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LIMIT THEOREMS FOR A $GI/G/\infty$ QUEUE[1]

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The $GI/G/\infty$ queue is studied. For the stable case (ν = expected service time $< \infty$), necessary and sufficient conditions are given for the process to to have a legitimate regeneration point. In the unstable case ($\nu = \infty$), several limit theorems are established. Let $X(t)$ equal the number of servers busy at time t. It is proven that when $\nu = \infty$,

i) $$\frac{X(t)}{\lambda(t)} \Rightarrow 1$$

and

ii) $$\frac{X(t) - \lambda(t)}{\sqrt{\lambda(t)}} \Rightarrow N(0, 1)$$

where $\lambda(t)$ is a deterministic function. ($\Rightarrow$ means convergence in distribution). A Poisson type limit result is also proved when the arrival of a customer is a rare event.

1. Introduction. In this paper we study a queueing model in which customers arrive at the epochs of a renewal process, and are served immediately upon arrival by one of an infinite number of servers. Let $X(t)$ equal the number of servers busy at time t, $t \geqq 0$. This is the well-known $GI/G/\infty$ queue.

The above model appears in many contexts other than queueing theory. One example is the number of colonies still in existence at time t in a branching process with immigration [11]. Another is the number of particles still alive in an immigration-death process. More generally, one can consider a renewal point process, where each point undergoes an independent translation forward in time. The number of points, at time t, still undergoing translation, has the same distribution as $X(t)$. In this paper we choose to adopt the queueing terminology, since many of our results have direct analogs for $GI/G/s$ queues $(1 \leqq s < \infty)$.

The basic data for this process in a sequence of independent identically distributed random vectors, $\{(u_n, v_n)\}_{n\geqq 1}$, defined on some underlying probability space $(\Omega, \mathscr{B}, P)$. For $n \geqq 2$, the variable u_n represents the interarrival time between the $(n-1)$th and nth customer, and v_n represents the service time of the nth customer. The variable u_1 is the time until the first customer arrives (after $t = 0$), and v_1 is its service time. Unless stated otherwise, the $\{u_i\}_{i\geqq 1}$ and $\{v_1\}_{i\geqq 1}$ are assumed to be independent.

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It is convenient at this point to introduce some notation. We set

$$F(x) = P\{u_1 \leqq x\}, \qquad H(x) = P\{v_1 \leqq x\} \qquad x \geqq 0$$
$$\mu = E\{u_1\}, \qquad \nu = E\{v_1\}$$
$$\Phi(s, t) = E\{s^{X(t)}\}, \qquad t \geqq 0, \ |s| \leqq 1 .$$

We assume throughout the paper that F is nonlattice, $F(0+) = 0$, and $\mu < \infty$.

When $F(x) = 1 - e^{-\mu^{-1}x}$ ($\mu > 0$), we have the well studied $M/G/\infty$ queue [1], [2], [4], [10]. It is instructive to briefly consider this process since it will serve to motivate many of our results for the more general model. Bartlett [1] has shown in this case that the probability generating function (p.g.f.) of $X(t)$, $\Phi(s, t)$, satisfies the relation,

$$\Phi(s, t) = \exp[-\lambda(1 - s) \textstyle\int_0^t (1 - H(x))\, dx, \qquad t \geqq 0 . \tag{1.1}$$

Thus $X(t)$ is Poisson with parameter $\lambda \int_0^t (1 - H(x))\, dx$. It follows from (1.1) that as $t \to \infty$,

(1.2) if $\nu < \infty$, then $X(t)$ has a legitimate limiting distribution and
if $\nu = \infty$, then $X(t) \Rightarrow \infty$ ($\Rightarrow$ means convergence in distribution).

It has recently been shown [11] that (1.2) also holds for the $GI/G/\infty$ queue. We state this as our first result.

