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BOOK REVIEW

D. G. Kendall and E. F. Harding, eds., *Stochastic Analysis.* John Wiley and Sons, London and New York, 1973, xiii + 465 pp. 11 pounds sterling.

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Stochastic Analysis is one of two volumes dedicated to the memory of the late Rollo Davidson, an unusually gifted young English mathematician who died in 1970 in a mountain climbing accident at the age of twenty-five. The companion volume *Stochastic Geometry* (E. F. Harding and D. G. Kendall, eds., Wiley, 1974) develops some of the geometric aspects of probability theory, whereas *Stochastic Analysis* concentrates primarily on those analytical areas to which Davidson contributed substantially. Kendall, in his introduction to the subject, defines stochastic analysis thus:

"Stochastic analysis is the field of interest of the members of a loosely knit body called the Stochastic Analysis Group, which was formed in Oxford in December 1961 to promote interest in the analytical aspects of probability theory among mathematicians and statisticians in the United Kingdom."

Kendall credits Wiener (footnote, page 389) as the originator of stochastic analysis.

Although the volume is a collection of invited papers by a number of authors, it manages to achieve a considerable unity. Clearly the credit for this, including the noteworthy effort in editing and elucidating Davidson's unfinished manuscripts, belongs to the two editors. The hand of Kendall is particularly evident throughout; in addition to his contribution of three previously unpublished papers and his collaboration with Harding on a short biographical note about Davidson, he has substantially edited and extended three unpublished manuscripts of Davidson.

The book is divided into six sections. Section 6 consists of a letter from Davidson to F. Papangelou, written in 1970, together with the aforementioned biographical sketch; Sections 1–5 deal with the mathematics of stochastic analysis. With the exception of most of Section 2, all papers appearing here are published for the first time. Several are expository; most contain new results or improved proofs of known results.

Section 1 consists of Kendall's exceptional essay *An Introduction to Stochastic Analysis*, which focusses mainly on the work of Davidson and his British colleagues and defines more sharply the scope of stochastic analysis. It begins with a careful development of an appropriate foundation for (continuous parameter) stochastic processes, progresses through a brief treatment of weak convergence (with special mention of queueing theory), characteristic functions and renewal sequences, then unifies the arithmetic of these last two topics within the framework of "Delphic semigroups." (The mysterious name derives from an excursion to Delphi on the occasion of a symposium held at Loutraki. The theory of such semigroups was developed primarily by Kendall and Davidson). There is considerable discussion of regenerative phenomena and their p-functions, dealing both with their Delphic structure and the oscillatory properties of p-functions; Kingman's classification with respect to "depth" of theorems of p-functions is also explored (a theorem is *deep* if it holds only for p-functions which are the diagonal transition functions of Markov chains, and *shallow* if it is true for all p-functions). The essay closes with some results and open questions on Markovian transition functions and the Chapman–Kolmogorov equations, including Kendall's conjecture that the transition semigroup of a Markov chain is determined by its "germ" (meaning in this context, the specification of each $p_{ij}(t)$ on some interval $[0, t_{ij})$).

The sketchy description given here falls far short of capturing the full flavor of Kendall's writing. His perceptive overview of the field of stochastic analysis gives the reader considerable feeling for the current state of the art; it is clear that significant progress has been made on many fronts but numerous unresolved conjectures and open problems are elucidated. (Indeed, one of the impressive features of this book is the care taken by many of the authors to describe open problems and conjectures—Kendall's survey alone mentions ten such problems explicitly.)

Section 2 (with the exception of 2.8) brings together previously published work of Kendall and of Davidson from the period 1967–1970, dealing principally with renewal sequences and Delphic semigroups. Of the ten papers in the volume authored or co-authored by Davidson, six are in Section 2. The final paper in this section (2.8) is the first draft of a manuscript by Davidson (with considerable simplification and explication by Kendall) concerning the indecomposability of distributions satisfying axioms similar to those which characterize the important properties of the Wishart distribution. The reader gets a strong impression from this section of the great breadth of Davidson's mathematical interests.

Section 3 is devoted to regenerative phenomena and their p-functions. A regenerative phenomenon (r.p.) is the continuous-time analogue of Feller's "recurrent event." Specifically if $(\Omega, \mathscr{F}, P)$ is a probability triple and $\xi = \{E(t); t \geqq 0\}$ is a family of events of $\mathscr{F}$, and if $0 < t_1 < \cdots < t_k$ implies that $P(E(t_1) \cap E(t_2) \cap \cdots \cap E(t_k)) = P(E(t_1))P(E(t_2 - t_1) \cap \cdots \cap E(t_k - t_1))$, then ξ is called a r.p. The function $p(t) = P(E(t))$ is called the p-function of ξ. Section 3 consists of a series of papers by the English school concerning various aspects of the theory of regenerative phenomena as developed primarily by Kingman. The term "regenerative phenomenon" here refers to those regenerative events which take place on a set of positive (Lebesgue) measure, e.g., the occupation of a state of a Markov chain. There are of course "smaller" regenerative events (we might call these "generalized regenerative phenomena" or g.r.p.) which, like the set of zeros of Brownian motion, take place on a set of measure zero, and thus have identically vanishing p-functions. The theory of these events, initiated by Krylov and Yushkevitch ([7]) and developed primarily by Hoffman–Jorgensen, Horowitz, Meyer and Maisonneuve ([5], [6], [10], [8], [9]) is not treated in this volume.

