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POISSON CONVERGENCE AND FAMILY TREES

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Cells of certain variety live a random length of time and then split into two new cells. Let $t_1 < t_2 < t_3 < \cdots$, be an increasing sequence of positive numbers such that any given cell has probability λ/n with $\lambda > 0$, that its life span be at least t_n units of time. Starting with one cell, the nth generation will have 2^n cells and for each one we count the number of its ancestors and itself whose life span was at least t_n units of time. These numbers determine an empirical distribution (the nth empirical distribution). It is shown that for almost all cell cultives (starting each time with one cell) the sequence of these empirical distributions converges to the Poisson distribution with parameter λ.

1. Introduction. Consider an age-dependent binary tree with vertices labelled * (in the 0th generation) and by the 2^k binary sequences $\sigma_1 \cdots \sigma_k$ (each $\sigma_i = 0$ or 1) for the kth generation vertices. The ancestors of $\sigma \equiv \sigma_1 \cdots \sigma_k$ are *, $\sigma_1, \cdots, \sigma_1 \cdots \sigma_{k-1}$. Associate with the generic vertex σ the random variable (rv) $Y(\sigma)$, and assume $Y(*)$, $Y(0)$, $Y(1)$, $Y(00)$, $\cdots$, $Y(\sigma)$, $\cdots$ to be independent identically distributed (i.i.d.) rv's with continuous distribution function on some probability space $(\Omega, \mathscr{A}, P)$. $Y(\sigma)$ is interpreted as the lifetime of the cell at the vertex σ: if the initial cell at * is born at time 0, then the cell at σ is born at time $Y(*) + Y(\sigma_1) + \cdots + Y(\sigma_1 \cdots \sigma_{k-1})$ and splits (and dies) at time $Y(*) + \cdots + Y(\sigma_1, \cdots \sigma_k)$.

Fix $\lambda > 0$, and for each $n \geqq \lambda$, let the reals t_n be such that $P\{Y(\sigma) > t_n\} = \lambda/n$ so that $t_n < t_{n+1}$. Then the indicator rv's $X^n(\sigma) \equiv X^n(\sigma, \omega)$, defined by $X^n(\sigma) = 1$ if $Y(\sigma) > t_n$, $= 0$ otherwise, distinguish cells with lifetimes exceeding t_n from those with smaller lifetimes. Write

$$S^n(\sigma) = X^n(*) + X^n(\sigma_1) + \cdots + X^n(\sigma_1 \cdots \sigma_k)\,,$$
$$\eta_n(j, \omega) = 2^{-n} \textstyle\sum_n U_j(S^n(\sigma))$$

where $U_j(i) = 1$ if $j \geqq i$, $= 0$ otherwise, and $\sum_n$ denotes summation over the 2^n binary sequences $\sigma_1 \cdots \sigma_n$ of length n. Then $\eta_n(j, \omega)$ is the proportion of paths in the tree from the 0th to the nth generations inclusive such that at most j $(0 \leqq j \leqq n + 1)$ of the lifetimes on the path exceed t_n. Finally write

$$\beta(j, \omega; n, r) = 2^{-r} \textstyle\sum_r U_j(S^n(\sigma))\,,$$

so that $\eta_n(j, \omega) = \beta(j, \omega; n, n)$.

The aim of this note is to prove the

THEOREM. $P\{\omega \in \Omega : \eta_n(j, \omega) \to \sum_{i=0}^{j} e^{-\lambda}\lambda^i/i!\ (n \to \infty)$ *for all* $j\} = 1$.

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This result describes a somewhat different aspect from the limit theorems given in [2] where further references to these family tree models may be found.

2. Proof of the theorem. Our proof is mostly based on two lemmas, of which the first may be found in [1] but is here given with an alternative proof more akin to standard branching process techniques.

LEMMA 1. *For sequences* $X(*)$, $X(\sigma_1)$, $\cdots$ *of* i.i.d. rv's *defined on a binary tree with characteristic function* (ch.f.) $\phi(t)$, *define for each binary* $\sigma_1 \cdots \sigma_k$ *of length* k

$$S(\sigma_1 \cdots \sigma_k) = X(*) + X(\sigma_1) + \cdots + X(\sigma_1 \cdots \sigma_k)$$

and the sample path ch.f.

$$\psi_k(t, \omega) = 2^{-k} \sum_k \exp(itS(\sigma_1 \cdots \sigma_k; \omega)) .$$

Then provided that $2|\phi(t)|^2 > 1$,

$$E|\psi_k(t) - E\psi_k(t)|^2 \leqq 2^{-k} + 2(1 - |\phi(t)|^2)/(2|\phi(t)|^2 - 1) .$$

