The Annals of Probability
1975, Vol. 3, No. 5, 883-888

A CONDITIONAL LOCAL LIMIT THEOREM FOR RECURRENT RANDOM WALK

BY W. D. KAIGH

The University of Texas at El Paso

Let S_n, $n = 1, 2, 3, \cdots$ denote the recurrent random walk formed by the partial sums of i.i.d. lattice random variables with mean zero and finite variance. Let $T_{\{x\}} = \min [n \geqq 1 \mid S_n = x]$ with $T \equiv T_{\{0\}}$. We obtain a local limit theorem for the random walk conditioned by the event $[T > n]$. This result is applied then to obtain an approximation for $P[T_{\{x\}} = n]$ and the asymptotic distribution of $T_{\{x\}}$ as x approaches infinity.

1. Introduction. Let $X_1, X_2, \cdots$ be a sequence of i.i.d. random variables defined on a probability space $(\Omega, \mathscr{F}, P)$. We assume that the X_i are distributed on the lattice of integers with $EX_i = 0$ and $EX_i^2 = \sigma^2 < \infty$. For a fixed but arbitrary integer x_0, define $S_0 = x_0$, $S_n = x_0 + X_1 + \cdots + X_n$ for $n = 1, 2, \cdots$. The sequence $\{S_n\}$ is a *random walk* with *initial state* x_0. We employ the notation P^x for the underlying probability measure to indicate that $x_0 = x$. When $x_0 = 0$ we simply write P.

An integer x is a *recurrent state* if $P[S_n = x \text{ i.o.}] = 1$. It follows from the assumption $EX_i = 0$ that every integer is a recurrent state and the random walk itself is said to be *recurrent*. In all that follows we assume also that the random walk is aperiodic (see Spitzer [5] for a discussion of periodicity of random walk).

Define the stopping time T either to be the first $n \geqq 1$ such that $S_n = 0$ or to be $+\infty$ if no such n exists. In this paper we consider the chance behavior of random walk conditioned by the event $[T > n]$.

It is well known that T is finite with probability one and that

$$\lim_{n\to\infty} n^{\frac{1}{2}} P[T > n] = (2/\pi)^{\frac{1}{2}}\sigma . \tag{1}$$

It follows from a result of Kesten [4] that

$$\lim_{n\to\infty} n^{\frac{3}{2}} P[T = n] = \sigma/(2\pi)^{\frac{1}{2}} . \tag{2}$$

Belkin [1] has shown that

$$\lim_{n\to\infty} P[S_n/n^{\frac{1}{2}} \leqq x \mid T > n] = \int_{-\infty}^{x} (|y|/2\sigma^2) \exp(-y^2/2\sigma^2)\, dy . \tag{3}$$

Our major result is the conditional local limit theorem corresponding to (3). We remark that in all which follows suprema will be taken over the set of integers.

THEOREM 1. *Suppose the random variables* $X_1, X_2, \cdots$ *are* i.i.d. *on the lattice of*

Received August 9, 1974; revised January 15, 1975.
AMS 1970 *subject classifications.* Primary 60F15, 60G40, 60J15; Secondary 60F05.
Key words and phrases. Local limit theorem, stopping time, hitting time, conditioned random walk, random walk.

integers with $EX_i = 0$ *and* $EX_i^2 = \sigma^2 < \infty$. *Then*

$$\lim_{n\to\infty} \sup_x |n^{\frac{1}{2}}P[S_n = x \mid T > n] - (|x|/2\sigma^2 n^{\frac{1}{2}}) \exp(-x^2/2n\sigma^2)| = 0 . \tag{4}$$

It is readily seen that this result is a generalization of (3) just as the local central limit theorem is a generalization of the integral version.

For any integer x define the hitting time $T_{\{x\}}$ either to be the first $n \geqq 1$ such that $S_n = x$ or to be $+\infty$ if no such n exists. We state two interesting consequences of (4).

