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WEAK COMPARATIVE PROBABILITY ON INFINITE SETS

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Let $\mathscr{S}$ be a Boolean algebra of subsets of a state space S and let $\succ$ be a binary comparative probability relation on $\mathscr{S}$ with $A \succ B$ interpreted as "A is more probable than B." Axioms are given for $\succ$ on $\mathscr{S}$ which are sufficient for the existence of a finitely additive probability measure P on $\mathscr{S}$ which has $P(A) > P(B)$ whenever $A \succ B$. The axioms consist of a necessary cancellation or additivity condition, a simple monotonicity axiom, an axiom for the preservation of $\succ$ under common deletions, and an Archimedean condition.

1. Introduction and main theorem. Throughout, S is a non-empty set of states [13], $\mathscr{S}$ is a Boolean algebra of subsets of S which contains S, $\emptyset$ is the empty set, and $\succ$ ("is more probable than") is an asymmetric comparative probability relation on $\mathscr{S}$ with symmetric complement $\sim$, so that $A \sim B$ if neither $A \succ B$ nor $B \succ A$. A finitely additive probability measure P on $\mathscr{S}$

$$\textit{weakly agrees} \text{ with } \succ \quad \text{iff} \quad A \succ B \Rightarrow P(A) > P(B) ,$$
$$\textit{almost agrees} \text{ with } \succ \quad \text{iff} \quad P(A) > P(B) \Rightarrow A \succ B ,$$

for all $A, B \in \mathscr{S}$, and *strictly agrees* with $\succ$ iff it weakly agrees and almost agrees with $\succ$. The relation $\succ$ is transitive under strict agreement and noncyclic under weak agreement, but it can cycle under almost agreement as when $A \succ B \succ C \succ A$ and $P(A) = P(B) = P(C)$. On the other hand, $\sim$ is transitive (hence an equivalence) under almost agreement or strict agreement, but need not be transitive under weak agreement. Nontransitivity of $\sim$ accommodates Savage's notion of vagueness in judgments of personal probabilities, as when small successive but accumulating differences between events $A_1, A_2, \cdots, A_n$ give $A_1 \sim A_2, \cdots, A_{n-1} \sim A_n$ along with $A_n \succ A_1$, and interest in the notion of weak agreement has been expressed by several writers [2, 4, 7, 14, 15, 17]. The purpose of the present paper is to provide axioms for $\succ$ which imply the existence of a weakly agreeing measure when $\mathscr{S}$ is infinite.

Kraft, Pratt and Seidenberg [10] and others [4, 16] present axioms for $\succ$ which are necessary and sufficient for strict agreement when $\mathscr{S}$ is finite, and Fishburn [4] and Domotor and Stelzer [2] axiomatize weak agreement and intermediate cases when $\mathscr{S}$ is finite. Moreover, when $\mathscr{S}$ is finite with atoms $a_1, \cdots, a_n$, so that $A \in \mathscr{S}$ iff $A = \emptyset$ or A is the union of one or more a_i, the method of these papers shows that $\mathscr{S}$ has an almost agreeing measure if, and only if, there is no finite sequence $\{(A_k, B_k)\}_{k=1}^m$ of event pairs for which $A_k \succ B_k$ or $A_k \sim B_k$ for all

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k and the number of B_k which include a_i exceeds the number of A_k which include a_i for every $i \in \{1, \cdots, n\}$. Sufficient conditions for strict agreement when S is infinite are given by Koopman [8, 9], Savage [13], Luce [11], Fine [3], and Narens [12], among others, and Savage [13, pages 34–35] gives conditions (including transitivity of $\succ$) which are sufficient for almost agreement but not for strict agreement when S is infinite. A more recent almost agreeing axiomatization for arbitrary S is given by Narens [12].

An important omission from prior work is the absence of easily interpreted conditions for $\succ$ which imply the existence of a weakly agreeing probability measure without also implying the existence of a strictly agreeing measure when $\mathscr{S}$ is infinite. The following theorem, proved in the next section, is an attempt to remedy this omission. For every $A \in \mathscr{S}$, $A' : S \to \{0, 1\}$ is the indicator function for A with $A'(s) = 1$ iff $s \in A$; $A \backslash B = \{s : s \in A \text{ and } s \notin B\}$; and a partition of a subset of S is an $\mathscr{S}$ partition iff every set in the partition is in $\mathscr{S}$.

