$

Ibragimov [6] obtained necessary and sufficient conditions for

$$\text{(3)} \qquad \sup_x |F_n(x) - G_{kn}(x)| = O(n^{-\frac{1}{2}(k+\delta)}) \qquad \text{with} \quad 0 < \delta \leqq 1 \quad (\text{or} \quad o(n^{-\frac{1}{2}k}))$$

to hold for $n = 1, 2, 3, \cdots$. If in (3), instead of n taking successively the values $1, 2, 3, \cdots$ it takes the values $n_1, n_2, n_3, \cdots$, we show (Theorems 2 and 3) that Ibragimov's conditions continue to be necessary and sufficient for (3) to hold.

Finally we use Theorem 2 to complete some results appearing in a joint article by Heyde and the author [4] (as well as in a similar article by F. N. Galstian [1]). In [4], necessary and sufficient conditions are found for the series

$$\sum_{n=1}^{\infty} n^{-1+\frac{1}{2}(k+\delta)} \sup_x |F_n(x) - G_{kn}(x)|$$

to converge, provided $0 < \delta < 1$. The case $\delta = 0$ could not be treated with the same generality because the authors were unable to show that

$$\text{(4)} \qquad \sum n^{-1+\frac{1}{2}k} \sup_x |F_n(x) - G_{kn}(x)| < \infty$$

implied both the existence of $E|X_i|^{k+2}$ and that $EX_i^j = \alpha_j$ for $j = 1, 2, \cdots, k+2$, where $\alpha_1, \alpha_2, \cdots, \alpha_{k+2}$ is the 'moment' sequence corresponding to the 'cumulant' sequence $\beta_1, \beta_2, \cdots, \beta_{k+2}$. As a result, a separate theorem treating the case $\delta = 0$ needed to be given in which both $E|X_i|^{k+2} < \infty$ and $EX_i^j = \alpha_j$, $j = 1, 2, \cdots, k+2$ are assumed and then it is shown that providing k is a nonnegative *even* integer

$$\sum n^{-1+\frac{1}{2}k} \sup_x |F_n(x) - G_{kn}(x)| < \infty \Leftrightarrow E|X_i|^{k+2} \ln(1 + |X_i|) < \infty .$$

We rectify this anomaly by firstly showing that for positive integer k (not necessarily even), relation (4) $\Rightarrow$ both $E|X_i|^{k+2} < \infty$ and $EX_i^j = \alpha_j$ for $j = 1, 2, \cdots, k+2$. We then go on to show that when k is an odd integer, under certain conditions (4) $\Leftrightarrow E|X_i|^{k+2} \ln(1 + |X_i|) < \infty$ whilst when X_i is symmetrically distributed, (4) $\Leftrightarrow E|X_i|^{k+2} < \infty$. These show that for $\delta = 0$ and k an even integer, we can obtain results as general as those obtained for $0 < \delta < 1$; however, when $\delta = 0$ and k is odd, no such general results can be given.

2. Results.

THEOREM 1. *In order that*

(5) $$\sup_x |F_{n_i}(x) - \Phi(x)| = O(n_i^{-\frac{1}{2}\delta}), \qquad 0 < \delta \leqq 1, i = 1, 2, 3, \cdots$$

it is necessary and sufficient that

(6) 1) $\int_{|u|>z} u^2\, dF(u) = O(z^{-\delta})$ *as* $z \to \infty$ *and when* $\delta = 1$, *also*
2) $\lim_{z\to\infty} \int_{-z}^{z} u^3\, dF(u) = O(1)$.

THEOREM 2. *In order that for k a positive integer,*

(7) $$\sup_x |F_{n_i}(x) - G_{k,n_i}(x)| = o(n_i^{-\frac{1}{2}k}), \quad i = 1, 2, 3, \cdots$$

it is necessary and for distributions satisfying Cramer's condition (C)[1] *also sufficient that*

(8) 1) $E|X_i|^{k+1} < \infty$ *and* $EX_i^j = \alpha_j$ *for* $j = 1, 2, \cdots, k+1$,
2) $\int_{|u|>z} |u|^{k+1}\, dF(u) = o(z^{-1})$ *as* $z \to \infty$ *and*
3) $\lim_{z\to\infty} \int_{-z}^{z} u^{k+2}\, dF(u) = \alpha_{k+2}$.

