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LIMIT THEOREMS FOR EXTREME VALUES OF CHAIN-DEPENDENT PROCESSES[1]

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The principal results of Resnick and Neuts (1970) and Resnick (1971) concerning limiting distributions for the maxima of a sequence of random variables defined on a Markov chain have been extended to denumerable Markov chains. These results apply *a fortiori* to Markov renewal processes. The method of proof is to show that limit distributions are independent of the initial distribution of the chain and then to apply known results for stationary processes.

0. Introduction. Let $(J_n)_{n \geqq 0}$ be a time-homogeneous Markov chain with the positive integers as state space, and transition matrix $P = (P_{ij})$ which is irreducible, aperiodic, and ergodic. Let $(X_n)_{n \geqq 1}$ be a real valued process defined on the same probability space and linked with the chain as follows:

$$\text{For} \quad n \geqq 1\,, \qquad \mathbf{P}[J_n = j, X_n \leqq x \mid X_1, X_2, \cdots, X_{n-1}, J_0, J_1, \cdots, J_{n-1} = i] = \mathbf{P}[J_n = j, X_n \leqq x \mid J_{n-1} = i] = P_{ij} H_i(x)\,,$$

where H_i is a proper right-continuous distribution function for each i.

The random variables (X_n) are then conditionally independent given the values of the chain. In fact

$$\mathbf{P}[X_i \leqq x_i, i = 1, 2, \cdots, n \mid J_0, J_1, \cdots, J_{n-1}] = \prod_{i=1}^{n} \mathbf{P}[X_i \leqq x_i \mid J_{i-1}]\,.$$

Hence we use the name "chain-dependent" process, emphasizing the fact that this is a natural extension of an i.i.d. process. These processes have been studied by Resnick and Neuts (1970), and Resnick (1971), who use the expression "random variables defined on a Markov chain." These authors investigated the problems which we consider below, but restricted their attention to the situation where the underlying chain had a finite state space. Our results, obtained by much different techniques, include the principal conclusions of Neuts and Resnick, and Resnick, for finite chains.

For $n \geqq 1$ denote by M_n the maximum of $X_1, \cdots, X_n$. We have investigated the possible limiting distributions for the sequence $a_n^{-1}(M_n - b_n)$, where $a_n > 0$ and b_n are suitable normalizing constants, as well as the question of domains of attraction.

Gnedenko's (1943) fundamental work on the i.i.d. case has been partially extended to other processes by Berman (1962, 1964), Loynes (1965), Welsch

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(1972) and O'Brien (1974b, 1974c). The main result of the present work is to show (in Section 2) that the limiting behaviour of M_n is not dependent on the initial distribution of the underlying Markov chain. Thus we may start with the stationary distribution, in which case (X_n) is stationary. The results of O'Brien and Loynes are applied to this process, which is shown in Section 1 to satisfy sufficient mixing conditions. In Section 3, we investigate the domains of attraction of the possible limit laws. Section 4 contains some results on the stability of (M_n). Section 5 gives examples to show that the results for the finite Markov chain case do not extend to the countable case without some extra hypothesis. Finding suitable necessary and sufficient conditions for convergence remains an open question in the denumerable case.

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Throughout this work, we denote by $\pi = (\pi_i)$ the unique stationary distribution for the underlying chain, and by $H(x) = \Sigma_i \pi_i H_i(x)$ the distribution of X_n in the stationary case. Let $x_0 = \sup\{x: H(x) < 1\}$. We write $\mathbf{P}_\beta$ for the probability measure when the initial distribution of the chain is β, $\mathbf{P} \equiv \mathbf{P}_\pi$, $\mathbf{P}_{i\cdot} = \mathbf{P}_{\delta_i}$, where δ_i is unit mass at i, and $P_{ij}^k = (P^k)_{ij}$. If β is a vector, we put $||\beta|| = \Sigma_i|\beta_i|$.