At the risk of slighting several other interesting contributions in this section, I will comment briefly only on 3.3, which gives a good idea of Davidson's interests and abilities in stochastic analysis, and 3.6. The latter provides an example of the kind of interaction between p-function theory and the theory of g.r.p. which one might hope will eventually enrich both branches.

Selection 3.3 is an incomplete manuscript by Davidson, edited by J. F. C. Kingman, G. E. H. Reuter and D. S. Griffeath, concerning "Smith's phenomenon" (the existence of Markov chains with a state 0 such that $p'_{00}(0) = -\infty$ and $\limsup_{t\to 0} p'_{00}(t) = +\infty$) and the oscillation of p-functions. In the first part of 3.3 Davidson gives a short construction, using the theory of p-functions, of a Markov transition function with Smith's pathology. The second part treats a subject which had a considerable fascination for Davidson, the oscillation of p-functions. The first theorem presented here is extended in a nontrivial fashion by Reuter and (especially) Griffeath. The problem is essentially this (with modifications as described in Kendall's introduction): given a p-function and $M(p) = p(1)$, how much oscillation can p have on $[0, 1]$? The Davidson–Reuter–Griffeath result is that for $\varepsilon > 0$ and $M(p) > \frac{2}{3}$, there exists a $\delta(M, \varepsilon) > 0$ such that for all such p-functions, $p(t) \geqq (2M - 1)/M - \varepsilon$ for $0 \leqq t \leqq \delta$. The extension to $M(p) > \frac{1}{2}$ is still unresolved—it would have as one consequence that $m(p) \equiv \inf_{0\leq t\leq 1} p(t) \geqq \frac{1}{2}\{(16M(p) - 7)^{\frac{1}{2}} - 1\}$. (Peter Bloomfield's paper in Section 3.2 has further results on the oscillation problem; more recent work on the same subject appears in [4]; still later advances have recently been made by A. Cornish.)

N. H. Bingham's contribution (3.6) is devoted primarily to analogues for regenerative phenomena of the Dynkin–Lamperti theorems in renewal theory which describe the limiting distribution of the age process and excess-life process. An obvious analogue of these processes exists for regenerative phenomena (as well as for the g.r.p. described above). Let $r(\tau)$ be the Laplace transform of a p-function p; a result by Kingman states that there exists a measure μ such that

$$[r(\tau)]^{-1} = \tau + \int_{(0,\infty]} (1 - e^{-\tau x})\mu(dx) , \qquad \text{where} \quad \int_{(0,\infty]} (1 - e^{-x})\mu(dx) < \infty .$$

Bingham proves that the age and excess-life processes of a r.p. converge weakly when suitably normed, if and only if $\{\mu(x, \infty]\}^{-1}$ is regularly varying at infinity with index $\alpha \in (0, 1)$.

Although the limiting processes are not the age and excess-life processes of a r.p., Bingham shows that they are closely related to the r.p. with canonical measure $\mu_\alpha(dx) = \alpha\, dx/(x^{1+\alpha}\Gamma(1 - \alpha))$ $(0 < \alpha < 1)$. It may be of interest to note that Bingham's F^* and B^*, and therefore by his Proposition 1 the limiting processes as well, are in fact the age and excess life processes of a g.r.p., namely the range of the driftless stable subordinator of index α. (Meyer showed in [10] that the range of a subordinator is a g.r.p.) Viewed in this light Bingham's Theorem 2b proves the weak convergence of the age and excess-life processes (suitably normed) associated with a class of g.r.p. Now it is known ([9]) that every g.r.p. which has a local time can be identified with the range of its inverse local time; noting that for $\alpha \in (0, \frac{1}{2}]$ the stable subordinator of index α is the inverse local time of the symmetric stable process of index $(1 - \alpha)^{-1}$, it is immediate that the limiting age and excess-life processes are those associated with the g.r.p. formed by the zero set of this symmetric stable process. For $\alpha \in (\frac{1}{2}, 1)$ the corresponding "regenerative set" is the zero set of a semi-stable process.

