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COEFFICIENT PROPERTIES OF RANDOM VARIABLE SEQUENCES[1]

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Burkholder [3] gave a simple necessary and sufficient condition, in terms of concentration functions, for independent sequences of random variables to have the Stein property. Here we find sufficient conditions for the Stein property without assuming that the random variable sequence is independent. Our conditions are also in terms of concentration functions, but in our case they are conditional concentration functions which specialize to those used by Burkholder. For some of our results, the sequence of random variables may be quite arbitrary; however, we usually assume it to be a martingale difference sequence satisfying certain regularity conditions.

1. Introduction. Burkholder [3] gave a simple necessary and sufficient condition, in terms of concentration functions, for independent sequences of random variables to have the Stein property. Here we find sufficient conditions for the Stein property without assuming that the random variable sequence is independent. Our conditions are also in terms of concentration functions, but in our case they are conditional concentration functions (see Definition 2.1) which specialize to those used by Burkholder. For some of our results, the sequence of random variables may be quite arbitrary; however, we usually assume it to be a martingale difference sequence satisfying certain regularity conditions.

Let $Z = \{Z_k, k \geqq 1\}$ be a sequence of complex valued random variables defined on a probability space $(\Omega, \mathscr{A}, P)$, Γ the set of sequences $(a, a_1, a_2, \cdots)$ of complex numbers such that the series $\sum_{k=1}^{\infty} a_k Z_k$ converges almost everywhere (to a finite limit), l^2 the set of sequences $(a, a_1, a_2, \cdots)$ of complex numbers such that $\sum_{k=1}^{\infty} |a_k|^2 < \infty$, $\mathscr{C}$ the field of complex numbers, and $\mathscr{N}$ the set of positive integers. If $\mathscr{B}$ is a subalgebra of $\mathscr{A}$ let $\mathscr{B}^+ = \{B \in \mathscr{B} : P(B) > 0\}$. It will be convenient to have names for several properties which Z may have.

DEFINITION 1.1. The sequence Z has the *Stein property* if there exists a number $b > 0$ such that if $A \in \mathscr{A}^+$ then there exists an $n = n(b, A) \in \mathscr{N}$ such that $(a, a_1, a_2, \cdots) \in \Gamma$ implies

$$b\left(\sum_{k=n}^{\infty} |a_k|^2\right)^{\frac{1}{2}} \leqq \operatorname{ess\,sup}_{\omega \in A} \left|a + \sum_{k=1}^{\infty} a_k Z_k(\omega)\right| .$$

DEFINITION 1.2. The sequence Z has the *coefficient property* if $l^2 \subset \Gamma$.

DEFINITION 1.3. The sequence Z has the *converse coefficient property* if $\Gamma \subset l^2$.

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It is obvious from Definition 1.1 that the Stein property implies the converse coefficient property. Besides the converse coefficient property, the Stein property has another obvious consequence: if $(a, a_1, a_2, \cdots) \in \Gamma$ and $a_k \neq 0$ for infinitely many k, then

$$P(\sum_{k=1}^{\infty} a_k Z_k = a) = 0\,, \qquad a \in \mathscr{C}\,.$$

Thus the distribution of a convergent sum $\sum_{k=1}^{\infty} a_k Z_k$ is non-atomic, provided that $a_k \neq 0$ for infinitely many k and Z has the Stein property. The contrapositive also has an interesting statement (cf. [4]): if there is an $a \in \mathscr{C}$ such that $P(\sum_{k=1}^{\infty} a_k Z_k = a) > 0$, then $\sum_{k=1}^{\infty} a_k Z_k$ is a "Z polynomial," i.e. has only finitely many non-zero coefficients. In addition, if one assumes a bit more than the Stein property (see Corollary 2 to Theorem 3 in [3] and Theorem 3.2) then it is possible to prove a summability theorem which has an unusual feature: there is no assumption that the columns of the summability matrix must converge; instead the column convergence is a consequence of the theorem.

Perhaps the most important consequence of the Stein property is the one used by Stein [8], Burkholder [1], and Sawyer [7] for the case in which the sequence Z was the Rademacher sequence (defined on [0, 1] by $r_k(t) = \operatorname{sign}\sin 2^k\pi t$ if $\sin 2^k\pi t \neq 0$ and by right continuity elsewhere). Suppose that Z has the Stein property. If the series $\sum_{k=1}^{\infty} a_{nk} Z_k$ converges, $n \geqq 1$, and $\sup_{n\geqq 1} |\sum_{k=1}^{\infty} a_{nk} Z_k| < \infty$ on a set $A \in \mathscr{A}^+$ then $\limsup_{k\to\infty} \sup_{n\geqq 1} |a_{nk}| < \infty$. For let $B_\lambda = \{\sup_{n\geqq 1} |\sum_{k=1}^{\infty} a_{nk} Z_k| < \lambda\}$, $\lambda > 0$. Then $B_\lambda = B \in \mathscr{A}^+$ for some $\lambda > 0$. By the Stein property, there is a $b > 0$ and an $N = N(B, b) \in \mathscr{N}$ such that

$$b(\sum_{k=N}^{\infty} |a_{nk}|^2)^{\frac{1}{2}} \leqq \sup_{n\geqq 1} \operatorname{ess\,sup}_{\omega\in B} |\sum_{k=1}^{\infty} a_{nk} Z_k(\omega)| \leqq \lambda\,.$$