1. Mixing conditions. We say $\{X_n\}$ is *strongly* (or uniformly) *mixing* if there is a function g on the positive integers with $g(k) \to 0$ as $k \to \infty$ such that, if $A \in \mathscr{B}(X_1, \cdots, X_m)$ and $B \in \mathscr{B}(X_{m+k}, X_{m+k+1}, \cdots)$ for some k, $m \geqq 1$, then $|\mathbf{P}[A \cap B] - \mathbf{P}[A]\mathbf{P}[B]| \leqq g(k)$. We say $\{X_n\}$ is *φ-mixing* if there is a function φ with $\varphi(k) \to 0$ such that for A and B as above $|\mathbf{P}[A \cap B] - \mathbf{P}[A]\mathbf{P}[B]| \leqq \mathbf{P}[A]\varphi(k)$.

LEMMA 1.1. *Every stationary chain-dependent process $\{X_n\}$ is strongly mixing with $g(k) = \Sigma_i \pi_i ||P_{i\cdot}^k - \pi||$. If (P_{ij}) satisfies Doeblin's condition (for some k and m and some $\varepsilon > 0$, $\sum_{j=1}^m P_{ij}^k > \varepsilon$ for all i) then $\{X_n\}$ is φ-mixing with $\varphi(k) = \sup_i ||P_{i\cdot}^k - \pi||$.*

PROOF. For the first statement, see O'Brien (1974a). The second follows from a routine extension of the arguments on page 222 of Doob (1953) in the case of stationary chains. □

Evidently, Doeblin's condition holds if the (stationary) chain is finite. If $\{X_n\}$ is the Markov chain itself, then φ-mixing can be seen to imply Doeblin's condition. This is not true in general (for example, if $H_i = H$ for all i).

2. Independence of starting measure. The following theorem will permit us to assume in the sequel that the initial distribution for the chain is π. A similar result holds for "tail-event" types of sequences—see O'Brien (1974a)—but we were unable to obtain a common proof for the two cases.

THEOREM 2.1. *Let β be an arbitrary initial distribution for the chain. Then, for any sequence $\{c_n\}$, $\mathbf{P}_\beta[M_n \leqq c_n] - \mathbf{P}[M_n \leqq c_n] \to 0$.*

PROOF. Suppose the result fails. Then there is a subsequence n_k along which $\mathbf{P}_\beta[M_{n_k} \leqq c_{n_k}] - \mathbf{P}[M_{n_k} \leqq c_{n_k}] \to \Delta \neq 0$, $c_{n_k} \to a \in [-\infty, \infty]$, and $c_{n_k} < x_0$ for all k. We restrict our attention to such a subsequence and write n for n_k. First suppose that $c_n \to x_0$ and that H is continuous at x_0. A well-known result of Orey's (cf. Kemeny–Snell–Knapp (1966), page 153) states that $\|\beta P^n - \pi\| \to 0$. Given $\varepsilon > 0$, take N large enough so that $\|\beta P^k - \pi\| < \varepsilon/3$ for all $k \geqq N$. In addition, we may take $K \geqq N$ such that for $n \geqq K$ and $l \leqq N$ we have $\mathbf{P}_\beta[X_l > c_n] < \varepsilon/3N$ and $\mathbf{P}[X_1 > c_n] < \varepsilon/3N$. Then, with n, l as above,

$$\begin{aligned}\mathbf{P}_\beta[M_n \leqq c_n] &= \Sigma_i \mathbf{P}_\beta[M_n \leqq c_n, J_N = i] \\ &= \Sigma_i \mathbf{P}[X_{N+1} \leqq c_n, \cdots, X_n \leqq c_n \mid J_N = i] \\ &\qquad \times \mathbf{P}_\beta[X_1 \leqq c_n, \cdots, X_N \leqq c_n, J_N = i]\,.\end{aligned}$$