COROLLARY 1. *Under the hypotheses of Theorem* 1

$$\lim_{n\to\infty} \sup_x |nP[T_{\{x\}} = n] - (|x|/\sigma n^{\frac{1}{2}})\phi(x/\sigma n^{\frac{1}{2}})| = 0 \tag{5}$$

where $\phi(t)$ *denotes the standard normal probability density function.*

COROLLARY 2. *Under the hypotheses of Theorem* 1

$$\lim_{x\to\infty} P[\sigma^2 T_{\{x\}}/x^2 \leqq z] = 2[1 - \Phi(z^{-\frac{1}{2}})]$$

where $\Phi(t)$ *denotes the standard normal distribution function.*

2. A lemma. We adopt the notation $r_n = P[T > n]$, $f_n = P[T = n]$, $u_n(x) = P[S_n = x]$ and record the following decomposition of $P[S_n = x; T > n]$ obtained by Belkin [1].

$$P[S_n = x; T > n] = \sum_{k=0}^{n-1} r_k[u_{n-k}(x) - u_{n-k-1}(x)] . \tag{6}$$

It will be seen that (6) provides the key to the proof of Theorem 1. To obtain (4) we must first determine the asymptotic nature of the differences $u_{n-k}(x) - u_{n-k-1}(x)$ appearing in (6).

LEMMA. *Under the hypotheses of Theorem* 1

$$\lim_{n\to\infty} \sup_x |\sigma n^{\frac{3}{2}}[u_n(x) - u_{n-1}(x)] - \tfrac{1}{2}[(x^2/\sigma^2 n) - 1]\phi(x/\sigma n^{\frac{1}{2}})| = 0 . \tag{7}$$

PROOF. We follow the approach of Gnedenko [3] in his proof of the local central limit theorem.

Let X_1 have characteristic function f. Then employing the Fourier inversion formula and substitution we obtain

$$\sigma n^{\frac{3}{2}}[u_n(x) - u_{n-1}(x)] = (\sigma/2\pi) \int_{-n^{\frac{1}{2}}\pi}^{n^{\frac{1}{2}}\pi} e^{-iux/n^{\frac{1}{2}}} n[f(u/n^{\frac{1}{2}}) - 1] f^{n-1}(u/n^{\frac{1}{2}})\, du .$$

Since

$$(1/2\pi) \int_{-\infty}^{\infty} e^{-ity}(\sigma^2 t^2/2) e^{-\sigma^2 t^2/2}\, dt = (1/2\sigma)(1 - y^2/\sigma^2)\phi(y/\sigma)$$

it suffices to prove that $R_n(x)$ approaches zero uniformly in x where

$$R_n(x) = \int_{-n^{\frac{1}{2}}\pi}^{n^{\frac{1}{2}}\pi} e^{-iux/n^{\frac{1}{2}}} n[1 - f(u/n^{\frac{1}{2}})] f^{n-1}(u/n^{\frac{1}{2}})\, du \tag{8}$$
$$- \int_{-\infty}^{\infty} e^{-iux/n^{\frac{1}{2}}} (\sigma^2 u^2/2) e^{-\sigma^2 u^2/2}\, du \equiv I_1 + I_2 + I_3 + I_4 .$$

$$I_1 = \int_{|u|<A} e^{-iux/n^{\frac{1}{2}}} \{n[1 - f(u/n^{\frac{1}{2}})] f^{n-1}(u/n^{\frac{1}{2}}) - (\sigma^2 u^2/2) e^{-\sigma^2 u^2/2}\}\, du$$

$$I_2 = \int_{A \leqq |u| < \delta n^{\frac{1}{2}}} e^{-iux/n^{\frac{1}{2}}} n[1 - f(u/n^{\frac{1}{2}})] f^{n-1}(u/n^{\frac{1}{2}})\, du$$

$$I_3 = \int_{\delta n^{\frac{1}{2}} \leqq |u| \leqq \pi n^{\frac{1}{2}}} e^{-iux/n^{\frac{1}{2}}} n[1 - f(u/n^{\frac{1}{2}})] f^{n-1}(u/n^{\frac{1}{2}})\, du$$

$$I_4 = -\int_{|u| \geqq A} e^{-iux/n^{\frac{1}{2}}} (\sigma^2 u^2/2) e^{-\sigma^2 u^2/2}\, du .$$

Noting that $\lim_{n\to\infty} n[1 - f(u/n^{\frac{1}{2}})] = \sigma^2 u^2/2$ uniformly on finite intervals, a verification that each of the four integrals in (8) is uniformly small for sufficiently large n completes the proof. The essential arguments are similar to those which appear in Gnedenko [3].