THEOREM 1. *There exists a finitely additive probability measure on $\mathscr{S}$ that weakly agrees with $\succ$ if the following hold for all $A, B, C, A_i, B_i \in \mathscr{S}$ and all positive integers n:*

(A1) $(A_i \succ B_i$ *and* $A_i \cap B_i = \emptyset$ *for* $i = 1, \cdots, n) \Rightarrow \sum_{i=1}^n A'_i \neq \sum_{i=1}^n B_i'$.

(A2) $(A \succ B \supseteqq C$ *or* $A \supseteqq B \succ C) \Rightarrow A \succ C$.

(A3) $(A \succ B$ *and* $C \subseteqq A \cap B) \Rightarrow A\backslash C \succ B\backslash C$.

(A4) $A \succ B \Rightarrow$ *there is a finite $\mathscr{S}$ partition of S such that* $A \succ B \cup C$ *for every set C in the partition.*

Axiom (A1) is an additivity condition which, since $\sum A_i' = \sum B_i' \Rightarrow \sum P(A_i) = \sum P(B_i)$, is necessary for weak agreement: (A1) and (A3) forbid $\succ$ cycles but do not imply that $\succ$ is transitive. Axiom (A2) is an appealing monotonicity condition for $\succ$ preservation under inclusion. Axiom (A3) says that $\succ$ is preserved under removal of a subset C included in both A and B. It seems psychologically realistic since if A is judged to be more probable than B then the bases for this judgment should be even more evident when C is removed from A and B. Axioms (A1), (A2) and (A3) are sufficient [4] for weak agreement when $\mathscr{S}$ is finite, but neither (A2) nor (A3) is necessary. Kraft, Pratt and Seidenberg [10] show that some condition like (A1) is required in the general finite context, but some strict-agreement axiomatizations [11, 13] with infinite $\mathscr{S}$ avoid the complexities of (A1) by using weak or simple orders along with strong structural presuppositions.

Axiom (A4), used elsewhere [5, page 195] in a characterization of Savage's strict-agreement axioms, in an Archimedean condition suggested by de Finetti [1] and Savage [13]. It is stronger than necessary since, in conjunction with the other axioms, it requires $A \sim \emptyset$ for every atom $A \in \mathscr{S}$, and when $S \succ \emptyset$ it forces S to be infinite. However, I have not been able to obtain weak agreement under (A1), (A2) and (A3) with the use of a more palatable Archimedean axiom and invite others to attempt to remedy this shortcoming of the axiomatization.

For examples in which the axioms hold but do not imply strict agreement when S is countable, let S be the set of all rational numbers in $[0, 1]$, let $\mathscr{S}$ be the algebra consisting of $\varnothing$ and all finite unions of intervals in S, and for each $A \in \mathscr{S}$ let $\mu(A)$ be the Lebesgue measure of the closure of A in $[0, 1]$. Two simple models which satisfy the axioms are $A \succ B$ iff $\mu(A) > \lambda\mu(B)$ with $\lambda \geqq 1$, and $A \succ B$ iff $\mu(A) > \mu(B) + \delta$ with $\delta \geqq 0$. In the latter case another weakly agreeing measure for $\delta = \frac{1}{2}$ is $P(A) = \frac{2}{3}\mu(A) + \frac{1}{3}A'(s)$ with s any fixed point in S.

2. Proof of Theorem 1. My proof of Theorem 1 is based on Hausner and Wendel's theorem [6] for real lexicographic representations of ordered vector spaces. We call $(V, >)$ an *ordered vector space* when V is a real vector space with origin θ, $>$ is a linear order (irreflexive, transitive, complete) on V and, for all $x, y \in V$ and $\lambda \in \mathrm{Re}$: (i) $x > \theta$ and $\lambda > 0 \Rightarrow \lambda x > \theta$, (ii) $x > \theta$ and $y > \theta \Rightarrow x + y > \theta$, (iii) $x > y$ iff $x - y > \theta$. The positive cone $V^+ = \{x \in V: x > \theta\}$ completely describes $>$.