Hence

$$\sup_{n\geqq 1} |a_{nj}|^2 \leqq \sup_{n\geqq 1} \sum_{k=N}^{\infty} |a_{nk}|^2 \leqq \frac{\lambda^2}{b^2} < \infty\,, \quad j \geqq N\,,$$

and the desired conclusion follows easily.

These consequences justify a search for conditions on the sequence Z which are sufficient for Z to have the Stein property.

2. Definitions and preliminaries. In this section we collect some additional notation and definitions as well as a few results that will be needed in Section 3. All equalities and inequalities between random variables are to be interpreted as holding almost everywhere.

A sequence $Z = \{(Z_k, \mathscr{A}_k), k \geqq 1\}$ is said to be *adapted* if Z_k is measurable relative to the σ-algebra $\mathscr{A}_k \subset \mathscr{A}$, $k \geqq 1$, and $\mathscr{A}_1 \subset \mathscr{A}_2 \cdots$. If in addition each Z_k has finite expectation and $E(Z_k \mid \mathscr{A}_{k-1}) = 0$, $k \geqq 2$, then Z is a *martingale difference sequence* and the sequence of partial sums is a *martingale* relative to $\{\mathscr{A}_k, k \geqq 1\}$.

For Z an adapted sequence, we make the following definitions.

DEFINITION 2.1. If $b > 0$ and $k \in \mathscr{N}$ then

$$\pi_k(b \mid \mathscr{A}_{k-1}) = \operatorname{ess\,sup}_{a\in\mathscr{C}} P(|a + Z_k| < b \mid \mathscr{A}_{k-1})$$

and

$$\pi(b \mid \mathscr{A}_\infty) = \limsup_{k\to\infty} \pi_k(b \mid \mathscr{A}_{k-1}) .$$

Here and throughout, $\mathscr{A}_\infty$ denotes the σ-algebra generated by $\bigcup_{k=1}^\infty \mathscr{A}_k$ and $\mathscr{A}_0$ is the trivial σ-algebra $\{\emptyset, \Omega\}$. We have used the essential supremum in Definition 2.1 to be certain that the function $\pi_k(b \mid \mathscr{A}_{k-1})$ is $\mathscr{A}_{k-1}$ measurable, $k \geqq 1$.

DEFINITION 2.2. If $b > 0$ and $n \in \mathscr{N}$, then $p_n(b \mid \mathscr{A}_{n-1})$ is the essential supremum, over all sequences $(a, a_1, a_2, \cdots)$ of complex numbers and all $N \in \mathscr{N}$ with $N \geqq n$, of

$$P(|a + \textstyle\sum_{k=n}^N a_k Z_k| \leqq b(\sum_{k=n}^N |a_k|^2)^{\frac{1}{2}} \mid \mathscr{A}_{n-1}) .$$

Also,

$$p(b \mid \mathscr{A}_\infty) = \limsup_{n\to\infty} p_n(b \mid \mathscr{A}_{n-1}) .$$

If $\mathscr{A}_{k-1}$ is independent of $Z_k, Z_{k+1}, \cdots, k \geqq 1$, then the functions of Definitions 2.1 and 2.2 specialize to the concentration functions in [3].

(The remainder of Section 2 consists of preliminary results. The reader may wish to skip to Section 3 now, and refer back to these results as they are used.)

Next, for ease of reference, we list two lemmas which will be of fundamental importance in Section 3. A convenient reference for the first one is [9], Lemma 8.26, page 216.

LEMMA 2.3 (Paley and Zygmund). *If $X \geqq 0$ is a random variable satisfying*

$$E(X) \geqq \alpha > 0 \qquad \textit{and} \qquad E(X^2) \leqq \beta < \infty$$

then, for any δ satisfying $0 < \delta < 1$,

$$P(X \geqq \delta\alpha) \geqq \frac{(1-\delta)^2\alpha^2}{\beta} .$$

LEMMA 2.4. *To each $\delta > 0$ corresponds an $\alpha > 0$ with the following property: If $f = \{f_1, f_2, \cdots\}$ is a martingale and $E(|d_k|) \geqq \delta E^{\frac{1}{2}}(|d_k|^2)$, $k \geqq 1$, where $\{d_1 = f_1, d_2 = f_2 - f_1, d_3 = f_3 - f_2, \cdots\}$ is the difference sequence of f, then $E(|f_n|) \geqq \alpha E^{\frac{1}{2}}(|f_n|^2)$, $n \geqq 1$.*

This is Burkholder's Lemma 4 in [3], and we refer the reader there for the proof.