THEOREM 0. *Suppose* $\mu < \infty$ *and* F *is nonlattice. Then,* $\lim_{t\to\infty} \Phi(s, t) = \Phi(s)$, $|s| \leqq 1$, *exists. If*

(i) $\nu = \infty$, *then* $\Phi(s) \equiv 0$, *and if*

(ii) $\nu < \infty$, *then* $\Phi(s)$ *is a legitimate* p.g.f. *with*

$$\Phi(s) = 1 - \mu^{-1} \textstyle\int_0^\infty \Phi(s, t)[1 - \Theta(s, t)]\, dt , \tag{1.3}$$

where

$$\Theta(s, t) = H(t) + (1 - H(t))s .$$

The proof of Theorem 0 is given in [11]. We call the $\{X(t)\}$ process "stable" if $\nu < \infty$ and "unstable" if $\nu = \infty$.

In Section 2 we consider the stable case. A detailed study of this subject for the $GI/G/s$ queue has recently been given by Iglehart ($s = 1$) [7] and Whitt ($s > 1$) [14]. The key to their analysis was the fact that the queue returns to the idle state infinitely often w.p. 1, and hence the time of arrival immediately preceding a busy period is a regeneration point. Our next result shows that the same is true for the $GI/G/\infty$ queue.

THEOREM 1. *Assume* $\nu < \infty$. *Then,*

$$P\{X(t) = 0 \ \textit{for arbitrarily large} \ t\} = 1 \tag{1.4}$$

iff

$$P\{u_1 > v_1\} > 0 .$$

A proof of Theorem 1 and some of its consequences are indicated in Section 2.

The unstable case is studied in Section 3. Karlin and McGregor [9] have shown for the $M/G/\infty$ queue, that $X(t)$, properly normalized, converges in distribution to a normal. The same result holds in the more general case.

THEOREM 2. *Assume* $\nu = \infty$ *and* $E\{u_1^2\} < \infty$. *Let* $\lambda(t) = \mu^{-1} \int_0^t (1 - H(x))\, dx$. *Then,*

i) $$\frac{X(t)}{\lambda(t)} \Rightarrow 1\,,$$

and

ii) $$\frac{X(t) - \lambda(t)}{(\lambda(t))^{\frac{1}{2}}} \Rightarrow N(0, 1)\,.$$

Our final result treats the case when the arrival rate of customers (the reciprocal of the expected interarrival time) is very low. For $n \geqq 1$ define $F^n(x) = F(x/n)$, and let $\{X^n(t)\}_{t \geqq 0}$ be the corresponding $GI/G/\infty$ queue. The arrival rate for the nth process is $(\mu n)^{-1}$. The limit theorem, which we now state, can be viewed as an approximation to a $GI/G/\infty$ queue with a very low arrival rate.

THEOREM 3. *Assume* $\nu = \infty$. *For a fixed* $\lambda_0 > 0$ *define* $\{t_n\}$ *so that*

$$\lim_{n\to\infty} \left[\frac{\mu^{-1}}{n} \int_0^{t_n} (1 - H(x))\, dx\right] = \lambda_0\,.$$

Then, $X^n(t_n) \Rightarrow P(\lambda_0)$, *a Poisson random variable with parameter* λ_0.

This result is proved in Section 4.

2. The stable case. In this section we prove Theorem 1 and indicate some of its implications.

One direction of the theorem is clear. If $P\{u_1 > v_1\} = 0$, then w.p. 1, $X(t) \neq 0$ for all $t > u_1$. Hence

$$P\{X(t) = 0 \text{ for arbitrarily large } t\} = 0\,.$$

For the other direction, we first extend the $\{(u_n, v_n)\}_{n \geqq 1}$ sequence, using standard arguments, to a doubly infinite sequence of independent vectors. Define

$$\tau_k = \sum_{i=1}^{k} u_i \qquad k \geqq 1\,,$$
$$= -\sum_{i=-k}^{0} u_i \qquad k \leqq 0\,.$$