Now it follows that

$$\begin{aligned}&|\mathbf{P}_\beta[M_n \leqq c_n] - \mathbf{P}[M_n \leqq c_n]| \\ &\quad \leqq \Sigma_i \mathbf{P}[X_{N+1} \leqq c_n, \cdots, X_n \leqq c_n \mid J_N = i] \\ &\qquad \times |\mathbf{P}_\beta[X_1 \leqq c_n, \cdots, X_N \leqq c_n, J_N = i] \\ &\qquad - \mathbf{P}[X_1 \leqq c_n, \cdots, X_N \leqq c_n, J_N = i]| \\ &\quad \leqq \Sigma_i |\mathbf{P}_\beta[J_N = i, X_1 \leqq c_n, \cdots, X_N \leqq c_n] - \mathbf{P}_\beta[J_N = i]| \\ &\qquad + \Sigma_i |\mathbf{P}_\beta[J_N = i] - \mathbf{P}[J_N = i]| \\ &\qquad + \Sigma_i |\mathbf{P}[J_N = i] - \mathbf{P}[J_N = i, X_1 \leqq c_n, \cdots, X_N \leqq c_n]| \\ &\quad \leqq (1 - \mathbf{P}_\beta[X_1 \leqq c_n, \cdots, X_N \leqq c_n]) + \|(\beta - \pi)P^N\| \\ &\qquad + (1 - \mathbf{P}[X_1 \leqq c_n, \cdots, X_N \leqq c_n]) \\ &\quad \leqq \textstyle\sum_{l=1}^N \mathbf{P}_\beta[X_l > c_n] + \|(\beta - \pi)P^N\| + \sum_{l=1}^N \mathbf{P}[X_l > c_n] < \varepsilon\,.\end{aligned}$$

If $c_n \to a < x_0$ or H is discontinuous at $x_0 < \infty$ and $c_n \to x_0$ from below then it can easily be shown that $\mathbf{P}_\beta[M_n \leqq c_n] \to 0$ for any β. □

Since the possible limit laws for $a_n^{-1}(M_n - b_n)$, where $a_n > 0$, will not depend on the starting distribution of the chain, we may assume that (X_n) is stationary. The results of Loynes (1965) then apply, and the only nondegenerate limit laws are the same three which can arise as limits of $H^n(a_n x + b_n)$ (cf. Gnedenko, 1943).

If the numbers $c_n(\xi)$, for $\xi > 0$, are chosen so that $H(c_n(\xi)\,-) \leqq 1 - \xi/n \leqq H(c_n(\xi))$ then, again applying Loynes' results we have that $\mathbf{P}_\beta[M_n \leqq c_n(\xi)]$ can only converge to $e^{-k\xi}$, where $0 \leqq k \leqq 1$. We give examples in Section 5 showing that all values of k are obtained for suitable chain-dependent processes.

Similar techniques can be used to obtain results on the possible limiting joint distribution of $a_n^{-1}(M_n - b_n)$ and $a_n^{-1}(S_n - b_n)$, where S_n is the second largest of $X_1, \cdots, X_n$. These results rely on a theorem of Welsch (1972) for the uniformly mixing stationary case.

3. Domains of attraction. In this section we apply the results of O'Brien (1974c) for stationary processes to obtain general sufficient conditions for the convergence of normalized maxima of a chain-dependent process to the various possible limit laws. Our first result shows that under fairly general conditions the limiting behavior of the maxima for chain dependent processes is identical to that for the i.i.d. process with the same marginal distributions.