3. Proof of Theorem 1. Using (1) it follows that the assertion of the theorem is equivalent to the statement

$$(9) \qquad \lim_{n\to\infty} \sup_x |nP[S_n = x; T > n] - (|x|/\sigma n^{\frac{1}{2}})\phi(x/\sigma n^{\frac{1}{2}})| = 0 .$$

We proceed to verify (9). From (1) and (2) it follows that there exist positive real numbers B_1 and B_2 such that

$$(10) \qquad r_n < B_1 n^{-\frac{1}{2}} \qquad \text{for} \quad n = 1, 2, \cdots$$

and

$$(11) \qquad f_n < B_2 n^{-\frac{3}{2}} \qquad \text{for} \quad n = 1, 2, \cdots .$$

The local version of the central limit theorem gives

$$(12) \qquad \lim_{n\to\infty} \sup_x |\sigma n^{\frac{1}{2}} P[S_n = x] - \phi(x/\sigma n^{\frac{1}{2}})| = 0 .$$

Observing that each of the sequences $\phi(x/\sigma n^{\frac{1}{2}})$ and $\frac{1}{2}(x^2/\sigma^2 n - 1)\phi(x/\sigma n^{\frac{1}{2}})$ is uniformly bounded in x, and employing (12) and (7) we are guaranteed the existence of positive real numbers B_3 and B_4 such that

$$(13) \qquad \sup_x u_n(x) < B_3 n^{-\frac{1}{2}} \qquad \text{for} \quad n = 1, 2, \cdots$$

and

$$(14) \qquad \sup_x |u_n(x) - u_{n-1}(x)| < B_4 n^{-\frac{3}{2}} \qquad \text{for} \quad n = 1, 2, \cdots .$$

Let Δ be any real number satisfying $0 < \Delta < \frac{1}{2}$. Using (6) and then summation by parts we obtain for every $x \neq 0$ (if $x = 0$, $P[S_n = x; T > n] = 0$ and there is nothing to prove)

$$(15) \qquad \begin{aligned} nP[S_n = x; T > n] = n[u_n(x) - u_{n-1}(x)] &+ n \textstyle\sum_{k=1}^{n\Delta-1} r_k[u_{n-k}(x) - u_{n-k-1}(x)] \\ &+ n \textstyle\sum_{k=n\Delta}^{n(1-\Delta)-1} r_k[u_{n-k}(x) - u_{n-k-1}(x)] \\ &+ nr_{n(1-\Delta)} u_{n\Delta}(x) - n \textstyle\sum_{k=n(1-\Delta)+1}^{n-1} f_k u_{n-k}(x) . \end{aligned}$$

Define $\sum(\Delta, n, x)$ and $I(\Delta, n, x)$ by

$$(16) \qquad \begin{aligned} \textstyle\sum(\Delta, n, x) &= \textstyle\sum_{k=n\Delta}^{n(1-\Delta)-1} (2/\pi)^{\frac{1}{2}}(k/n)^{-\frac{1}{2}}(1 - k/n)^{-\frac{3}{2}} n^{-1} \\ &\qquad \times \frac{1}{2}\left[\frac{x^2}{\sigma^2 n(1 - k/n)} - 1\right]\phi\left(\frac{x}{\sigma n^{\frac{1}{2}}(1 - k/n)^{\frac{1}{2}}}\right) \\ &= n \textstyle\sum_{k=n\Delta}^{n(1-\Delta)-1} \sigma(2/\pi)^{\frac{1}{2}} k^{-\frac{1}{2}}(n - k)^{-\frac{3}{2}}(1/2\sigma) \\ &\qquad \times \left[\frac{x^2}{\sigma^2(n - k)} - 1\right]\phi\left(\frac{x}{\sigma(n - k)^{\frac{1}{2}}}\right) \end{aligned}$$

$$(17) \qquad \begin{aligned} I(\Delta, n, x) &= \textstyle\int_{\Delta}^{1-\Delta} (2/\pi)^{\frac{1}{2}} t^{-\frac{1}{2}}(1 - t)^{-\frac{3}{2}} \\ &\qquad \times \frac{1}{2}\left[\frac{x^2}{\sigma^2 n(1 - t)} - 1\right]\phi\left(\frac{x}{\sigma n^{\frac{1}{2}}(1 - t)^{\frac{1}{2}}}\right) dt . \end{aligned}$$