THEOREM 3. *In order that for positive integer k*

(9) $$\sup_x |F_{n_i}(x) - G_{k,n_i}(x)| = O(n_i^{-\frac{1}{2}(k+2)}), \qquad i = 1, 2, 3, \cdots; 0 < \delta \leqq 1$$

it is necessary and for distributions satisfying (C) also sufficient that

(10) 1) $E|X_i|^{k+2} < \infty$ *and* $EX_i^j = \alpha_j$, *for* $j = 1, 2, \cdots, k+2$
2) $\int_{|u|>z} |u|^{k+2}\, dF(u) = O(z^{-\delta})$, *as* $z \to \infty$ *and when* $\delta = 1$, *also*
3) $\lim_{z\to\infty} \int_{-z}^{z} u^{k+2}\, dF(u) = O(1)$.

THEOREM 4. *In order that for positive integer k*

(11) $$\sum_{n=1}^{\infty} n^{-1+\frac{1}{2}k} \sup_x |F_n(x) - G_{kn}(x)| < \infty$$

(A) *if k is even, it is necessary and, for distributions satisfying* (C), *also sufficient that* $E|X_i|^{k+2} \ln(1 + |X_i|) < \infty$ *and* $EX_i^j = \alpha_j, j = 1, 2, \cdots, k+2$.

(B) *if k is odd and if there exists a positive finite constant D such that* $P(X_i < -D) = 0$, *it is necessary and, for distributions satisfying* (C) *also sufficient that* $E|X_i|^{k+2} \ln(1 + |X_i|) < \infty$ *and* $EX_i^j = \alpha_j, j = 1, 2, \cdots, k+2$.

(C) *if k is odd and if* X_i *is symmetically distributed, it is necessary and, for distributions satisfying* (C), *also sufficient that* $E|X_i|^{k+2} < \infty$ *and* $EX_i^j = \alpha_j, j = 1, 2, \cdots, k+2$.

REMARK. Theorems 1, 2 and 3 are interesting in so far as we need know only that (1) or (3) are satisfied at a set of geometrically increasing points to ensure that they are satisfied for all integer n.

3. Proofs.

PROOFS OF THEOREMS 1, 2 AND 3. As would be expected, the proofs follow

[1] Cramer's condition (C): $\limsup_{|t|\to\infty} |f(t)| < 1$.

essentially verbatim their counterparts in [5], [6] and [7]. Furthermore, the sufficiency of conditions (6), (8) and (10) is a direct consequence of the Ibragimov theorems. Thus it remains only to outline the adjustments to the proofs in [5], [6] and [7] which have to be carried out to establish the necessity of conditions (6), (8) and (10).

From (5) by Parseval's identity we have

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}t^2}(f_{n_i}(t) - e^{-\frac{1}{2}t^2})\,dt = O(n_i^{-\frac{1}{2}\delta})\,, \qquad i = 1, 2, 3, \cdots \tag{12}$$

where $f_n(t)$ is the characteristic function of $S_n/n^{\frac{1}{2}}$. In [7], Ibragimov shows that

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}t^2}(f_n(t) - e^{-\frac{1}{2}t^2})\,dt = O(n^{-\frac{1}{2}\delta}) \tag{13}$$

implies for large n,

$$\int_{|u|>cn^{\frac{1}{2}}} u^2\,dF(u) = O(n^{-\frac{1}{2}\delta})\,, \quad c \quad \text{some positive constant.}$$

In precisely the same way (12) implies

$$\int_{|u|>cn_i^{\frac{1}{2}}} u^2\,dF(u) = O(n_i^{-\frac{1}{2}\delta}) \qquad \text{as } i \to \infty\,. \tag{14}$$

It should be noted that in Ibragimov's proof, the discrete variable $n^{-\frac{1}{2}}$ is replaced by a continuous variable x. In our case, this transformation is invalid, however, as Ibragimov introduces the transformation principally for notational convenience, we can regard x merely as representing $n^{-\frac{1}{2}}$. Using (12) in place of (13) allows us to regard x as representing $n_i^{-\frac{1}{2}}$. Thus we arrive at (14).