To prove (1.4) it suffices to show since $X(t)$ is integer valued that,

(2.1) $$P\{\liminf_{k\to\infty} X(\tau_k -) = 0\} = 1\,.$$

However,

$$X(\tau_k -) = \sum_{i=1}^{k-1} I_{\{\tau_i + v_i > \tau_k\}} \leqq \sum_{i=-\infty}^{k-1} I_{\{\tau_i + v_i > \tau_k\}} = \tilde{X}_k\,.$$

So to prove (2.1) it is enough to have

(2.2) $$P\{\liminf_{k\to\infty} \tilde{X}_k = 0\} = 1\,.$$

In view of our construction, $\{\tilde{X}_k\}$ is a stationary ergodic sequence. Stationarity is clear and ergodicity can be proved using a method similar to that in [4, pages 149–151]. Thus, (2.2) will follow providing $P\{\tilde{X}_1 = 0\} > 0$. But

$$P\{\tilde{X}_1 = 0\} = E\{\prod_{i=-\infty}^{0} H(\tau_1 - \tau_i)\} .$$

It suffices therefore, to prove that $\prod_{i=-\infty}^{0} H(\tau_1 - \tau_i)$ is positive with positive probability. Since $P\{u_1 > v_1\} > 0$, we can find a set A of positive measure such that on A, $H(\tau_1) > 0$.

It remains only to show that on A, the infinite product does not converge to zero. This follows at once if we prove that

$$\sum_{i=-\infty}^{0} (1 - H(\tau_1 - \tau_i)) < \infty \quad \text{w.p. } 1 . \tag{2.3}$$

Let $0 < \varepsilon < \mu$. By the ergodic theorem, there exists a random variable I such that $|i| \geqq I$ implies that

$$\tau_1 - \tau_i > (\mu - \varepsilon)(|i| + 2) \quad \text{w.p. } 1 .$$

Hence,

$$\begin{aligned} \sum_{|i|=I}^{\infty} (1 - H(\tau_1 - \tau_i)) &\leqq \sum_{|i|=I}^{\infty} (1 - H(\mu - \varepsilon)(|i| + 2)) \\ &\leqq (\mu - \varepsilon)^{-1} \int_{(\mu-\varepsilon)(I+1)}^{\infty} (1 - H(u))\, du . \end{aligned}$$

The last integral is finite since $E\{v_1\} < \infty$. (2.3) now follows.

REMARKS. (1) It is clear from the proof of Theorem 1 that the assumption that the $\{u_i\}$ be i.i.d. is much more than needed. It is enough to assume tnat the $\{u_i\}$ be stationary ergodic.

(2) In the i.i.d. case, an alternative proof to Theorem 1 can be given using the fact that the $\{X(t)\}$ process has a regeneration point.

(3) Define

$$T_1 = \inf\{t\colon t > u_1 \text{ and } X(t) = 0\} ,$$
$$T_2 = \inf\{t\colon t > T_1 \text{ and } X(t) = 1\} .$$

In queueing language T_1 is the length of a busy period, $T_2 - T_1$ is the length of an idle period and T_2 is the length of a busy cycle. Theorem 1 implies that the length of a busy cycle is finite w.p. 1. One can now proceed as in [7] to exploit this fact to obtain results for $\int_0^t X(s)\, ds$.

3. The unstable case. The proof of Theorem 2 will be developed in a series of lemmas. Throughout this section we assume $E\{u_1^2\} < \infty$.

Let F equal the σ-field generated by the $\{\tau_i\}$ process. The first step in the proof of Theorem 2 is to analyze the behavior of $E\{X(t) \mid F\}$. It is not difficult to see that

$$E\{X(t) \mid F\} = M(t) = \sum_{i=1}^{N(t)} p(t - \tau_i) = \int_0^t p(t - u) N(du)$$

where

$$p(t) = 1 - H(t)$$

and $N(t)$ is the renewal function, i.e. $N(t) = k$ iff $\tau_k \leqq t < \tau_{k+1}$.