The following lemma and its corollary are of a technical nature, and will be needed in Section 3.

LEMMA 2.5. *Let X and Y be complex valued random variables and let $b > 0$. Then, for every λ satisfying $0 < \lambda < b$, there is a disjoint collection $\{D_j, j \geqq 1\}$ of measurable rectangles in $\mathscr{C}$ with union $\mathscr{C}$ satisfying*

$$I_{\{|X+Y|<\lambda\}} \leqq \textstyle\sum_{j=1}^\infty I_{\{X \in D_j\}} I_{\{|d_j+Y|<b\}} ,$$

where d_j is the center of D_j, $j \geqq 1$.

PROOF. Let

$$D_{m,n} = \left\{x + iy \in \mathscr{C} : (2m-1)\frac{b-\lambda}{2} \leqq x < (2m+1)\frac{b-\lambda}{2}, \right.$$
$$\left. (2n-1)\frac{b-\lambda}{2} \leqq y < (2n+1)\frac{b-\lambda}{2}\right\}$$

and $d_{m,n} = (m + in)(b - \lambda)$ for each pair of integers m and n. Then the $D_{m,n}$ are disjoint and $\bigcup_{(m,n)} D_{m,n} = \mathscr{C}$. If $X \in D_{m,n}$ then $|X - d_{m,n}| < b - \lambda$, so $X \in D_{m,n}$ and $|X + Y| < \lambda$ imply

$$\begin{aligned}\lambda > |X + Y| &= |X - d_{m,n} + d_{m,n} + Y| \geqq |d_{m,n} + Y| - |X - d_{m,n}| \\ &\geqq |d_{m,n} + Y| - (b - \lambda)\end{aligned}$$

from which $|d_{m,n} + Y| < \lambda + (b - \lambda) = b$ follows. Therefore, for all (m, n),

$$I_{\{|X+Y|<\lambda\}} I_{\{X \in D_{m,n}\}} \leqq I_{\{X \in D_{m,n}\}} I_{\{|d_{m,n}+Y|<b\}} ;$$

hence

$$\begin{aligned}I_{\{|X+Y|<\lambda\}} &= \textstyle\sum_{(m,n)} I_{\{|X+Y|<\lambda\}} I_{\{X \in D_{m,n}\}} \\ &\leqq \textstyle\sum_{(m,n)} I_{\{X \in D_{m,n}\}} I_{\{|d_{m,n}+Y|<b\}} .\end{aligned}$$

The desired conclusion follows immediately.

COROLLARY 2.6. *Let X, Y, and b be as in Lemma 2.5 and let $\mathscr{B} \subset \mathscr{A}$ be a σ-algebra such that X is $\mathscr{B}$-measurable. Then*

$$P(|X + Y| < b \,|\, \mathscr{B}) \leqq \text{ess sup}_{a \in \mathscr{C}} P(|a + Y| < b \,|\, \mathscr{B}) .$$

PROOF. By Lemma 2.5, if $0 < \lambda < b$ then

$$I_{\{|X+Y|<\lambda\}} \leqq \textstyle\sum_{j=1}^{\infty} I_{\{X \in D_j\}} I_{\{|d_j+Y|<b\}} .$$

Taking the conditional expectation given $\mathscr{B}$ of both sides yields

$$\begin{aligned}P(|X + Y| < \lambda \,|\, \mathscr{B}) &\leqq E(\textstyle\sum_{j=1}^{\infty} I_{\{X \in D_j\}} I_{\{|d_j+Y|<b\}} \,|\, \mathscr{B}) \\ &= \textstyle\sum_{j=1}^{\infty} I_{\{X \in D_j\}} P(|d_j + Y| < b \,|\, \mathscr{B}) \\ &\leqq \textstyle\sum_{j=1}^{\infty} I_{\{X \in D_j\}} \text{ ess sup}_{a \in \mathscr{C}} P(|a + Y| < b \,|\, \mathscr{B}) \\ &= \text{ess sup}_{a \in \mathscr{C}} P(|a + Y| < b \,|\, \mathscr{B})\end{aligned}$$

since the D_j are disjoint and have union $\mathscr{C}$. Now, letting $\lambda \uparrow b$ and using left continuity, we obtain the desired result.

The next lemma characterizes a condition on concentration functions that we use in Section 3 in terms of a condition used by Gundy [6].

LEMMA 2.7. *Let $Z = \{(Z_k, \mathscr{A}_k), k \geqq 1\}$ be a martingale difference sequence satisfying $E(Z_1) = 0$ and $E(|Z_k|^2 \,|\, \mathscr{A}_{k-1}) = 1$, $k \geqq 1$. Then the following assertions are equivalent*:

(i) $||\sup_{k \geqq 1} \pi_k(\alpha \,|\, \mathscr{A}_{k-1})||_\infty < 1$ *for some* $\alpha > 0$.
(ii) $E(|Z_k| \,|\, \mathscr{A}_{k-1}) \geqq \beta$, $k \geqq 1$, *for some* $\beta > 0$.