THEOREM 3.1. *Let $\{X_n\}$ be a chain-dependent process satisfying Doeblin's condition, and such that*

(1) $$\mathbf{P}[X_{i+1} > \xi \mid X_1 > \xi] \to 0 \quad \text{as} \quad \xi \to x_0 \quad \text{for} \quad i \geqq 1\,.$$

Then if $\{d_n\}$ is a sequence of real numbers and $0 < l < 1$,

(2) $$\mathbf{P}[M_n \leqq d_n] \to l \Leftrightarrow H^n(d_n) \to l\,.$$

PROOF. We have that the process is φ-mixing by Lemma 1.1, and the condition (1) guarantees that the $\lim_n \mathbf{P}[M_n \leqq c_n(\xi)] = e^{-\xi}$ for all $\xi > 0$ (see the remark after the proof the Theorem 1 of O'Brien (1974c)). The desired result is now immediate from Theorem 4a of the same reference. □

REMARKS. It can be easily shown that if $\mathbf{P}[M_n \leqq d_n] \to l$, and $H^n(d_n) \to l$, as in the above theorem, then $\mathbf{P}[X_{i+1} > d_n \mid X_1 > d_n] \to 0$ for $i = 1, 2, \cdots$, as $n \to \infty$. We are interested in two special sequences. The first is $d_n = c_n(\xi)$, $\xi > 0$. In this case, under Doeblin's condition, one can show that $\mathbf{P}[M_n \leqq c_n(\xi)] - H^n(c_n(\xi)) \to 0 \Leftrightarrow \mathbf{P}[X_{i+1} > c_n(\xi) \mid X_1 > c_n(\xi)] \to 0$ for $i \geqq 1$. We remark that $H^n(c_n(\xi))$ converges if and only if $\mathbf{P}[X_1 > x \mid X_1 \geqq x] \to 1$ as $x \to x_0$ in which case the limit is $e^{-\xi}$ (see O'Brien (1974b)). The second case of interest is $d_n = a_n x + b_n$, $a_n > 0$.

The condition $\mathbf{P}[X_{i+1} > x \mid X_1 > x] \to 0$ as $x \to x_0$ is not easily verified for most chain-dependent processes. We therefore give a condition which is based on the natural parameters of the process. For notational convenience, let $\tilde{F}(x) = 1 - F(x)$ for a distribution function F.

COROLLARY 3.2. *If $\{X_n\}$ is a chain-dependent process satisfying Doeblin's condition, and if $\Sigma_j P_{ij} \tilde{H}_j(x) \to 0$ uniformly in i as $x \to x_0$, then for $\{d_n\}$, l as above, (2) holds.*

PROOF. For $c < x_0$, let $\theta_j(c) = \pi_j \tilde{H}_j(c)\tilde{H}(c)^{-1}$. Then

$$\begin{aligned}
\mathbf{P}[X_{i+1} > c \mid X_1 > c] &= \tilde{H}(c)^{-1}\mathbf{P}[X_{i+1} > c,\, X_1 > c] \\
&= \tilde{H}(c)^{-1}\Sigma_j \Sigma_k \mathbf{P}[X_{i+1} > c,\, J_i = k,\, X_1 > c,\, J_0 = j] \\
&= \tilde{H}(c)^{-1}\Sigma_j \Sigma_k \pi_j P_{jk}^i \tilde{H}_j(c)\tilde{H}_k(c) \\
&= \Sigma_j \theta_j(c) \Sigma_k P_{jk}^i \tilde{H}_k(c)\,.
\end{aligned}$$

The right hand side must go to zero as $c \to x_0$. This is obvious for $i = 1$, and follows for $i > 1$ from the fact that $P_{jk}^i = \Sigma_l P_{jl}^{i-1} P_{lk}$. Thus the conditions of Theorem 3.1 are satisfied. □

By the above corollary, a sufficient condition for (2) to hold is that (P_{ij}) has uniformly convergent row sums. In contrast, the next theorem gives a condition depending only on the distribution functions H_j, $j = 1, 2, \cdots$.