From (15), (16), (17) and the triangle inequality we obtain

$$\begin{aligned}
&|nP[S_n = x;\, T > n] - (|x|/\sigma n^{\frac{1}{2}})\phi(x/\sigma n^{\frac{1}{2}})| \\
&\quad \leqq n|u_n(x) - u_{n-1}(x)| \\
&\qquad + n \textstyle\sum_{k=1}^{n\Delta-1} r_k|u_{n-k}(x) - u_{n-k-1}(x)| + n \sum_{k=n(1-\Delta)+1}^{n-1} f_k u_{n-k}(x) \\
&\qquad + |nr_{n(1-\Delta)} u_{n\Delta}(x) - (2/\pi)^{\frac{1}{2}}[\Delta(1-\Delta)]^{-\frac{1}{2}}\phi(x/\sigma(n\Delta)^{\frac{1}{2}})| \\
&\qquad + |n \textstyle\sum_{k=n\Delta}^{n(1-\Delta)-1} r_k[u_{n-k}(x) - u_{n-k-1}(x)] - \sum(\Delta, n, x)| \\
&\qquad + |\textstyle\sum(\Delta, n, x) - I(\Delta, n, x)| \\
&\qquad + |I(\Delta, n, x) + (2/\pi)^{\frac{1}{2}}[\Delta(1-\Delta)]^{-\frac{1}{2}}\phi(x/\sigma(n\Delta)^{\frac{1}{2}}) \\
&\qquad - (|x|/\sigma n^{\frac{1}{2}})\phi(x/\sigma n^{\frac{1}{2}})|\,.
\end{aligned} \tag{18}$$

We now consider the asymptotic behavior of each term in (18).

From (14) we obtain

$$\lim_{n\to\infty} \sup_x n|u_n(x) - u_{n-1}(x)| = 0\,. \tag{19}$$

Applying (10) and (14) with the fact that $\sum_{k=1}^{n\Delta} k^{-\frac{1}{2}} \leqq 2(n\Delta)^{\frac{1}{2}}$ gives

$$\limsup_{n\to\infty} \sup_x n \textstyle\sum_{k=1}^{n\Delta-1} r_k|u_{n-k}(x) - u_{n-k-1}(x)| \leqq 2B_1B_4\Delta^{\frac{1}{2}}(1-\Delta)^{-\frac{3}{2}}\,. \tag{20}$$

Similarly we obtain from (11) and (13)

$$\limsup_{n\to\infty} \sup_x n \textstyle\sum_{k=n(1-\Delta)+1}^{n-1} f_k u_{n-k}(x) \leqq 2B_2B_3\Delta^{\frac{1}{2}}(1-\Delta)^{-\frac{3}{2}}\,. \tag{21}$$

Applying (13) and then (1) and (12) will show that

$$\limsup_{n\to\infty} \sup_x |nr_{n(1-\Delta)} u_{n\Delta}(x) - (2/\pi)^{\frac{1}{2}}[\Delta(1-\Delta)]^{-\frac{1}{2}}\phi(x/\sigma(n\Delta)^{\frac{1}{2}})| = 0\,. \tag{22}$$

Using the inequality

$$\begin{aligned}
&\sup_x |n \textstyle\sum_{k=n\Delta}^{n(1-\Delta)-1} r_k[u_{n-k}(x) - u_{n-k-1}(x)] - \sum(\Delta, n, x)| \\
&\quad \leqq n \textstyle\sum_{k=n\Delta}^{n(1-\Delta)-1} |k^{\frac{1}{2}}r_k - (2/\pi)^{\frac{1}{2}}\sigma k^{-\frac{1}{2}} \sup_x |u_{n-k}(x) - u_{n-k-1}(x)| \\
&\qquad + n \textstyle\sum_{k=n\Delta}^{n(1-\Delta)-1} (2/\pi)^{\frac{1}{2}}k^{-\frac{1}{2}}(n-k)^{-\frac{3}{2}} \sup_x |\sigma(n-k)^{\frac{3}{2}}[u_{n-k}(x) - u_{n-k-1}(x)] \\
&\qquad - \tfrac{1}{2}[x^2/\sigma^2(n-k) - 1]\phi(x/\sigma(n-k)^{\frac{1}{2}})|
\end{aligned}$$