PROOF. Assume that (i) holds, choose $\varepsilon > 0$ such that $||\sup_{k\geq 1} \pi_k(\alpha \mid \mathscr{A}_{k-1})||_\infty \leqq 1 - \varepsilon$, and write $\beta = \alpha\varepsilon$. Then

$$\begin{aligned} E(|Z_k| \mid \mathscr{A}_{k-1}) &\geqq \alpha P(|Z_k| \geqq \alpha \mid \mathscr{A}_{k-1}) = \alpha[1 - P(|Z_k| < \alpha \mid \mathscr{A}_{k-1})] \\ &\geqq \alpha[1 - \pi_k(\alpha \mid \mathscr{A}_{k-1})] \geqq \alpha[1 - \sup_{k\geq 1} \pi_k(\alpha \mid \mathscr{A}_{k-1})] \\ &\geqq \alpha[1 - ||\sup_{k\geq 1} \pi_k(\alpha \mid \mathscr{A}_{k-1})||_\infty] \geqq \alpha\varepsilon = \beta\,, \quad k \geqq 1\,. \end{aligned}$$

For the converse, we assume that $||\sup_{k\geq 1} \pi_k(\alpha \mid \mathscr{A}_{k-1})||_\infty = 1$, $\alpha > 0$, and show (ii) cannot hold. Then

$$1 = ||\sup_{k\geq 1} \pi_k(\alpha \mid \mathscr{A}_{k-1})||_\infty = \sup_{k\geq 1} \sup_{a \in \mathscr{C}} ||P(|a + Z_k| < \alpha \mid \mathscr{A}_{k-1})||_\infty\,, \quad \alpha > 0\,.$$

Hence, given $\alpha > 0$ and $\delta > 0$, we can choose $j = j(\alpha, \delta) \in \mathscr{N}$ and $c = c(\alpha, \beta) \in \mathscr{C}$ such that $\{P(|c + Z_j| < \alpha \mid \mathscr{A}_{j-1}) > 1 - \delta^2\} = A_{j-1} \in \mathscr{A}_{j-1}^+$. Let $\beta > 0$ be given and choose $\alpha > 0$ and $\delta > 0$ such that $4(\alpha + \delta) < \beta$. For the $j \in \mathscr{N}$ and $c \in \mathscr{C}$ chosen as above, write $Z_j = X_j + iY_j$ and $c = a + ib$. Then

$$\begin{aligned} \tfrac{1}{2}E(|Z_j| \mid \mathscr{A}_{j-1}) &\leqq \tfrac{1}{2}E(|X_j| \mid \mathscr{A}_{j-1}) + \tfrac{1}{2}E(|Y_j| \mid \mathscr{A}_{j-1}) \\ &= \tfrac{1}{2}[E(X_j^+ \mid \mathscr{A}_{j-1}) + E(X_j^- \mid \mathscr{A}_{j-1}) + E(Y_j^+ \mid \mathscr{A}_{j-1}) \\ &\qquad + E(Y_j^- \mid \mathscr{A}_{j-1})]\,. \end{aligned} \tag{1}$$

Using the Schwarz inequality and the assumption that $E(|Z_j|^2 \mid \mathscr{A}_{j-1}) = 1$,

$$\begin{aligned} E(X_j^+ \mid \mathscr{A}_{j-1}) &= E(X_j^+ I_{\{|c+Z_j|<\alpha\}} \mid \mathscr{A}_{j-1}) + E(X_j^+ I_{\{|c+Z_j|\geqq\alpha\}} \mid \mathscr{A}_{j-1}) \\ &\leqq E(X_j^+ I_{\{|c+Z_j|<\alpha\}} \mid \mathscr{A}_{j-1}) + P^{\frac{1}{2}}(|c + Z_j| \geqq \alpha \mid \mathscr{A}_{j-1}) \\ &< E(X_j^+ I_{\{|c+Z_j|<\alpha\}} \mid \mathscr{A}_{j-1}) + \delta \qquad \text{on} \quad A_{j-1}\,. \end{aligned}$$

But $E(X_j^+ I_{\{|c+Z_j|<\alpha\}} \mid \mathscr{A}_{j-1}) \leqq E(X_j^+ I_{\{|a+X_j|<\alpha\}} \mid \mathscr{A}_{j-1})$ since $\{|c + Z_j| < \alpha\} \subset \{|a + X_j| < \alpha\} \cap \{|b + Y_j| < \alpha\} \subset \{|a + X_j| < \alpha\}$. Moreover, if $a \geqq 0$ then $E(X_j^+ I_{\{|a+X_j|<\alpha\}} \mid \mathscr{A}_{j-1}) \leqq E(X_j^+ I_{\{X_j^+<\alpha-a\}} \mid \mathscr{A}_{j-1}) \leqq \alpha - a \leqq \alpha$. Thus we have $a \geqq 0$ implies $E(X_j^+ \mid \mathscr{A}_{j-1}) \leqq \alpha + \delta$ on A_{j-1}, $a < 0$ implies $E(X_j^- \mid \mathscr{A}_{j-1}) \leqq \alpha + \delta$ on A_{j-1}, $b \geqq 0$ implies $E(Y_j^+ \mid \mathscr{A}_{j-1}) \leqq \alpha + \delta$ on A_{j-1}, and $b < 0$ implies $E(Y_j^- \mid \mathscr{A}_{j-1}) \leqq \alpha + \delta$ on A_{j-1}. Therefore, by (1) and the choice of α and δ, $E(|Z_j| \mid \mathscr{A}_{j-1}) \leqq 4(\alpha + \delta) < \beta$ on A_{j-1}.