LEMMA 3.1.

$$\Theta(s, t) = E\{e^{-sM(t)}\} = \int_0^t e^{-sp(t-u)}\Theta(s, t-u)F(du) . \tag{3.1}$$

PROOF. The above relation is found by conditioning on the time of arrival of the first customer.

Differentiating (3.1) successively, with respect to s, and setting $s = 1$, we obtain renewal equations for $E\{M(t)\}$ and $E\{M^2(t)\}$. Indeed,

$$E\{M(t)\} = \int_0^t p(t-u)F(du) + \int_0^t E\{M(t-u)\}F(du) \tag{3.2}$$

and

$$E\{M^2(t)\} = \int_0^t p^2(t-u)F(du) + 2\int_0^t p(t-u)\}E\{M(t-u)\}F(du) + \int_0^t E\{M^2(t-u)\}F(du) . \tag{3.3}$$

Put

$$V(t) = \sum_{k=1}^\infty F_k(t)$$

where F_k is the kth-fold convolution of F with itself. Standard renewal arguments yield

$$E\{M(t)\} = \int_0^t p(t-u)V(du) \tag{3.4}$$

and

$$E\{M^2(t)\} = \int_0^t p^2(t-u)V(du) + 2\int_0^t p(t-u)E\{M(t-u)\}V(du) . \tag{3.5}$$

Our next lemma deals with the behavior of $E\{M(t)\}$.

LEMMA 3.2. *Assume* $E\{u_1^2\} < \infty$. *Let* $L > 0$. *Then,*

(i) *there exists a constant D independent of L such that for all $t > L$,*

$$|\int_0^{t-L} p(t-u)V(du) - \mu^{1-}\int_0^{t-L} p(t-u)\,du| \leqq Dp(L) ;$$

(ii) $$\lim_{t\to\infty} |\int_{t-L}^t p(t-u)V(du) - \mu^{-1}\int_{t-L}^t p(t-u)\,du| = 0 .$$

PROOF. We first prove (i). Define:

$$\begin{aligned} p_t(u) &= p(t-u) \qquad \text{for} \quad 0 \leqq u \leqq t , \\ &= p(0) \qquad\qquad\quad u \geqq t . \end{aligned}$$

Note that q_t is increasing in u for t fixed. Integrating by parts, we obtain

$$\int_0^{t-L} p_t(u)V(du) + \int_0^{t-L} V(u)p_t(du) = V(t-L)p(L) . \tag{3.6}$$

Similarly,

$$\mu^{-1}\int_0^{t-L} p_t(u)\,du + \mu^{-1}\int_0^{t-L} up_t(du) = \mu^{-1}(t-L)p(L) . \tag{3.7}$$

Combining (3.6) and (3.7),

$$\begin{aligned} &|\int_0^{t-L} p_t(u)V(du) - \mu^{-1}\int_0^{t-L} p_t(u)\,du| \\ &\qquad \leqq \int_0^{t-L} |V(u) - \mu^{-1}u|p_t(du) + p(L)|V(t-L) - \mu^{-1}(t-L)| . \end{aligned}$$

It is well-known [6, page 357], that $E\{u_1^2\} < \infty$ implies

$$\lim_{t\to\infty} [V(t) - \mu^{-1}t] = \kappa < \infty . \tag{3.8}$$

Thus,

$$\sup_{t>0} |V(t) - \mu^{-1}t| = \frac{D}{2} < \infty . \tag{3.9}$$

Hence,

$$|\int_0^{t-L} p_t(u)V(du) - \mu^{-1} \int_0^{t-L} p_t(u)\, du| \leqq Dp(L) .$$

This proves (i). To prove (ii) we appeal to the key renewal theorem [6, page 349]. The details are omitted.

We recall that $\lambda(t) = \mu^{-1} \int_0^t p(u)\, du$. As an immediate consequence of Lemma 3.2 we have:

COROLLARY 3.1. $\lim_{t\to\infty} |E\{M(t)\} - \lambda(t)| = 0$.