it follows from (1) and (7) that

$$\limsup_{n\to\infty} \sup_x |n \textstyle\sum_{k=n\Delta}^{n(1-\Delta)-1} r_k[u_{n-k}(x) - u_{n-k-1}(x)] - \sum(\Delta, n, x)| = 0\,. \tag{23}$$

Since the function $f\colon [\Delta, 1-\Delta] \times \mathbb{R} \to \mathbb{R}$ defined by

$$f(t, y) = (2/\pi)^{\frac{1}{2}}t^{-\frac{1}{2}}(1-t)^{-\frac{3}{2}}\tfrac{1}{2}[y^2/\sigma^2(1-t) - 1]\phi(y/\sigma(1-t)^{\frac{1}{2}})$$

is uniformly continuous, it follows that

$$\limsup_{n\to\infty} \sup_x |\textstyle\sum(\Delta, n, x) - I(\Delta, n, x)| = 0\,. \tag{24}$$

We consider now the final term in (18). The substitution $u = (x^2/\sigma^2 n)[(1-t)^{-1} - 1]$ followed by integration by parts gives

$$\begin{aligned}
I(\Delta, n, x) &= (1/(2\pi)^{\frac{1}{2}})(|x|/\sigma n^{\frac{1}{2}})\phi(x/\sigma n^{\frac{1}{2}}) \textstyle\int_{L_1}^{L_2} u^{-\frac{1}{2}}e^{-u/2}\,du \\
&\quad + (2/\pi)^{\frac{1}{2}}\{[\Delta/(1-\Delta)]^{\frac{1}{2}}\phi(x/\sigma(n(1-\Delta))^{\frac{1}{2}}) \\
&\quad - [(1-\Delta)/\Delta]^{\frac{1}{2}}\phi(x/\sigma(n\Delta)^{\frac{1}{2}})\}\,.
\end{aligned} \tag{25}$$

where $L_1 = x^2\Delta/\sigma^2 n(1-\Delta)$ and $L_2 = x^2(1-\Delta)/\sigma^2 n\Delta$.

From (25) we obtain the inequality

$$\begin{aligned}(26)\qquad &|I(\Delta, n, x) + (2/\pi)^{\frac{1}{2}}[\Delta/(1-\Delta)]^{-\frac{1}{2}}\phi(x/\sigma(n\Delta)^{\frac{1}{2}}) - (|x|/\sigma n^{\frac{1}{2}})\phi(x/\sigma n^{\frac{1}{2}})| \\ &\quad \leqq (|x|/\sigma n^{\frac{1}{2}})\phi(x/\sigma n^{\frac{1}{2}})|(1/(2\pi)^{\frac{1}{2}}) \textstyle\int_{L_1}^{L_2} u^{-\frac{1}{2}}e^{-u/2}\,du - 1| \\ &\qquad + (2/\pi)[\Delta/(1-\Delta)]^{\frac{1}{2}} .\end{aligned}$$

Combining (18), (19), (20), (21), (22), (23), (24) and (26) we have that for all x and $\Delta \varepsilon (0, \frac{1}{2})$

$$\begin{aligned}(27)\qquad &\limsup_{n\to\infty} |nP[S_n = x; T > n] - (|x|/\sigma n^{\frac{1}{2}})\phi(x/\sigma n^{\frac{1}{2}})| \\ &\quad \leqq 2(B_1B_4 + B_2B_3)\Delta^{\frac{1}{2}}(1-\Delta)^{-\frac{3}{2}} + (2/\pi)[\Delta/(1-\Delta)]^{\frac{1}{2}} \\ &\qquad + \limsup_{n\to\infty} (|x|/\sigma n^{\frac{1}{2}})\phi(x/\sigma n^{\frac{1}{2}})|(1/(2\pi)^{\frac{1}{2}}) \textstyle\int_{L_1}^{L_2} u^{-\frac{1}{2}}e^{-u/2}\,du - 1| .\end{aligned}$$