3. Principal results.

THEOREM 3.1. *Let $Z = \{(Z_k, \mathscr{A}_k), k \geqq 1\}$ be a martingale difference sequence satisfying $E(|Z_k|^2 \mid \mathscr{A}_{k-1}) = 1$, $k \geqq 1$. If there is an $\alpha > 0$ such that $E(|Z_k| \mid \mathscr{A}_{k-1}) \geqq \alpha$, $k \geqq 1$, then Z has the Stein property.*

PROOF. We will prove:

(i) Under the hypotheses of the theorem, there is a number $b > 0$ such that $||p(b \mid \mathscr{A}_\infty)||_\infty \leqq ||\sup_{n\geq 1} p_n(b \mid \mathscr{A}_{n-1})||_\infty < 1$.

(ii) Let Z be an adapted sequence. For any $b > 0$, if $p(b \mid \mathscr{A}_\infty) < 1$ then Z has the Stein property with constant b.

Proof of (i): Let $a, a_1, a_2, \cdots$ be complex numbers, let $n \geqq 0$, $A_n \in \mathscr{A}_n^+$, and $N \geqq n+1$. Consider the process $\{(I_{A_n} Z_{n+k}, \mathscr{A}_{n+k}), k \geqq 1\}$ on the probability space $(\Omega, \mathscr{A}, P_{A_n})$, where P_{A_n} denotes the measure $P/P(A_n)$. We also use the notation E_{A_n} for expectations relative to P_{A_n}. This process is easily seen to be a martingale difference sequence on $(\Omega, \mathscr{A}, P_{A_n})$ which satisfies

$$(1) \qquad E_{A_n}(|I_{A_n} Z_{n+k}|) \geqq \alpha E_{A_n}^{\frac{1}{2}}(|I_{A_n} Z_{n+k}|^2)\,, \quad k \geqq 1\,,$$

since $E_{A_n}(|I_{A_n} Z_{n+k}|^2) = 1$. Write $f_j = a + \sum_{k=1}^{j} a_{n+k} Z_{n+k}$, $1 \leqq j \leqq N$. Then, by (1) and Lemma 2.4, there exists a constant $\gamma > 0$ such that

$$E_{A_n}(|I_{A_n} f_N|) \geqq \gamma E_{A_n}^{\frac{1}{2}}(|I_{A_n} f_N|^2)\,.$$

Note that the constant γ does not depend on $(a, a_1, a_2, \cdots)$, n, N, or A_n (see the proof of Theorem 9 in [2]). By the Paley-Zygmund lemma and the fact that $E_{A_n}(|I_{A_n} f_N|^2) = |a|^2 + \sum_{k=1}^{N} |a_{n+k}|^2$ we have

$$P\left(A_n \cap \left\{|a + \textstyle\sum_{k=n+1}^{N} a_k Z_k| \geqq \frac{\gamma}{2}(\sum_{k=n+1}^{N} |a_k|^2)^{\frac{1}{2}}\right\}\right) \geqq \frac{\gamma^2}{4} P(A_n)\,.$$

Since $A_n \in \mathscr{A}_n^+$ was arbitrary, this implies that

$$P\left(|a + \textstyle\sum_{k=n+1}^{N} a_k Z_k| < \frac{\gamma}{2}(\sum_{k=n+1}^{N} |a_k|^2)^{\frac{1}{2}} \,\middle|\, \mathscr{A}_n\right) \leqq 1 - \frac{\gamma^2}{4}\,,$$

from which

$$(2) \qquad \sup_{n \geqq 0} p_{n+1}\left(\frac{\gamma}{2} \,\middle|\, \mathscr{A}_n\right) \leqq 1 - \frac{\gamma^2}{4}$$

follows, in view of the above remarks about γ. Writing $b = \gamma/2$, we then have

$$\|\sup_{n \geqq 1} p_n(b \,|\, \mathscr{A}_{n-1})\|_\infty \leqq 1 - b^2 < 1$$

at once from (2), and this completes the proof of (i).