The proof of the next result is omitted.

LEMMA 3.3. $E\{M^2(t)\} = E^2\{M(t)\} + o(\lambda(t))$.

REMARKS. (1) It follows immediately from Lemma 3.3 that $\sigma^2(M(t)) = o(\lambda(t))$.
(2) A result similar to Lemma 3.3 can be found in [12].

We are now ready to prove Theorem 2. The details will be carried out in the next two lemmas.

LEMMA 3.4. $X(t)\lambda^{-1}(t) \Rightarrow 1$.

PROOF. Let

$$\begin{aligned} B(s, t) &= E\{e^{isX(t)} \mid F\} \\ &= \prod_{j=1}^{N(t)} [p(t - \tau_j)e^{is} + 1 - p(t - \tau_j)] , \qquad t > 0,\ s \text{ real.} \end{aligned}$$

To prove the result it suffices to show

$$B(s\lambda^{-1}(t), t) \Rightarrow e^{is} . \tag{3.10}$$

Once (3.10) is proved, dominated convergence can be used to obtain the lemma. It is proved in [1] that for any complex z, $|z| \leqq \frac{1}{2}$

$$\log (1 + z) = z(1 + \varepsilon(z)) \tag{3.11}$$

where $|\varepsilon(z)| \leqq |z|$. For fixed s, and t sufficiently large, we can use (3.11) to obtain

$$\begin{aligned} \log B(s\lambda^{-1}(t), t) &= \sum_{j=1}^{N(t)} p(t - \tau_j)(e^{is\lambda^{-1}(t)} - 1)[1 + \varepsilon(p(t - \tau_j)(e^{is\lambda^{-1}(t)} - 1))] \\ &= \frac{is}{\lambda(t)} M(t) + R(t) \end{aligned}$$

where

$$|R(t)| \leqq \frac{s^2 M(t)}{\lambda^2(t)} + \sum_{j=1}^{N(t)} [p(t - \tau_j)(e^{is\lambda^{-1}(t)} - 1)]^2 .$$

The result now follows since $M(t)/\lambda(t) \Rightarrow 1$ and $|R(t)| \Rightarrow 0$.

LEMMA 3.5.

$$\frac{X(t) - \lambda(t)}{(\lambda(t))^{\frac{1}{2}}} \Rightarrow N(0, 1) .$$

PROOF. Arguing as in Lemma 3.4, it suffices to show

(3.12) $$-i(\lambda(t))^{\frac{1}{2}}s + \log B(s\lambda^{-\frac{1}{2}}(t), t) \Rightarrow -\frac{s^2}{2}.$$

If t is sufficiently large, we can use (3.11) to obtain

$$\log B(s\lambda^{-\frac{1}{2}}(t), t) = M(t)\left(\frac{is}{(\lambda(t))^{\frac{1}{2}}} - \frac{s^2}{2\lambda(t)}\right) + S(t)$$

where

$$|S(t)| \leqq M(t)\frac{s^3}{6\lambda(t)^{\frac{3}{2}}} + \sum_{i=1}^{N(t)} [p(t-\tau_j)(e^{is\lambda^{-\frac{1}{2}}(t)} - 1)]^2 .$$

Since

(3.12) $$\frac{M(t)-\lambda(t)}{(\lambda(t))^{\frac{1}{2}}} \Rightarrow 0 \quad \text{and} \quad \frac{M(t)}{\lambda(t)} \Rightarrow 1,$$

will follow providing $|S(t)| \Rightarrow 0$. To prove that $|S(t)| \Rightarrow 0$ we argue as follows. Let $\varepsilon > 0$ and choose L so large that $4p(L)s^2 < \varepsilon/3$. Using the simple inequality $|e^{i\theta} - 1| \leqq 2|\theta|$ for $|\theta|$ small, it is not hard to check that for t large (necessarily greater than L)