THEOREM 3.3. *If* $\Sigma_j P_{ij}\tilde{H}_j \leqq L\tilde{H}$, $i = 1, 2, \cdots$ *for some* $L > 0$ (*in particular, if each* $\tilde{H}_j \leqq L\tilde{H}$) *then* (2) *holds.*

PROOF. Let $t(m)$ be any sequence of positive integers such that $m = o(t(m))$ as $m \to \infty$. Writing c for $c_{t(m)}(\xi)$ where $\xi > 0$, it suffices by O'Brien (1974c) to show

$$\limsup_{m\to\infty} \mathbf{P}[\max(X_2, \cdots, X_m) > c \mid X_1 > c] = 0 .$$

As in the preceding proof, $\mathbf{P}[X_{i+1} > c \mid X_1 > c] \leqq \Sigma_j \theta_j L\tilde{H}(c) \leqq L\xi/t(m)$. Thus, the above lim sup $\leqq L\xi(m-1)/t(m) \to 0$ as required. □

The above results all involve some sort of uniform convergence of $\Sigma_j P_{ij}\tilde{H}_j(c)$ to 0. If the Markov chain has only finitely many states, the conditions of both Corollary 3.2 and Theorem 3.3 are satisfied trivially. Thus we have generalized the principal results of Resnick (1971) and Neuts and Resnick (1970).

4. Stability. Known results on stability for stationary processes lead to similar ones for chain-dependent processes. A process (Y_n) is said to satisfy the *law of large numbers* relative to a sequence $\{A_n\}$ of real numbers if $\mathbf{P}[|Y_n - A_n| > \varepsilon] \to 0$ for all $\varepsilon > 0$. It is said to be *stable* relative to a sequence $\{B_n > 0\}$ if $\mathbf{P}[|B_n^{-1}Y_n - 1| > \varepsilon] \to 0$ for all $\varepsilon > 0$. Gnedenko gave necessary and sufficient conditions for these properties to hold for the maxima of i.i.d. processes. O'Brien (1974c) obtained conditions under which these results hold for stationary processes. Theorem 2.1 allows us to extend these to chain-dependent processes. The following theorem, which includes the results of Resnick (1972) for finite chains, is proved by methods similar to those of Section 3.

THEOREM 4.1. *If* $\{X_n\}$ *is a chain-dependent process satisfying Doeblin's condition or if* $\Sigma_j P_{ij}^k \tilde{H}_j(x) \leqq L\tilde{H}(x)$, $i = 1, 2, \cdots$, *for some* k *and some* $L > 0$, *then* M_n *satisfies the law of large numbers relative to* $\{A_n\}$ *if and only if the associated independent process does, and similarly for stability.*

5. Examples. Let H be any continuous distribution, and $k \in (0, 1]$. We construct a chain-dependent process such that $\mathbf{P}[M_n \leqq c_n(\xi)] \to e^{-k\xi}$, whereas $H^n(c_n(\xi)) \to e^{-\xi}$.

Let $\pi = (1 - 2^{-\frac{1}{2}}, 2^{-\frac{1}{2}} - 3^{-\frac{1}{2}}, \cdots, n^{-\frac{1}{2}} - (n+1)^{-\frac{1}{2}}, \cdots)$. Let y_i satisfy $y_0 = -\infty$, $H(y_1) = \pi_1$, $H(y_n) = \pi_1 + \pi_2 + \cdots + \pi_n = 1 - (n+1)^{-\frac{1}{2}}$. Define the matrix B to have all rows equal to π, and $P = kB + (1-k)I$, where I is the identity matrix. Then $\pi P = \pi$. This Markov chain satisfies Doeblin's condition, in fact $||P_{i\cdot}^n - \pi|| \leqq (1-k)^n$. If

$$\begin{aligned} H_n(x) &= 0 && \text{for } x \leqq y_{n-1} \\ &= \pi_n^{-1}(H(x) - H(y_{n-1})) && \text{for } y_{n-1} < x \leqq y_n \\ &= 1 && \text{for } x > y_n \end{aligned}$$

then $\Sigma_i \pi_i H_i(x) = H(x)$ for all x, and we have a chain-dependent process with the given H.