Proof of (ii): Let $B = B_n = \{\omega : |a + \sum_{k=1}^{\infty} a_k Z_k(\omega)| < b(\sum_{k=n}^{\infty} |a_k|^2)^{\frac{1}{2}}\}$. Since $P(A \,|\, \mathscr{A}_n)$ converges to $P(A \,|\, \mathscr{A}_\infty)$, both almost everywhere and in L^1 (see [5], Chapter 7, Theorems 4.3 and 4.1), we have that if $\delta > 0$ is given then there is an $m = m(\delta) \in \mathscr{N}$ such that, for $n \geqq m$,

$$(3) \qquad \begin{aligned} P(A \cap B) &= E[I_B P(A \,|\, \mathscr{A}_\infty)] < E[I_B P(A \,|\, \mathscr{A}_{n-1})] + \delta \\ &= E[P(B \,|\, \mathscr{A}_{n-1}) P(A \,|\, \mathscr{A}_{n-1})] + \delta\,. \end{aligned}$$

Now $P(B \,|\, \mathscr{A}_{n-1}) = 0$ if $\sum_{k=n}^{\infty} |a_k|^2 = 0$, $n \geqq 1$, so we assume that $\sum_{k=n}^{\infty} |a_k|^2 > 0$, $n \geqq 1$. Then, by Corollary 2.6 we have

$$(4) \qquad P(B \,|\, \mathscr{A}_{n-1}) \leqq \operatorname{ess\,sup}_{a \in \mathscr{C}} P(|a + \textstyle\sum_{k=n}^{\infty} a_k Z_k| < b(\sum_{k=n}^{\infty} |a_k|^2)^{\frac{1}{2}} \,|\, \mathscr{A}_{n-1})\,, \quad n \geqq 1\,.$$

Since $(a, a_1, a_2, \cdots) \in \Gamma$

$$\begin{aligned} &\operatorname{ess\,sup}_{a \in \mathscr{C}} P(|a + \textstyle\sum_{k=n}^{\infty} a_k Z_k| < b(\sum_{k=n}^{\infty} |a_k|^2)^{\frac{1}{2}} \,|\, \mathscr{A}_{n-1}) \\ &\quad \leqq \operatorname{ess\,sup}_a \liminf_{N \to \infty} P(|a + \textstyle\sum_{k=n}^{N} a_k Z_k| < b(\sum_{k=n}^{N} |a_k|^2)^{\frac{1}{2}} \,|\, \mathscr{A}_{n-1}) \\ &\quad \leqq p_n(b \,|\, \mathscr{A}_{n-1})\,, \quad n \geqq 1\,. \end{aligned}$$

This, together with (4), gives $P(B \mid \mathscr{A}_{n-1}) \leqq p_n(b \mid \mathscr{A}_{n-1})$, $n \geqq 1$, which, together with (3), implies that there is an $m \in \mathscr{N}$ such that for $n \geqq m$ we have (using the dominated convergence theorem)

$$(5) \qquad \begin{aligned} P(A \cap B) &< E[P(A \mid \mathscr{A}_{n-1}) p_n(b \mid \mathscr{A}_{n-1})] + \delta \\ &\leqq E[P(A \mid \mathscr{A}_\infty) p(b \mid \mathscr{A}_\infty)] + 2\delta \,, \end{aligned}$$

where $\delta > 0$ is arbitrary and m depends on δ.

Now let $A \in \mathscr{A}^+$. Since $P[p(b \mid \mathscr{A}_\infty) < 1] = 1$, we have

$$E[P(A \mid \mathscr{A}_\infty) p(b \mid \mathscr{A}_\infty)] < E[P(A \mid \mathscr{A}_\infty)] = P(A) \,.$$

Hence we can choose $\delta > 0$ such that $E[P(A \mid \mathscr{A}_\infty) p(b \mid \mathscr{A}_\infty)] + 2\delta < P(A)$ and then choose $n \in \mathscr{N}$ as at (5). Then, for any $(a, a_1, a_2, \cdots) \in \Gamma$, we have

$$\begin{aligned} P(\omega \in A : |a + \textstyle\sum_{k=1}^\infty a_k Z_k(\omega)| &< b(\textstyle\sum_{k=n}^\infty |a_k|^2)^{\frac{1}{2}}) \\ &< E[P(A \mid \mathscr{A}_\infty) p(b \mid \mathscr{A}_\infty)| + 2\delta < P(A) \,. \end{aligned}$$

Therefore $P(\omega \in A : |a + \sum_{k=1}^\infty a_k Z_k(\omega)| \geqq b(\sum_{k=n}^\infty |a_k|^2)^{\frac{1}{2}}) > 0$ and the Stein property with constant b follows at once.

The next theorem is patterned after Corollary 2 to Theorem 3 in [3], but we have relaxed the independence condition.