$$|S(t)| \leqq \frac{M(t)}{\lambda(t)}\frac{s^3}{6(\lambda(t))^{\frac{1}{2}}} + \frac{4s^2p(L)}{\lambda(t)}\sum_{i=1}^{N(t-L)} p(t-\tau_i) + \frac{4s^2}{\lambda(t)}(N(t)-N(t-L))$$
$$\leqq \frac{M(t)}{\lambda(t)}\left(\frac{s^3}{6(\lambda(t))^{\frac{1}{2}}} + \frac{\varepsilon}{3}\right) + \frac{4s^2}{\lambda(t)}(N(t)-N(t-L)).$$

Thus

$$\limsup_{t\to\infty} P\{|S(t)| > \varepsilon\} \leqq \limsup_{t\to\infty} P\left\{\frac{M(t)}{\lambda(t)}\left(\frac{s^3}{6(\lambda(t))^{\frac{1}{2}}} + \frac{\varepsilon}{3}\right) > \frac{\varepsilon}{2}\right\}$$
$$+ \limsup_{t\to\infty} P\left\{\frac{4s^2}{\lambda(t)}(N(t)-N(t-L)) > \frac{\varepsilon}{2}\right\}.$$

Since $M(t)/\lambda(t) \Rightarrow 1$, the first term on the right is zero. To handle the second term, we use Chebychev's Inequality and the renewal theorem.

4. Proof of Theorem 3. In this section we prove Theorem 3. For the readers' benefit we recall the setup. Define: $X^n(t)$ = number of servers busy at time t when the interarrival times have distribution function $F^n(x) = F(x/n)$ $(n \geqq 1)$. $\{\tau_i^n\}_{i\geqq 1}$ denote the times customers arrive in the nth process. $N^n(t)$ is the associated renewal function and $U^n(t)$ is the associated renewal measure, i.e. $U^n(t) = \sum_{k=0}^{\infty} F_k^n(t)$. Note that $U^n(t) = U(t/n)$ where $U(t)$ is the renewal measure associated with F. Let $\{t_n\}$ be a sequence of numbers chosen so that

$$\lim_{n\to\infty} \frac{\mu^{-1}}{n} \int_0^{t_n} [1 - H(u)]\,du = \lambda_0 > 0 .$$

The $\{t_n\}$ are well-defined since $\int_0^\infty [1 - H(u)]\,du = \infty$.

To prove Theorem 3 we need a Poisson convergence theorem. Arguing as in the classical case [5, pages 263–264], it is not difficult to prove that the next two conditions are sufficient to imply the result.

$$\sup_{1 \leq k \leq N^n(t_n)} p(t_n - \tau_k^n) \Rightarrow 0 \tag{4.1}$$

and

$$\sum_{k=1}^{N^n(t_n)} p(t_n - \tau_k^n) \Rightarrow \lambda_0 . \tag{4.2}$$

The remainder of this section is used to prove (4.1) and (4.2).

We first consider (4.1). By monotonicity,

$$\sup_{1 \leq k \leq N^n(t_n)} p(t_n - \tau_k^n) = p(t_n - \tau^n_{N^n(t_n)}) . \tag{4.3}$$

Let $\zeta > 0$. A simple renewal argument yields,

$$\begin{aligned} P\{t_n - \tau^n_{N^n(t_n)} < \zeta\} &= \int_{t_n-\zeta}^{t_n} U^n(du)(1 - F^n(t_n - u)) \\ &= \int_{(t_n-\zeta)/n}^{t_n/n} U(du)\left(1 - F\left(\frac{t_n}{n} - u\right)\right). \end{aligned}$$

Assuming that $t_n/n \to \infty$, we can apply the key renewal theorem to find,

$$\lim_{n\to\infty} (t_n - \tau^n_{N^n(t_n)}) \Rightarrow \infty . \tag{4.4}$$

Since $p(u) \to 0$ as $u \to \infty$, (4.3) and (4.4) imply (4.1).