If we now take $cn_i^{\frac{1}{2}} < z \leqq cn_{i+1}^{\frac{1}{2}}$,

$$\int_{|u|>z} u^2\,dF(u) \leqq \int_{|u|>cn_i^{\frac{1}{2}}} u^2\,dF(u) = O(n_i^{-\frac{1}{2}\delta})\,.$$

But $n_i^{-\frac{1}{2}\delta} \leqq z^{-\delta}(n_i^{-\frac{1}{2}}cn_{i+1}^{\frac{1}{2}})^{\delta} \leqq z^{-\delta}(cA)^{\delta}$ where A is an absolute constant whose existence is assured by the definition of $\{n_i\}$ and hence

$$\int_{|u|>z} u^2\,dF(u) = O(z^{-\delta}) \qquad \text{as } z \to \infty\,. \tag{15}$$

Similarly, when $\delta = 1$, by using (15) we can extract directly from [5] that

$$\int_{-n_i^{\frac{1}{2}}}^{n_i^{\frac{1}{2}}} u^3\,dF(u) = O(1) \qquad \text{as } i \to \infty\,.$$

Again, taking $n_i^{\frac{1}{2}} < z \leqq n_{i+1}^{\frac{1}{2}}$

$$\begin{aligned} \left|\int_{-z}^{z} u^3\,dF(u)\right| &\leqq \left|\int_{|u|<n_i^{\frac{1}{2}}} u^3\,dF(u)\right| + \left|\int_{n_i^{\frac{1}{2}}<|u|\leqq z} u^3\,dF(u)\right| \\ &\leqq O(1) + z\int_{n_i^{\frac{1}{2}}<|u|\leqq z} u^2\,dF(u) \\ &= O(1) \quad (\text{by (15) and the definition of } \{n_i\})\,. \end{aligned}$$

Theorem 1 is now complete.

Theorems 2 and 3 follow in a similar manner from their counterparts in [6].

PROOF OF THEOREM 4. Firstly we show that (11) $\Rightarrow$ both $E|X_i|^{k+2} < \infty$ and $EX_i^j = \alpha_j$, $j = 1, 2, \cdots, k+2$. From the lemma in Heyde [3], we know that for any nonnegative function $g(n)$ such that $\sum_{n=1}^{\infty} n^{-1}g(n) < \infty$, there exists an infinite subsequence $\{n_i\}$ of the positive integers such that $g(n_i) \to 0$ as $i \to \infty$,

$n_{i+1} > n_i$ for all i and $\lim_{i\to\infty} n_{i+1}/n_i = 1$. Hence from (11) we know such a subsequence exists satisfying

$$\sup_x |F_{n_i}(x) - G_{kn_i}(x)| = o(n_i^{-\frac{1}{2}k}), \qquad i \to \infty . \tag{16}$$

By Theorem 2, (16) implies

i) $E|X_i|^{k+1} < \infty$ and $EX_i^j = \alpha_j$, $j = 1, 2, \cdots, k+1$,
ii) $\int_{|u|>z} |u|^{k+1}\, dF(u) = o(z^{-1})$ as $z \to \infty$ and
iii) $\lim_{z\to\infty} \int_{-z}^{z} u^{k+2}\, dF(u) = \alpha_{k+2}$.

Clearly when k is even, we have $E|X_i|^{k+2} < \infty$ as desired. However, when k is odd, we need to return to Ibragimov [6] using his equation (3.6) to prove that

$$\sum_{n=1}^{\infty} n^{-1+\frac{1}{2}k} \left| \int_0^1 \left(1 - \exp\left\{(it)^{k+2} n^{-\frac{1}{2}k} \omega_k\left(\frac{t}{n^{\frac{1}{2}}}\right)\right\}\right)(1-t)\, dt \right| < \infty \tag{17}$$

where by Theorem 3 of [6],

$$f(t) = \exp\left\{-\tfrac{1}{2}t^2 + \sum_{s=3}^{k+2} \frac{(it)^s}{s!}\gamma_s + (it)^{k+2} n^{-\frac{1}{2}k} \omega_k\left(\frac{t}{n^{\frac{1}{2}}}\right)\right\}$$

and $|\omega_k(t)| = o(1)$ as $t \to 0$. As $|a + ib| \geqq |a|$ or $|b|$, we find that

$$\sum_{n=1}^{\infty} n^{-1+\frac{1}{2}k} \left| \int_0^1 t^{k+2} n^{-\frac{1}{2}k} Re\left(\omega_k\left(\frac{t}{n^{\frac{1}{2}}}\right)\right) \cdot (1-t)\, dt \right| < \infty . \tag{18}$$

Using now a technique similar to that in [6], we have

$$\sum_{n=1}^{\infty} n^{-\frac{1}{2}} \int_{|u|>n^{\frac{1}{2}}T_k} |u|^{k+1}\, dF(u) < \infty$$

where $T_k = (2(k+3)(k+7))^{-\frac{1}{2}}$. Furthermore, it is well known that for fixed $a > 0$,

$$E|X|^{k+2} < \infty \Leftrightarrow \sum_{n=1}^{\infty} n^{\frac{1}{2}k} P\{|X_i| > an^{\frac{1}{2}}\} < \infty .$$