THEOREM 3.2. *Let $Z = \{(Z_k, \mathscr{A}_k), k \geqq 1\}$ be a martingale difference sequence satisfying $E(|Z_k|^2 \mid \mathscr{A}_{k-1}) = 1$, $k \geqq 1$, $E(Z_1) = 0$, and $E(|Z_k| \mid \mathscr{A}_{k-1}) \geqq \alpha$, $k \geqq 1$ for some $\alpha > 0$. Let $a_{11}, a_{12}, \cdots, a_{21}, \cdots$ be complex numbers such that for each $n \in \mathscr{N}$ the series $\sum_{k=1}^\infty a_{nk} Z_k$ converges almost everywhere, write $g_n = \sum_{k=1}^\infty a_{nk} Z_k$, $n \geqq 1$, and assume that the sequence $\{g_n, n \geqq 1\}$ converges almost everywhere. Then there exist complex numbers $a_1, a_2, \cdots$ such that*

(i) $\lim_{n\to\infty} \sum_{k=1}^\infty |a_{nk} - a_k|^2 = 0$ *and*

(ii) $\sum_{k=1}^\infty |a_k|^2 \leqq \liminf_{n\to\infty} \sum_{k=1}^\infty |a_{nk}|^2 \leqq \sup_{n \geqq 1} \sum_{k=1}^\infty |a_{nk}|^2 < \infty$.

PROOF. To prove (i), it is sufficient to prove $\lim_{m,n\to\infty} \sum_{k=1}^\infty |a_{nk} - a_{mk}|^2 = 0$. For assume that this has been proved. Then given $\varepsilon > 0$, there is an $M \in \mathscr{N}$ such that $n > m > M$ implies

$$|a_{nk} - a_{mk}|^2 \leqq \textstyle\sum_{j=1}^\infty |a_{nj} - a_{mj}|^2 < \varepsilon \,, \quad k \geqq 1 \,.$$

Thus $\{a_{nk}, n \geqq 1\}$ is a Cauchy sequence, $k \geqq 1$, so that $\lim_{n\to\infty} a_{nk} = a_k \in \mathscr{C}$ exists, $k \geqq 1$. Hence, by Fatou's lemma,

$$\textstyle\sum_{k=1}^\infty |a_k - a_{mk}|^2 \leqq \liminf_{n\to\infty} \sum_{k=1}^\infty |a_{nk} - a_{mk}|^2 < \varepsilon$$

if $m > M$ and (i) follows immediately.

Now we show that $\lim_{m,n\to\infty} \sum_{k=1}^\infty |a_{nk} - a_{mk}|^2 = 0$. Let $\varepsilon > 0$ be given. By Theorem 3.1, Z has the Stein property with constant b for some $b > 0$. Choose $\delta > 0$ satisfying $\delta^2/b^2 < \varepsilon$. Since $\{g_n, n \geqq 1\}$ converges almost everywhere, there is an $M \in \mathscr{N}$ such that $A = \{\sup_{n>m>M} |g_n - g_m| < \delta\} \in \mathscr{A}^+$. Therefore, by the Stein property with constant b, there is a $K = K(b, A) \in \mathscr{N}$ such that if

$n > m > M$ then

$$b(\sum_{k=K}^{\infty} |a_{nk} - a_{mk}|^2)^{\frac{1}{2}} \leqq \text{ess sup}_{\omega \in A} |g_n(\omega) - g_m(\omega)| \leqq \delta .$$

Thus

$$\sum_{k=K}^{\infty} |a_{nk} - a_{mk}|^2 \leqq \frac{\delta^2}{b^2} < \varepsilon \tag{6}$$

if $n > m > M$ and therefore

$$\lim_{m,n\to\infty} \sum_{k=K}^{\infty} |a_{nk} - a_{mk}|^2 = 0 . \tag{7}$$

Let $B = \{j \in \mathscr{N}: \lim_{m,n\to\infty} \sum_{k=j}^{\infty} |a_{nk} - a_{mk}|^2 = 0\}$. B is nonempty by (7). Hence B has a least element, say J. We claim that $J = 1$. For suppose that $J > 1$. Then, using the orthonormality of the sequence Z, we get

$$E(|\sum_{k=J}^{\infty} (a_{nk} - a_{mk})Z_k|) \leqq E^{\frac{1}{2}}(|\sum_{k=J}^{\infty} (a_{nk} - a_{mk})Z_k|^2) = (\sum_{k=J}^{\infty} |a_{nk} - a_{mk}|^2)^{\frac{1}{2}}$$

and this last sequence converges to zero as $m, n \to \infty$. Thus

$$\sum_{k=J}^{\infty} (a_{nk} - a_{mk})Z_k \to 0 \qquad \text{in } L_1, \quad \text{as} \quad m, n \to \infty .$$

By hypothesis we have

$$\sum_{k=1}^{\infty} (a_{nk} - a_{mk})Z_k \to 0 \qquad \text{almost everywhere as} \quad m, n \to \infty .$$

Thus both sequences converge to zero in probability as $m, n \to \infty$. Subtraction of the limits yields

$$\sum_{k=1}^{J-1} (a_{nk} - a_{mk})Z_k \to 0 \qquad \text{in probability as} \quad m, n \to \infty . \tag{8}$$

