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A STRONG LAW OF LARGE NUMBERS FOR RANDOM COMPACT SETS

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A strong law of large numbers is shown for random sets taking values in the nonempty, compact subsets of $\mathbb{R}^n$.

1. Introduction. In the study of probabilities on geometrical objects, there have been some recent attempts to formulate general theories of *random sets*, notably by Kendall [8] and Matheron [10]. It is our purpose here to make a contribution in this direction by demonstrating the existence of a strong law of large numbers for random sets taking values in the class of compact subsets of $\mathbb{R}^n$. The result is proved first under the assumption of convexity and then extended to the general case. In the spirit of previous (nonprobabilistic) work by Castaing [6], Debreu [7], and Rockafellar [12] among others, we find it useful to proceed from the definition of a random set as a *measurable set-valued function*.

Our particular concern with the behavior of sums of random sets arose in the formulation of a stochastic model of growth which seems useful in certain applications where enlargement occurs by surface accretion. Mathematically it is appealing to model such a dynamic by the set addition of random "growth elements." The question of asymptotic *shape* then leads naturally to the consideration of normalized sums of random sets.

2. Random sets. We regard a random set X as a measurable map defined on an abstract probability space (Ω, Σ, P) and taking values in the collection $\mathscr{C}$ of nonempty, compact subsets of $\mathbb{R}^n$. With the introduction of the Hausdorff distance

$$d(C_1, C_2) = \inf\{\lambda : C_1 \subseteqq C_2 + \lambda B, C_2 \subseteqq C_1 + \lambda B\},$$

$\mathscr{C}$ can be made into a separable, locally compact metric space. Here B is the closed unit ball in $\mathbb{R}^n$ and scalar multiplication and addition are defined as usual by $\lambda C = \{\lambda c : c \in C\}$ and $C_1 + C_2 = \{c_1 + c_2 : c_1 \in C_1, c_2 \in C_2\}$. We denote by $||C||$, the distance $d(C, \{0\})$, which is equal to $\sup\{||c|| : c \in C\}$.

If C belongs to $\mathscr{C}$, then we denote by $\operatorname{co} C$ the convex hull of C and by $\operatorname{co}\mathscr{C}$ the collection of all such nonempty, convex, and compact subsets of $\mathbb{R}^n$. It is

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well-known that $\mathrm{co}\,\mathscr{C}$ is a closed subset of $\mathscr{C}$ and that the map $C \to \mathrm{co}\,C$ is continuous.

In light of the metric structure of $\mathscr{C}$, measurability of X will be taken in the Borel sense (Billingsley [5]). It should be pointed out that an apparent alternative is to focus on the *geometry* of the values of X and to require Σ-measurability of sets of the form $\{\omega \in \Omega : X(\omega) \cap C_0 \neq \phi\}$, $C_0 \in \mathscr{C}$ (Kuratowski and Ryll-Nardzewski [9], Rockafellar [12]). Debreu [7, page 355] has shown that this is in fact an equivalent formulation.

Viewing random sets as random elements in a metric space, we note that well-known results imply that, if X is a random set, than αX and $\mathrm{co}\,X$ are random sets and $||X||$ is a random variable. Moreover, if $X_1, X_2, \cdots, X_N$ are random sets, then so is $\sum_1^N X_i$. If the former compose an independent collection, then this implies the same for $\mathrm{co}\,X_1, \mathrm{co}\,X_2, \cdots, \mathrm{co}\,X_N$.

3. Law of large numbers. We begin by defining the expectation of a random set. A *selection* of the random set X is a random vector x such that $x(\omega) \in X(\omega)$ a.s. Selections exist (see Aumann [4], Kuratowski and Ryll-Nardzewski [9]).

DEFINITION. Let X be a random set such that each selection x has finite expectation Ex. The *expectation of* X, written EX, is the set $\{Ex \mid x \text{ is a selection of } X\}$.

We regard the expectation as well-defined if $EX \in \mathscr{C}$, for which a necessary and sufficient condition is that $E||X|| < \infty$. Moreover, if P is nonatomic, then $EX = E\,\mathrm{co}\,X$ (see, for instance, Artstein [2], Aumann [4] for these respective results in different notations).

An equivalent formulation of the expectation is to consider the integration of a random set viewed as a *point-valued* function valued in $\mathscr{C}$ (Artstein and Burns [3], Debreu [7]). The resulting expectation is the same and we do not present the precise formulation here.

We turn now to the law of large numbers.

THEOREM. *Let X_i, $i = 1, 2, \cdots$ be independent, identically distributed random sets such that $E||X_i|| < \infty$. Then*

$$S_N = \frac{X_1 + X_2 + \cdots + X_N}{N} \to E\,\mathrm{co}\,X \quad \text{a.s.}$$

Note that by Aumann's result $E\,\mathrm{co}\,X$ is equal to EX if the probability space is nonatomic. In the context of the law of large numbers for nonconstant X_i, the appropriate (product) probability space is always nonatomic.