We already have, however, that

$$\infty > \sum_{n=1}^{\infty} n^{-\frac{1}{2}} \int_{|u|>n^{\frac{1}{2}}T_k} |u|^{k+1}\, dF(u) \geqq c \sum_{n=1}^{\infty} n^{\frac{1}{2}k} \int_{|u|>n^{\frac{1}{2}}T_k} dF(u)$$

and hence $E|X_i|^{k+2} < \infty$. Since both odd and even k have now been considered, we have (11) $\Rightarrow$ both $E|X_i|^{k+2} < \infty$ and $EX_i^j = \alpha_j$, $j = 1, 2, \cdots, k+2$ for all k. In view of this result and using Theorem 3 of [4], Part A of Theorem 4 is true.

To prove the rest of Theorem 4, it remains to consider the case of k odd. For X_i symmetric, we have (11) $\Rightarrow E|X_i|^{k+2} < \infty$ and (from expression (12) of [4]) that

$$E|X_i|^{k+2} < \infty \Rightarrow \int_0^A |\omega_k(t)| t^{-1}\, dt < \infty,\ A > 0 .$$

Also from Section 3 of [4], $\int_0^A |\omega_k(t)| t^{-1}\, dt < \infty \Rightarrow$ (11). Hence, Part C of Theorem 4 is established.

With X_i now satisfying the conditions of Part B (X_i no longer necessarily symmetric), we define

$$t^{k+2}\beta(t) = f(t) - \sum_{s=0}^{k+2} \frac{(it)^s}{s!} (EX_i^s) .$$

From Lemma 2 of [4], we know for $A > 0$,

$$\int_0^A |\beta(t)|t^{-1}\, dt < \infty \Leftrightarrow \int_0^A |\omega_k(t)|t^{-1}\, dt < \infty .$$

Treating $Im(\beta(t))$ in the same way as $Re(\beta(t))$ is treated in the proof of Lemma 2 of [4], and using the fact that $b_{2k}(u) \equiv \sin u - \sum_{s=0}^{k} (-1)^s(u^{2s+1}/(2s+1)!)$ is of constant sign for $u > 0$, that

$$\int_0^A t^{-(k+3)}|\int_{-D}^0 b_{k+1}(xt)\, dF(x)|\, dt < \infty$$

and that

$$\int_0^A t^{-1}|Re\beta(t)|\, dt < \infty ,$$

it readily follows that $E|X_i|^{k+2} \ln(1 + |X_i|) < \infty \Leftrightarrow \int_0^A t^{-1}|\beta(t)|\, dt < \infty$. Furthermore, directly from [4], we have $\int_0^A t^{-1}|\omega_k(t)|\, dt < \infty \Rightarrow (11)$ and hence $E|X_i|^{k+2} \ln(1 + |X_i|) < \infty \Rightarrow (11)$.

Finally we show that $(11) \Rightarrow \int_0^A t^{-1}|Im\beta(t)|\, dt < \infty$ by the use of expression (31) of [4], and the subsequent work with $f_n(t)$ and $g_{kn}(t)$ in place of $|f_n(t)|^2$ and $|g_{kn}(t)|^2$. This completes the proof of Theorem 4.

REFERENCES

[1] GALSTIAN, F. N. (1971). On asymptotic expansions in the central limit theorem. *Theor. Probability Appl.* **16** 3, 528–533.

[2] GNEDENKO, B. A. and KOLMOGOROV, A. N. (1954). *Limit Distributions for Sums of Independent Random Variables*. Addison-Wesley, Cambridge, Mass.

[3] HEYDE, C. C. (1973). On the uniform metric in the context of convergence to normality. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **25** 83–95.

[4] HEYDE, C. C. and LESLIE, J. R. (1972). On the influence of moments on approximations by portion of a Chebyshev series in central limit convergence. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **21** 255–268.

[5] IBRAGIMOV, I. A. (1966). On the accuracy of Gaussian approximation to the distribution functions of sums of independent variables. *Theor. Probability Appl.* **11** 4, 559–579.

[6] IBRAGIMOV, I. A. (1967). On the Chebyshev-Cramér asymptotic expansions. *Theor. Probability Appl.* **12** 3, 455–469.

[7] IBRAGIMOV, I. A. and LINNIK, YU. V. (1971). *Independent and Stationary Sequences of Random Variables*. Wolters-Noordhoff, Groningen, 104–111.

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