Sections 4 and 5 are more diverse; Section 4 concentrates mainly on Markov processes, while Section 5 contains three surveys of special topics as well as an algebraic note by Davidson on confounding in complete factorial experiments. Again with apologies to the other authors, I will restrict my remarks to 4.3, 5.3 and 5.4, all of which are "surveys," and 4.2 and 4.4.

The contributions 4.2 and 4.4 defy easy classification, as both deal with "nonstandard" problems in the theory of Markov chains. Kingman's paper 4.2 is an interesting extension of some ideas of Heller concerning the algebraic structure of stochastic processes with a discrete time parameter. (The problem examined by Heller was essentially that of determing when a finite-state stochastic process could be derived by combining states in some finite Markov chain.) In brief, one forms the free associative algebra generated by the elements of the state space and expresses the joint distributions of the process in terms of linear functionals on this algebra. It turns out that many of the important properties of discrete-parameter processes, including stationarity and the martingale property, can be simply formulated in algebraic terms. Kingman pays particular attention to a class of Markov processes, introduced by Runnenberg and Steutel, whose transition functions have a simple form ("finite rank"); the structure of these processes is determined, using the Kingman–Heller theory and some results from the theory of positive operators.

In 4.4 David Williams takes up the theory of harnesses, initiated by Hammersley as a model for spatial variation. He associates with a wide class of discrete parameter Markov chains with countable state spaces A a stochastic process $\{f(x); x \in A\}$ parametrized by A; f is a *Q-harness* if $E[f(x) \mid f(y), y \neq x] = \sum_{y \in A} Q(x, y)f(y)$ where Q is the one-step transition matrix. Among other results he gives an explicit construction of a Gaussian harness associated with a random walk and obtains a decomposition of the general Q-harness as the sum of a Q-harmonic part plus a harness with the properties of a potential. It happens that if Q is recurrent, every Q-harness is Q-harmonic; Williams poses the question of finding a necessary and sufficient condition on Q for the existence of a non-Q-harmonic Q-harness.

The three survey papers 4.3, 5.3, and 5.4 provide an interesting contrast, since they deal with subjects in very different stages of development. Ron Pyke's contribution (4.3) surveys recent progress in multiparameter stochastic processes, a rather new area which is developing at a rapid rate. The main conern is with sums of independent random variables indexed by multidimensional integer lattices and with the "Brownian sheet" (a Gaussian process $Z(\mathbf{s})$, $\mathbf{s} \in R^k$, with mean zero and $\text{Cov}\,(Z(\mathbf{s}), Z(\mathbf{t})) = |(s_1 \wedge t_1)(s_2 \wedge t_2) \cdots (s_k \wedge t_k)|$). Since this article was written much progress has been made on several fronts, including Orey and Pruitt's results on local sample function behavior for the Brownian sheet ([11]), the as yet unpublished work of P. Revesz and his colleagues on the embedding of multiparameter sums of independent random variables, and Cairoli and Walsh's theory of planar stochastic integrals ([1]). An important class of problems still open concerns limit results when the definitions of the partial sum processes and Brownian sheet (as random signed measures) are extended to a suitably large index class of sets.

Taylor's survey 5.3 concerns a more established area, processes with stationary independent increments. Taylor gives a lucid and thorough exposition of the theory of these processes, concentrating almost entirely on sample path behavior (this survey is an expanded version of a talk presented at the Joint Statistical Meetings at Fort Collins, Colorado in August 1971). This paper provides an excellent introduction to the subject and includes a useful bibliography of 99 items. (Kendall has added a 100th entry as a compactification point; actually, it is a two-point compactification, listing the autobiographies of Wiener and Lévy).

Section 5 closes with Kendall's survey (5.4) of separability and measurability for stochastic processes. The word "survey" is something of a misnomer here; the paper does not so much survey the existing literature on the subject as it re-derives and extends Doob's classical results, which are fundamental to a rigorous theory of continuous-parameter processes. The primary goal here is the extension of Doob's results to the case where the state space Z for the process is any compact second-countable Hausdorff space, and the parameter set A is second-countable and Hausdorff. Perhaps it is appropriate to mention here another approach to separability and measurability questions in the special case when $A = R^+$, that of using the "essential topology" with respect to Lebesgue measure (cf. [2], [3], and [12], for example). This approach has proved quite useful in probabilistic potential theory. It does, however, entwine the topological aspects of the subject with the measure-theoretic properties, and Kendall's avowed aim in 5.4 is to separate these two elements as much as possible.

Any reader of this review who has persevered this far will doubtless have formed his own opinion of the book's value from the above description; nonetheless it is the reviewer's solemn duty to pass a weighty judgment on the worth of the object under scrutiny. In this case duty becomes pleasure, for *Stochastic Analysis* is a volume of high quality. The mathematics is consistently interesting, the readability is excellent, and none of the relatively few typographical errors that were found presents any serious obstacle to understanding. In short, this book is a fitting tribute to a fine mathematician.

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