The proof of the theorem will be given in two steps. First it is specialized to the case of convex-set-valued random sets (Section 4) and then extended to the general case (Section 5).

4. The convex case. We note that $\mathrm{co}\,\mathscr{C}$ can be embedded in the Banach space of continuous functions defined on the unit sphere in $\mathbb{R}^n$ by identifying a set $C \in \text{co}\mathscr{C}$ with its support function

$$s(p, C) = \sup \{p \cdot c : c \in C\} .$$

Since this mapping preserves both metric and linear structure, a direct appeal to a known strong law in a Banach space (see, for instance, Mourier [11]) yields the a.s. convergence of the sequence $s(\cdot, S_N)$. In particular for each $p \in \mathbb{R}^n$, $s(p, S_N) \rightarrow Es(p, X)$ a.s. which is equal to $s(p, EX)$ (see Artstein [2]). In view of the isometry between convex sets and their support functions, we conclude $S_N \rightarrow EX$ a.s.

5. The general case. In relaxing the assumption of convexity, we note that an embedding such as the one mentioned in the previous section cannot be done except for extremely specialized situations. Here, it seems, we are forced to recognize the geometrical content of the problem. The following result will provide a convenient way to bridge the gap.

PROPOSITION. (Shapley–Folkman; see Arrow and Hahn [1, page 396]): *Let* $C_i \in \mathscr{C}$, 1, 2, $\cdots$, N, *such that* $||C_i|| \leqq M$. *Then*

$$d(\textstyle\sum_{i=1}^N C_i, \text{co} \sum_{i=1}^N C_i) \leqq n^{\frac{1}{2}} M$$

(note the lack of dependence on N*).*

We use this to show that in some sense averaging is asymptotically "convexifying."

LEMMA. *Let* $C_i \in \mathscr{C}$, $i = 1, 2, \cdots$, *be such that* $\text{co}[(C_1 + C_2 + \cdots + C_N)/N] \rightarrow C$ *(necessarily convex). Then* $(C_1 + C_2 + \cdots + C_N)/N \rightarrow C$.

PROOF. With the triangle inequality, we have

$$d\left(\frac{C_1 + C_2 + \cdots + C_N}{N}, C\right) \leqq d\left(\frac{C_1 + \cdots + C_N}{N}, \text{co}\left[\frac{C_1 + \cdots + C_N}{N}\right]\right) + d\left(\text{co}\left[\frac{C_1 + \cdots + C_N}{N}\right], C\right).$$

The second term goes to zero with increasing N by assumption. As for the first term, we note that for fixed $p \in \mathbb{R}^n$

$$s\left(p, \frac{C_1 + \cdots + C_N}{N}\right) = \frac{1}{N} \sum s(p, C_i)$$

converges to $s(p, C)$, which implies *a fortiori* $s(p, C_N) = o(N)$. Since this holds for each p among the $2n$ signed unit vectors in $\mathbb{R}^n$, we clearly have $||C_N|| = o(N)$. It is straightforward to conclude that $\max \{||C_i||/N \mid i = 1, 2, \cdots, N\} \rightarrow 0$ and hence by the Shapley–Folkman result

$$d\left(\frac{C_1 + C_2 + \cdots + C_N}{N}, \text{co}\left[\frac{C_1 + C_2 + \cdots + C_N}{N}\right]\right) \leqq n \cdot \max \left\{\frac{||C_i||}{N} \,\middle|\, i = 1, 2, \cdots, N\right\} \rightarrow 0 .$$

With this preparation, the extension of the law of large numbers to the general case is easily effected. Indeed under the assumptions made, the sequence $\mathrm{co}\,X_1$, $\mathrm{co}\,X_2, \cdots$ satisfies the hypotheses of the theorem for the convex case. Hence

$$\frac{\mathrm{co}\,X_1 + \mathrm{co}\,X_2 + \cdots + \mathrm{co}\,X_N}{N} \to E\,\mathrm{co}\,X \quad \text{a.s.}$$

By the lemma, S_N converges on the same ω set to the identical limit, and we are done.

It should be pointed out that the hypotheses of the theorem can be relaxed to state simply that $\mathrm{co}\,X_1, \mathrm{co}\,X_2, \cdots$ satisfy the conditions of the theorem. Indeed, it is possible to construct examples where the individual X_i *are not even random sets* in the sense that they fail the measurability criterion. However, we have chosen to state the theorem in the more natural context.

We conclude by noting that other results can probably be obtained by using an outline of the development employed above; in short, by first applying a standard limit theorem for random variables to support functions—yielding a limit result for random convex sets—and then using the lemma above (or another extension of the Shapley–Folkman result) to cover the general case.

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