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NOTE ON A LIMIT THEOREM

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A simple derivation of a limiting distribution of Logan, Mallows, Rice and Shepp is found. This is the limiting distribution of a sequence of partial sums of independent, identically distributed random variables in the domain of attraction of a stable law, normalized by the partial sums of the absolute values of these variables.

Let $X_1, X_2, \cdots$ be independent, identically distributed random variables, belonging to the domain of attraction of a stable distribution with index α, $0 < \alpha < 1$. In [2], Logan, Mallows, Rice and Shepp found, among many other interesting results, the limiting distribution of

$$S_n = (\sum_{i=1}^n X_i)/(\sum_{i=1}^n |X_i|) . \tag{1}$$

They did this by calculating the joint limiting characteristic function of the numerator and denominator in (1). In this note we utilize a property of stable laws, perhaps of independent interest, to obtain a simple proof of their result.

Write $X_i = X_i^+ - X_i^-$, where X_i^+ and X_i^- are positive random variables: $X_i^+ = \max(X_i, 0)$, $X_i^- = -\min(X_i, 0)$.

LEMMA. *Let X_i belong to the domain of attraction of a stable law of index α, $0 < \alpha < 1$. Then there exist positive, independent stable laws, U^+, U^- of index α, and normalizing constants b_n, such that the joint limiting distribution of*

$$U_n^+ = b_n \sum_{i=1}^n X_i^+ \tag{2}$$

$$U_n^- = b_n \sum_{i=1}^n X_i^- \tag{3}$$

is that of (U^+, U^-).

In particular X_i^+ and X_i^- are attracted to the domain of a common stable law U.

To prove the lemma, by Doeblin's condition (cf. [1], page 175) we have

$$P(X_i^+ > x) + P(X_i^- > x) \sim x^{-\alpha}L(x) , \qquad x \to \infty \tag{4}$$

$$\frac{P(X_i^- > x)}{P(X_i^+ > x)} \to \frac{C_1}{C_2} , \qquad x \to \infty \tag{5}$$

where $L(x)$ is a slowly varying function, and $C_1 \geqq 0$, $C_2 \geqq 0$, $C_1 + C_2 > 0$. If $C_2 = 0$, then (5) is to be interpreted in the obvious way.

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The two "equations" (4) and (5) can be solved to give

$$P(X_i^- > x) \sim \frac{C_1}{C_1 + C_2} x^{-\alpha} L(x) , \qquad x \to \infty$$

$$P(X_i^+ > x) \sim \frac{C_2}{C_1 + C_2} x^{-\alpha} L(x) , \qquad x \to \infty$$

and in addition, if $\xi_1 > 0$, $\xi_2 > 0$, the random variables $Y_i = \xi_1 X_i^- + \xi_2 X_i^+$ belong to the domain of attraction of U also, with the same normalizing constants b_n. This follows from the fact that for Z_i in the domain of attraction of a stable law the scale constants b_n can be chosen to satisfy $P(Z_i > 1/b_n) \sim c/n$, and from

$$\begin{aligned} P(Y_i > x) &= P(X_i^- > x/\xi_1) + P(X_i^+ > x/\xi_2) \\ &\sim \left(\frac{C_1 \xi_1^\alpha}{C_1 + C_2} + \frac{C_2 \xi_2^\alpha}{C_1 + C_2}\right) x^{-\alpha} L(x) . \end{aligned}$$

It now follows that the joint characteristic function of U_n^+ and U_n^-, namely $E(\exp(ib_n \sum_{i=1}^n Y_i))$, converges to the product of the limiting characteristic functions of U_n^+ and U_n^-. It follows also that the same conclusion is reached for arbitrary real ξ_1, ξ_2, and the lemma is proved.

Since from (1)

$$S_n = \frac{U_n^+ - U_n^-}{U_n^+ + U_n^-}$$

where U_n^+ and U_n^- are given by (2) and (3), S_n converges in distribution to

$$S = \frac{U^+ - U^-}{U^+ + U^-} .$$

To find the distribution of S we shall use the following result. Let X be a random variable with a characteristic function $\Psi(\xi) = E(\exp(i\xi X))$ and such that $P(X = 0) = 0$, $E(|X|) < \infty$. Then

$$P(X > 0) = \tfrac{1}{2} + \lim_{T\to\infty} \frac{1}{\pi} \int_0^T \operatorname{Im} \Psi(\xi) \frac{d\xi}{\xi} . \tag{6}$$

The same formula holds if $E(|X|) = \infty$, but then the lower limit in (6) is also singular. (6) can be proved by using the well-known formula

$$\lim_{T\to\infty} \frac{2}{\pi} \int_0^T \frac{\sin at}{t} dt = \operatorname{sgn} a .$$

Let $\varphi_+(z)$ and $\varphi_-(z)$ be the characteristic functions of the two independent random variables U^+ and U^-, so that

$$\varphi_\pm(z) = \exp\{-c_\pm(\cos \tfrac{1}{2}\pi\alpha + i \operatorname{sgn} z \sin \tfrac{1}{2}\pi\alpha)|z|^\alpha\}$$

where $c_- = C_1/(C_1 + C_2)$, $c_+ = C_2/(C_1 + C_2)$.

Then the distribution function of S is

$$F(x) = P(S < x) = P\left(\frac{U^+ - U^-}{U^+ + U^-} < x\right)$$
$$= P(U^-(1 + x) - U^+(1 - x) > 0)$$
$$= \tfrac{1}{2} + \frac{1}{\pi} \int_0^\infty \operatorname{Im}[\varphi_-((1 + x)z)\varphi_+(-(1 - x)z)] \frac{dz}{z}$$
$$= \tfrac{1}{2} + \frac{1}{\pi} \int_0^\infty e^{-az^\alpha} \sin bz^\alpha \frac{dz}{z},$$

where, as easy algebra shows,

$$a = \cos \tfrac{1}{2}\pi\alpha(c_-(1 + x)^\alpha + c_+(1 - x)^\alpha),$$
$$b = -\sin \tfrac{1}{2}\pi\alpha(c_-(1 + x)^\alpha - c_+(1 - x)^\alpha).$$

The preceding integral is easy to evaluate, giving

$$F(x) = \tfrac{1}{2} - \frac{1}{\pi\alpha} \tan^{-1} \frac{b}{a}$$
$$= \tfrac{1}{2} + \frac{1}{\pi\alpha} \tan^{-1}\left(\left[\frac{c_-(1 + x)^\alpha - c_+(1 - x)^\alpha}{c_-(1 + x)^\alpha + c_+(1 - x)^\alpha}\right] \tan \tfrac{1}{2}\pi\alpha\right), \qquad |x| < 1.$$

For $\alpha = \frac{1}{2}$, $c_+ = c_-$, this becomes $F(x) = \frac{1}{2} + 1/\pi \sin^{-1} x$, a form of the "arcsin law."

REFERENCES

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