Let $(V, >)$ be an ordered vector space and define binary relations $\gg$ and $\approx$ on V^+ by $x \gg y$ iff $x > \lambda y$ for all $\lambda > 0$, and $x \approx y$ iff $\lambda x > y > \mu x$ for some $\lambda, \mu > 0$. Then $\approx$ is an equivalence and, with $[x]$ the equivalence class in $V^+/\approx$ which contains $x \in V^+$, the relation $<_0$ on $V^+/\approx$, defined by $[x] <_0 [y]$ iff $x \gg y$, is a linear order on $V^+/\approx$. A set $W \subseteq V$ is *Archimedean* iff $x, y \in W \Rightarrow \lambda x - y \in W$ and $y - \mu x \in W$ for some $\lambda, \mu > 0$. The classes in $V^+/\approx$ are the maximal Archimedean sets in V^+.

A function $F: V \to U$, where U also is a real vector space, is *linear* iff $F(\lambda x + \mu y) = \lambda F(x) + \mu F(y)$ for all $x, y \in V$ and $\lambda, \mu \in \mathrm{Re}$.

THEOREM 2 (Hausner and Wendel). *Let $(V, >)$ be an ordered vector space with $T = V^+/\approx$ and $[x] <_0 [y]$ iff $x \gg y$. Define $(V_T, >_L)$ as the ordered vector space of all real-valued functions on T which are nonzero on at most a well ordered subset of $(T, <_0)$, with $f >_L g$ when $f, g \in V_T$ iff $f \neq g$ and $f(t) > g(t)$ for the first t in T at which $f(t) \neq g(t)$. Select $e_t \in t$ for each $t \in T$ and define $f_t \in V_T$ by $f_t(t) = 1$ and $f_t(s) = 0$ for all $s \in T\backslash\{t\}$. Then there exists a linear $F: V \to V_T$ with $F(e_t) = f_t$ for all $t \in T$ such that $x > y$ iff $F(x) >_L F(y)$, for all $x, y \in V$.*

Henceforth, let V be the real vector space of all real-valued functions on S, let $V_0 = \{A' - B' : A, B \in \mathscr{S}$ and $A \succ B\}$ and let $V_1 = \{\sum_{i=1}^{n} \lambda_i x_i : n \in \{1, 2, \cdots\}$, $\lambda_i > 0$ and $x_i \in V_0\}$, the convex cone in V generated by V_0. We presume axioms (A1) through (A4) and $S \succ \varnothing$, for otherwise $V_0 = \varnothing$ by (A2).

LEMMA 1. *$\theta \notin V_1$ and V_1 is Archimedean.*

PROOF. Suppose $\theta \in V_1$ with $A_i, B_i \in \mathscr{S}$, $A_i \succ B_i$ and $\lambda_i > 0$ for $i = 1, \cdots, n$, and $\sum \lambda_i(A_i' - B_i') = \theta$. Using (A3), $A_i \cap B_i = \varnothing$ can be assumed without loss of generality. Since $A_i'(s) - B_i'(s) \in \{1, 0, -1\}$ for all i and s, $\sum \lambda_i(A_i' - B_i') = \theta$ is tantamount to a finite system $(\lambda_1, \cdots, \lambda_n) \cdot p^j = 0$ for a subset of p^j in $\{1, 0, -1\}^n$. Since the p^j are integral vectors there are integral $\lambda_i^* > 0$ such that $\sum \lambda_i^*(A_i' - B_i') = \theta$. Then λ_i^* replications of (A_i, B_i) gives $\sum_{i=1}^{m} C_i' = \sum_{i=1}^{m} D_i'$

with $C_i \succ D_i$ for $i = 1, \cdots, m(= \sum \lambda_i^*)$ and $C_i \cap D_i = \emptyset$ for each i. But this contradicts (A1). Hence $\theta \notin V_1$.