Now note that $\Sigma_j P_{ij}\tilde{H}_j(x) = k\tilde{H}(x) + (1-k)\tilde{H}_i(x)$, which for $k < 1$ does not converge uniformly to zero as $x \to x_0$. In fact, it is even the case that for $k < 1$, $\mathbf{P}[X_2 > c \mid X_1 > c]$ does not converge to zero as $c \to x_0$. To see this, one need only compute, as in the proof of Corollary 3.2 that

$$\mathbf{P}[X_2 > c \mid X_1 > c] = k\tilde{H}(c) + (1-k)\Sigma_j \pi_j(\tilde{H}_j(c))^2(\tilde{H}(c))^{-1},$$

which for $c = y_n$ has the value $k\tilde{H}(y_n) + (1-k)$.

To show the convergence of $\mathbf{P}[M_n \leqq c_n(\xi)]$, fix $\xi > 0$ and let $j(n) = [n^2/\xi^2]$, and $d_n(\xi) = y_{j(n)}$. Then

$$H(d_n(\xi)) = 1 - (1 + j(n))^{-\frac{1}{2}} \geqq 1 - \xi/n = H(c_n(\xi)),$$

and hence $c_n(\xi) \leqq d_n(\xi)$. Now $\mathbf{P}[M_n \leqq d_n(\xi)] = \mathbf{P}[\max(J_0, \cdots, J_{n-1}) \leqq j(n)]$, but $\mathbf{P}[J_n \leqq l \mid J_{n-1}] = k(\pi_1 + \cdots + \pi_l) + (1-k) = 1 - k(1+l)^{-\frac{1}{2}}$ if $J_{n-1} \leqq l$. Thus it follows that $\mathbf{P}[\max(J_0, \cdots, J_{n-1}) \leqq j(n)] = \{1 - k(1 + j(n))^{-\frac{1}{2}}\}^{n-1}\{1 - (1 + j(n))^{-\frac{1}{2}}\}$. An easy computation shows that $(1 + j(n))^{-\frac{1}{2}} = \xi/n + o(1/n)$, and thus we have

$$\mathbf{P}[M_n \leqq c_n(\xi)] \leqq \mathbf{P}[M_n \leqq d_n(\xi)] \to e^{-k\xi}.$$

A similar argument, replacing $j(n)$ with $j(n) - 1$ wherever it appears, shows that $\liminf_n \mathbf{P}[M_n \leqq c_n(\xi)] \geqq e^{-k\xi}$, and we are done.

We note that if the above H is such that $H^n(a_n x + b_n) \to \Phi(x)$, then it can be shown that $\mathbf{P}[M_n \leqq a_n x + b_n] \to \Phi(x)^k$.

As a second example we construct a chain dependent process with the property that $\mathbf{P}[M_n \leqq c_n(\xi)] \to 1$ for $\xi > 0$, i.e. $k = 0$. Take a chain P with $P_{i,i+1} = p_{i+1} > 0$, $P_{i0} = 1 - p_{i+1}$, for $i \geqq 0$. Let $\beta_0 = 1$, and $\beta_i = \prod_{k=1}^{i} p_k$. If $s = \sum_{i=0}^{\infty} \beta_i < \infty$, the chain is positive recurrent with stationary measure π given by $\pi_i = s^{-1}\beta_i$ (cf. Kemeny, Snell, Knapp (1966, page 161).) We choose $p_k = k(k+2)^{-1}$ so that $\pi_k = (k+1)^{-1} - (k+2)^{-1} = (k+1)^{-1}(k+2)^{-1}$, and $\sum_{i=k}^{\infty} \pi_i = (k+1)^{-1}$. For simplicity of computation we take $H_i(x) \equiv U_i(x)$, where

$$\begin{aligned} U_i(x) &= 1, \qquad x \geqq i \\ &= 0, \qquad x < i, \end{aligned}$$

so that $\tilde{H}(x) = \Sigma_i \pi_i \tilde{H}_i(x) = \Sigma_{i=[x]+1}\pi_i = ([x] + 2)^{-1}$. Then $c_n(\xi) = [n/\xi - 2]'$ ($[x]'$ = smallest integer $\geqq x$). For convenience, we restrict attention to the case $\xi = 1$.