It still remains to prove that

$$\lim_{n\to\infty} \frac{t_n}{n} = \infty . \tag{4.5}$$

Suppose not. Then there exists a subsequence $\{t_{n_k}\}$ such that ,

$$\liminf_{k\to\infty} \left(\frac{t_{n_k}}{n_k}\right) = B < \infty .$$

Since p decreases to zero, there exists a t_0 such that $t \geqq t_0 \Rightarrow p(t) < \lambda_0/2B$. Thus for $t_{n_k} > t_0$,

$$\begin{aligned} \frac{1}{n_k} \int_0^{t_{n_k}} p(u)\,du &= \frac{1}{n_k} \left[\int_0^{t_0} p(u)\,du + \int_{t_0}^{t_{n_k}} p(u)\,du\right] \\ &\leqq \frac{1}{n_k} \int_0^{t_0} p(u)\,du + \frac{\lambda_0}{2B} \frac{t_{n_k}}{n_k} \end{aligned}$$

and so

$$\lambda_0 = \liminf_{k\to\infty} \left(\frac{1}{n_k} \int_0^{t_{n_k}} p(u)\,du\right) \leqq \frac{\lambda_0}{2}$$

which is a contradiction. Therefore (4.5) holds.

Let $\Lambda_n = \sum_{k=1}^{N^n(t_n)} p(t_n - \tau_k^n)$, $n \geqq 1$. To prove (4.2) it is enough to prove:

$$\lim_{n\to\infty} E\{\Lambda_n\} = \lambda_0 \tag{4.6}$$

and

$$\lim_{n\to\infty} \sigma^2(\Lambda_n) = 0 . \tag{4.7}$$

The proofs of (4.6) and (4.7) are identical to those of Lemmas 3.3 and 3.4, providing we can prove that for any $L > 0$

$$\lim_{n\to\infty} \int_{t_n-L}^{t_n} p(t_n - u)\, dV^n(u) = 0\,, \tag{4.8}$$

where

$$V^n(u) = \sum_{k=1}^{\infty} F_k^{\,n}(u) = V\left(\frac{u}{n}\right).$$

In Section 3, we proved (4.8) by appealing to the key renewal theorem. Since the measure now depends on n, this argument does not work.

PROOF OF (4.8). Integrating by parts,

$$\begin{aligned}\int_{t_n-L}^{t_n} p(t_n - u)d\left(V^n(u) - \mu^{-1}\frac{u}{n}\right) &+ \int_{t_n-L}^{t_n}\left(V^n(u) - \mu^{-1}\frac{u}{n}\right)p_{t_n}(du)\\ &= P(0)\left(V^n(t_n) - \mu^{-1}\frac{t_n}{n}\right) - p(L)\left(V^n(t_n - L) - \mu^{-1}\frac{(t_n - L)}{n}\right)\\ &= \Theta_n\,.\end{aligned}$$

Using (3.8), it is not hard to show

$$\begin{aligned}\lim_{n\to\infty}\int_{t_n-L}^{t_n}\left(V^n(u) - \mu^{-1}\frac{u}{n}\right)p_{t_n}(du) &= \lim_{n\to\infty}\int_{t_n-L}^{t_n}\left(V\left(\frac{u}{n}\right) - \mu^{-1}\frac{u}{n}\right)p_{t_n}(du)\\ &= \kappa[p(0) - p(L)]\,.\end{aligned}$$

Similarly, $\lim_{n\to\infty}\Theta_n = \kappa[p(0) - p(L)]$. Hence,

$$\lim_{n\to\infty}\int_{t_n-L}^{t_n} p(t_n - u)V^n(du) = \lim_{n\to\infty}\frac{\mu^{-1}}{n}\int_{t_n-L}^{t_n} p(t_n - u)\, du = 0\,.$$

This proves (4.8).

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