We prove (9) by contradiction. If the assertion is false, then for some $\delta > 0$ there exist both an increasing sequence of integers $\{n_j\}$ and a sequence of integers $\{x_j\}$ such that

$$(28)\qquad |n_jP[S_{n_j} = x_j; T > n_j] - (|x_j|/\sigma(n_j)^{\frac{1}{2}})\phi(x_j/\sigma(n_j)^{\frac{1}{2}})| \geqq \delta \quad \text{for} \quad j = 1, 2, \cdots .$$

Without loss of generality we assume that $\lim_{j\to\infty} |x_j|/\sigma(n_j)^{\frac{1}{2}}$ exists (possibly infinite).

First suppose $\lim_{j\to\infty} |x_j|/\sigma(n_j)^{\frac{1}{2}} = 0$ or $+\infty$. For every n, x, and $\Delta\varepsilon(0, \frac{1}{2})$ we have

$$0 \leqq (1/(2\pi)^{\frac{1}{2}}) \textstyle\int_{L_1}^{L_2} u^{-\frac{1}{2}}e^{-u/2}\,du < (1/(2\pi)^{\frac{1}{2}}) \int_0^\infty u^{-\frac{1}{2}}e^{-u/2}\,du = 1 .$$

Then $|(1/(2\pi)^{\frac{1}{2}}) \int_{L_1}^{L_2} u^{-\frac{1}{2}}e^{-u/2}\,du - 1| \leqq 1$ and (27) and (28) give

$$\begin{aligned}0 < \delta &\leqq \limsup_{j\to\infty} |n_jP[S_{n_j} = x_j; T > n_j] - (|x_j|/\sigma(n_j)^{\frac{1}{2}})\phi(x_j/\sigma(n_j)^{\frac{1}{2}})| \\ &\leqq 2(B_1B_4 + B_2B_3)\Delta^{\frac{1}{2}}(1-\Delta)^{-\frac{3}{2}} + (2/\pi)[\Delta/(1-\Delta)]^{\frac{1}{2}} \\ &\qquad + \limsup_{j\to\infty} (|x_j|/\sigma(n_j)^{\frac{1}{2}})\phi(x_j/\sigma(n_j)^{\frac{1}{2}}) \\ &= 2(B_1B_4 + B_2B_3)\Delta^{\frac{1}{2}}(1-\Delta)^{-\frac{3}{2}} + (2/\pi)[\Delta/(1-\Delta)]^{\frac{1}{2}} .\end{aligned}$$

Allowing Δ to approach zero we obtain a contradiction.

Now suppose $\lim_{j\to\infty} |x_j|/\sigma(n_j)^{\frac{1}{2}} = a$ where $0 < a < \infty$. Employing the substitution $t = a^2u\sigma^2n_j/2x_j^2$ we have

$$\begin{aligned}(1/(2\pi)^{\frac{1}{2}}) \textstyle\int_{L_1}^{L_2} u^{-\frac{1}{2}}e^{-u/2}\,du &= (1/(2\pi)^{\frac{1}{2}}) \textstyle\int_{x_j^2\Delta/\sigma^2n_j(1-\Delta)}^{x_j^2(1-\Delta)/\sigma^2n_j\Delta} u^{-\frac{1}{2}}e^{-u/2}\,du \\ &= (1/\pi^{\frac{1}{2}})(a^{-1}|x_j|/\sigma(n_j)^{\frac{1}{2}}) \textstyle\int_{a^2\Delta/2(1-\Delta)}^{a^2(1-\Delta)/2\Delta} t^{-\frac{1}{2}}e^{-tx_j^2/a^2\sigma^2n_j}\,dt \\ &\to (1/\pi^{\frac{1}{2}}) \textstyle\int_{a^2\Delta/2(1-\Delta)}^{a^2(1-\Delta)/2\Delta} t^{-\frac{1}{2}}e^{-t}\,dt \qquad \text{as} \quad j \to \infty .\end{aligned}$$