LEMMA 5.1. *For the chain dependent process defined above*, $\mathbf{P}[M_n \leqq c_n(1)] \to 1$.

PROOF. Let $\mathbf{P}[J_0 = 0] = 1$. Let T_1 = first $n > 0$ s.t. $J_n = 0$, and $Y_1 = J_{T_1-1}$. Then $\mathbf{P}[Y_1 \geqq k] = \beta_k$. Let T_{n+1} = first $j > T_n$ s.t. $J_j = 0$, and $Y_{n+1} = J_{T_{n+1}-1}$. Finally, define N_j to be the largest n s.t. $T_n \leqq j$. Then

$$\mathbf{P}[M_n \leqq n - 2 \mid N_n] = (1 - \beta_{n-1})^{N_n} = (1 - 2\pi_{n-1})^{N_n}, \quad \text{and}$$

$$\mathbf{P}[M_n \leqq n - 2] = \mathbf{E}[\mathbf{P}[M_n \leqq n - 2 \mid N_n]] = \mathbf{E}[(1 - 2\pi_{n-1})^{N_n}].$$

But $N_n/n \to \pi_0 = \frac{1}{2}$ a.s. as $n \to \infty$ by the SLLN, and $\pi_{n-1} = o(1/n)$, and so the result follows.

We observe that the above process does not satisfy Doeblin's condition, and also $\Sigma_j P_{ij} \tilde{H}_j(x)$ has the value $p_{i+1} = (i+1)(i+3)^{-1}$ for $0 \leqq x < i+1$, and hence fails to converge to zero uniformly in i.

These two examples serve as well for insight into stongly mixing stationary processes. Again the conclusion is the necessity of some uniformity in the (conditional) distributions involved.

REFERENCES

[1] Berman, S. M. (1962). Limiting distribution of the maximum term in sequences of dependent random variables. *Ann. Math. Statist.* **33** 894–908.

[2] Berman, S. M. (1964). Limit theorems for the maximum term in stationary sequences. *Ann. Math. Statist.* **35** 502–516.

[3] Doob, J. L. (1953). *Stochastic Processes.* Wiley, New York.

[4] Gnedenko, B. (1943). Sur la distribution limite du terms maximum d'une serie aleatoire. *Ann. of Math.* **44** 423–453.

[5] Kemeny, J., Snell, J. L. and Knapp, A. W. (1966). *Denumerable Markov Chains.* Van Nostrand, Princeton.

[6] Loynes, R. M. (1965). Extreme values in uniformly mixing stationary stochastic processes. *Ann. Math. Statist.* **36** 993–999.

[7] O'Brien, G. L. (1974a). Limit theorems for sums of chain-dependent processes. *J. Appl. Probability* **11** 582–587.

[8] O'Brien, G. L. (1974b). Limit theorems for the maximum term of a stationary process. *Ann. Probability* **2** 540–545.

[9] O'Brien, G. L. (1974c). The maximum term of uniformly mixing stationary processes. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **30** 57–63.

[10] Resnick, S. I. and Neuts, M. F. (1970). Limit laws for maxima of a sequence of random variables defined on a Markov chain. *Advances in Appl. Probability* **2** 323–343.

[11] Resnick, S. I. (1971). Tail equivalence and its applications. *J. Appl. Probability* **8** 135–156.

[12] Resnick, S. I. (1972). Stability of maxima of random variables defined on a Markov chain. *Advances in Appl. Probability* **4** 285–295.

[13] Welsch, R. E. (1972). Limit laws for extreme order statistics from strong-mixing processes. *Ann. Math. Statist.* **43** 439–446.

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