Now choose $\gamma > 0$ such that $\gamma/\beta < (\varepsilon/2)^{\frac{1}{2}}$, where β is such that $||\sup_{k\geqq 1} \pi_k(\beta \,|\, \mathscr{A}_{k-1})||_\infty < 1$. This is possible by Lemma 2.7. By Corollary 2.6 we have

$$\begin{aligned} &P(|\sum_{k=1}^{J-1} (a_{nk} - a_{mk})Z_k| < \gamma \,|\, \mathscr{A}_{J-2}) \\ &\qquad \leqq \text{ess sup}_{a \in \mathscr{C}} P(|a + (a_{n,J-1} - a_{m,J-1})Z_{J-1}| < \gamma \,|\, \mathscr{A}_{J-2}) \\ &\qquad = \pi_{J-1}\left(\frac{\gamma}{|a_{n,J-1} - a_{m,J-1}|} \,\middle|\, \mathscr{A}_{J-2}\right) . \end{aligned} \tag{9}$$

By (8) there is an $N \in \mathscr{N}$ such that $n > m > N$ implies that $P(|\sum_{k=1}^{J-1} (a_{nk} - a_{mk})Z_k| < \gamma \,|\, \mathscr{A}_{J-2})$ is arbitrarily close to one on some set C of positive probability. Since $||\sup_{k\geqq 1} \pi_k(\beta \,|\, \mathscr{A}_{k-1})||_\infty < 1$, we can choose $N \in \mathscr{N}$ such that $n > m > N$ implies

$$P(|\sum_{k=1}^{J-1} (a_{nk} - a_{mk})Z_k| < \gamma \,|\, \mathscr{A}_{J-2}) > ||\sup_{k\geqq 1} \pi_k(\beta \,|\, \mathscr{A}_{k-1})||_\infty \geqq \pi_{J-1}(\beta \,|\, \mathscr{A}_{J-2})$$

on the set C. By this and (9) we then have

$$\pi_{J-1}(\beta \,|\, \mathscr{A}_{J-2}) < \pi_{J-1}\left(\frac{\gamma}{|a_{n,J-1} - a_{m,J-1}|} \,\middle|\, \mathscr{A}_{J-2}\right)$$

on C. Since $\pi_k(\lambda \,|\, \mathscr{A}_{k-1})$ is non-decreasing in λ, $k \geqq 1$, this implies that if $n > m > N$ then

$$|a_{n,J-1} - a_{m,J-1}| \leqq \frac{\gamma}{\beta} < \left(\frac{\varepsilon}{2}\right)^{\frac{1}{2}} . \tag{10}$$

But since $J \in B$, there exists an $M \in \mathcal{N}$ with $M > N$ such that $n > m > M$ implies

$$\sum_{k=J}^{\infty} |a_{nk} - a_{mk}|^2 < \frac{\varepsilon}{2} \,. \tag{11}$$

Therefore, combining (10) and (11) yields $\sum_{k=J-1}^{\infty} |a_{nk} - a_{mk}|^2 < \varepsilon$ if $n > m > M$. Thus $\lim_{m,n\to\infty} \sum_{k=J-1}^{\infty} |a_{nk} - a_{mk}|^2 = 0$, a contradiction of the choice of J. Hence $J = 1$ and we have

$$\lim_{m,n\to\infty} \sum_{k=1}^{\infty} |a_{nk} - a_{mk}|^2 = 0 \,, \tag{12}$$

completing the proof of (i).

Now we prove (ii): By (12) there is an $N \in \mathcal{N}$ such that if $n > N$ then

$$\varepsilon > \left(\sum_{k=1}^{\infty} |a_{nk} - a_{Nk}|^2\right)^{\frac{1}{2}} \geqq \left(\sum_{k=1}^{\infty} |a_{nk}|^2\right)^{\frac{1}{2}} - \left(\sum_{k=1}^{\infty} |a_{Nk}|^2\right)^{\frac{1}{2}}$$

so that

$$\left(\sum_{k=1}^{\infty} |a_{nk}|^2\right)^{\frac{1}{2}} < \varepsilon + \left(\sum_{k=1}^{\infty} |a_{Nk}|^2\right)^{\frac{1}{2}} .$$

By the converse coefficient property (a consequence of Theorem 3.1), $\sum_{k=1}^{\infty} |a_{Nk}|^2 < \infty$. Hence $\sup_{n>N} \sum_{k=1}^{\infty} |a_{nk}|^2 < \infty$.

Clearly $\sup_{1\leqq n\leqq N} \sum_{k=1}^{\infty} |a_{nk}|^2 < \infty$ since $\sum_{k=1}^{\infty} |a_{nk}| < \infty$, $1 \leqq n \leqq N$, by the converse coefficient property again. Thus

$$\sum_{k=1}^{\infty} |a_k|^2 \leqq \liminf_{n\to\infty} \sum_{k=1}^{\infty} |a_{nk}|^2 \leqq \sup_{n\geqq 1} \sum_{k=1}^{\infty} |a_{nk}|^2 < \infty \,,$$

and this completes the proof of the theorem.

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