From (27) and (28)

$$\begin{aligned}0 < \delta &\leqq \limsup_{j\to\infty} |n_jP[S_{n_j} = x_j; T > n_j] - (|x_j|/\sigma(n_j)^{\frac{1}{2}})\phi(x_j/\sigma(n_j)^{\frac{1}{2}})| \\ &\leqq 2(B_1B_4 + B_2B_3)\Delta^{\frac{1}{2}}(1-\Delta)^{-\frac{3}{2}} + (2/\pi)[\Delta/(1-\Delta)]^{\frac{1}{2}} \\ &\qquad + \limsup_{j\to\infty} (|x_j|/\sigma(n_j)^{\frac{1}{2}})\phi(|x_j|/\sigma(n_j)^{\frac{1}{2}})|(1/(2\pi)^{\frac{1}{2}}) \textstyle\int_{L_1}^{L_2} u^{-\frac{1}{2}}e^{-u/2}\,du - 1| \\ &= 2(B_1B_4 + B_2B_3)\Delta^{\frac{1}{2}}(1-\Delta)^{-\frac{3}{2}} + (2/\pi)[\Delta/(1-\Delta)]^{\frac{1}{2}} \\ &\qquad + a\phi(a)|(1/\pi^{\frac{1}{2}}) \textstyle\int_{a^2\Delta/2(1-\Delta)}^{a^2(1-\Delta)/2\Delta} t^{-\frac{1}{2}}e^{-t}\,dt - 1| .\end{aligned}$$

Again allowing Δ to approach zero we obtain a contradiction. Hence,

$$\lim_{n\to\infty} \sup_x |nP[S_n = x; T > n] - (|x|/\sigma n^{\frac{1}{2}})\phi(x/\sigma n^{\frac{1}{2}})| = 0$$

and the proof is complete.

4. Proof of Corollary 1. For $k = 0, 1, \cdots, n$ define $S_k{}^* = S_n - S_{n-k}$. The P distribution of the random vector $(S_0{}^*, S_1{}^*, \cdots, S_n{}^*)$ is the same as that of $(S_0, S_1, \cdots, S_n)$.

Then

$$\begin{aligned} P[T_{\{x\}} = n] &= P[S_0 = 0, S_1 \neq x, \cdots, S_{n-1} \neq x, S_n = x] \\ &= P[S_0{}^* = 0, S_1{}^* \neq 0, \cdots, S_{n-1}^* \neq 0, S_n{}^* = x] \\ &= P[S_n = x; T > n] \end{aligned}$$

and the assertion follows from Theorem 1.

5. Proof of Corollary 2. The proof involves an application of Theorem 7.8 of [2].

Suppose y is any real number and that the sequence $\{y_x\}$ varies with x in such a way that $y_x \to y$ as $x \to \infty$ (each of the terms y_x must be of the form $\sigma^2 k/x^2$ where k is an integer). It follows from Corollary 1 that

$$\lim_{x\to\infty} (x^2/\sigma^2)P[\sigma^2 T_{\{x\}}/x^2 = y_x] = y^{-\frac{3}{2}}\phi(y^{-\frac{1}{2}})$$

and hence that

$$\lim_{x\to\infty} P[\sigma^2 T_{\{x\}}/x^2 \leqq z] = \int_0^z y^{-\frac{3}{2}}\phi(y^{-\frac{1}{2}})\, dy = 2[1 - \Phi(z^{-\frac{1}{2}})]\,.$$

REFERENCES

[1] BELKIN, B. (1970). A limit theorem for conditioned recurrent random walk attracted to a stable law. *Ann. Math. Statist.* **41** 146–163.

[2] BILLINGSLEY, P. (1968). *Convergence of Probability Measures.* Wiley, New York.

[3] GNEDENKO, B. V. (1962). *Theory of Probability.* Chelsea, New York.

[4] KESTEN, H. (1963). Ratio theorems for random walk II. *J. Analyse Math.* **9** 323–379.

[5] SPITZER, F. (1964). *Principles of Random Walk.* Van Nostrand, Princeton.

DEPARTMENT OF MATHEMATICS
THE UNIVERSITY OF TEXAS
EL PASO, TEXAS 79968