To show that V_1 is Archimedean suppose first that $A \succ B$. Using (A3), we can presume that $A \cap B = \emptyset$. Then, using (A2) and (A4), there are partitions $\{C_i\}_{i=1}^n$ of A and $\{D_j\}_{j=1}^m$ of $S \backslash B$ such that $A \succ B \cup C_i$ and $A \succ B \cup D_j$ for all i and j, so that $A' - B' - C_i' \in V_1$ and $A' - B' - D_j' \in V_1$ for all i and j. Addition over all i and j then gives $(n + m)(A' - B') - A' - (S \backslash B)' = (n + m - 1)(A' - B') - S' \in V_1$ with $n + m - 1 > 0$, so that $N(A' - B') - S' \in V_1$ for positive N. By an analogous procedure (given $S \succ \emptyset$), partitions of A and $S \backslash B$ lead to $MS' - (A' - B') \in V_1$ for some positive M.

Therefore, if $A \succ B$ and $C \succ D$, $N(A' - B') - S' \in V_1$ and $S' - M^{-1}(C' - D') \in V_1$ for some positive N and M so that $NM(A' - B') - (C' - D') \in V_1$. To complete the Archimedean proof, suppose $x, y \in V_1$ with $x = \sum_{i=1}^n \lambda_i(A_i' - B_i')$ and $y = \sum_{j=1}^m \mu_j(C_j' - D_j')$ with $\lambda_i, \mu_j > 0$ and $A_i \succ B_i$, $C_j \succ D_j$ for all i and j. Then there exists N for which $N(A_i' - B_i') - (C_j' - D_j') \in V_1$ for all i and j. Multiplying $N(A_i' - B_i') - (C_j' - D_j')$ by $\lambda_i \mu_j$ and double summing over all i and j, we get $(N \sum_j \mu_j / \sum_i \lambda_i)x - y \in V_1$. This proves that V_1 is Archimedean.

To complete the proof of Theorem 1 let K be the set of all convex cones in V which include V_1, contain A' for every nonempty $A \in \mathscr{S}$, and do not contain θ. Using (A1), (A2) and (A3) it is easily checked that $K \neq \emptyset$. Zorn's lemma then implies that K contains a maximal such cone, say V^+. Defining $x > y$ iff $x - y \in V^+$, $(V, >)$ is easily seen to be an ordered vector space. Let $F: V \rightarrow V_T$ be as given by Theorem 2. Since $V_1 \subset V^+$ and V_1 is Archimedean by Lemma 1, V_1 is included in one of the equivalence classes in $T = V^+/\approx$, say $t \in T$. Since $e_t \in t$ can be chosen as we wish let $e_t = S'$, with $F(S') = f_t$. It is readily seen that, with $F_t(x)$ the value of $F(x)$ at t for $x \in V$, $F_t(x) > 0$ for all $x \in V_1$, and indeed for all $x \in t$. Hence if $A \in \mathscr{S}$ and $A \neq \emptyset$ then $F_t(A') > 0$ if $A' \in t$. Suppose however that $A \in \mathscr{S}$, $A \neq \emptyset$ and $A' \notin t$. Then, since $A' \in V^+$, A' is in some other class in T, say t^*. Since $S' - A'$ is the indicator function of $S \backslash A$, $S' - A' \in V^+$. Therefore, the definitions prior to Theorem 2 require $t <_0 t^*$. It then follows from Theorem 2 that $F_t(A') = 0$. Moreover, $F_t(\theta) = 0$ by linearity.

A finitely additive probability measure $P: \mathscr{S} \rightarrow \text{Re}$ which weakly agrees with $\succ$ is defined by $P(A) = F_t(A')$ for all $A \in \mathscr{S}$. As just noted, $P(A) \geqq 0$ for all $A \in \mathscr{S}$, $P(S) = F_t(S') = f_t(t) = 1$ by Theorem 2, and additivity for P follows from linearity for F_t. Moreover, if $A, B \in \mathscr{S}$ and $A \succ B$ then $A' - B' \in V_1$ so that $P(A) - P(B) = F_t(A') - F_t(B') = F_t(A' - B') > 0$.

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