OUTLINE PROOF. Omit the argument t for convenience. Observe that $E\psi_k = \phi^{k+1}$, and that $E|\psi_0|^2 = 1$. The joint distributions of $X(\sigma_1), \cdots, X(\sigma_1 \cdots \sigma_k)$ and $X(*), \cdots, X(\sigma_1 \cdots \sigma_{k-1})$ are the same, and if $\sigma_1' \neq \sigma_1$, then $X(\sigma_1'), \cdots, X(\sigma_1' \cdots \sigma_k')$ is independent of $X(\sigma_1), \cdots, X(\sigma_1 \cdots \sigma_k)$. Using a backwards decomposition,

$$\begin{aligned} E|\psi_k|^2 &= 2^{-1}|\phi^2|^k + 2^{-1}E|\psi_{k-1}|^2 \\ &= 2^{-1} \textstyle\sum_{i=0}^{k-1} 2^{-i}|\phi^2|^{k-i} + 2^{-k} , \end{aligned}$$

whence the assertion.

COROLLARY. *If* $P\{X(*) = 1\} = \lambda/r = 1 - P\{X(*) = 0\}$, *then for* $r > 16\lambda$,

$$E|\psi_k - E\psi_k|^2 \leqq 2^{-k} + 16\lambda/r \quad (\textit{all} \ \ t) .$$

PROOF. For $X(*)$ just described, $|\phi(t)|^2 = 1 - 2(\lambda/r)(1 - \lambda/r)(1 - \cos t)$, so $1 - |\phi(t)|^2 < 4\lambda/r$ for all t and all $r > 16\lambda$, and thus $2|\phi(t)|^2 - 1 > \frac{1}{2}$ for all t.

LEMMA 2.

$$\begin{aligned} &P\{\omega \in \Omega : \beta(j, \omega; n^2, (n+1)^2) \to \textstyle\sum_{i=0}^{j} e^{-\lambda}\lambda^i/i! \ (n \to \infty) \ \ \textit{for all} \ \ j\} \\ &\quad = P\{\omega \in \Omega : \beta(j, \omega; (n+1)^2, n^2) \to \textstyle\sum_{i=0}^{j} e^{-\lambda}\lambda^i/i! \ (n \to \infty) \ \ \textit{for all} \ \ j\} = 1 . \end{aligned}$$

PROOF. Let $\Psi_n(t, \omega) = 2^{-(n+1)^2} \sum_{(n+1)^2} \exp(itS^{n^2}(\sigma_1 \cdots \sigma_{(n+1)^2}; \omega))$ denote the empirical ch.f. of $\beta(j, \omega; n^2, (n+1)^2)$. Applying the corollary with $k = (n+1)^2$ and $r = n^2$, we have

$$E|\Psi_n - E\Psi_n|^2 \leqq 2^{-(n+1)^2} + 16\lambda/n^2 \qquad n^2 > 16\lambda , \ \text{all} \ \ t .$$

By Tchebycheff's inequality, for any given $\varepsilon > 0$, all t, and all $n^2 > 16\lambda$,

$$\begin{aligned} P\{\omega \in \Omega : |\Psi_n(t, \omega) - E\Psi_n(t)| > \varepsilon\} &\leqq E|\Psi_n - E\Psi_n|^2/\varepsilon^2 \\ &\leqq (2^{-(n+1)^2} + 16\lambda/n^2)/\varepsilon^2 . \end{aligned}$$

Hence by the Borel–Cantelli lemma,

$$P\{\omega \in \Omega : |\Psi_n(t, \omega) - E\Psi_n(t)| \to 0\} = 1 \qquad \text{all} \ \ t .$$

$$E\Psi_n(t) = [1 + \lambda(e^{it} - 1)/n^2]^{(n+1)^2} \to \exp(\lambda(e^{it} - 1)) \equiv f(t) \qquad n \to \infty ,$$

so

$$P\{\omega \in \Omega : \Psi_n(t, \omega) \to f(t) \quad (n \to \infty)\} = 1 \qquad \text{all } t .$$

Now consider the subset of $R \times \Omega$

$$A = \{(t, \omega) : \Psi_n(t, \omega) \nrightarrow f(t) \quad (n \to \infty)\} ,$$

which is certainly measurable with respect to $P \times m$ where $m(\cdot)$ denotes Lebesgue measure. Let A_t and A_ω be respectively the t and ω sections of A, and write $I_A(t, \omega)$ for the indicator function. We have just shown that $\int_\Omega I_A(t, \omega)\, dP = P(A_t) = 0$, and therefore

$$\begin{aligned} 0 &= \textstyle\sum_{r=-\infty}^{\infty} \int_r^{r+1} \int_\Omega I_A(t, \omega)\, dP\, dm = \int_\Omega \int_R I_A(t, \omega)\, dm\, dP \\ &= \textstyle\int_\Omega m(A_\omega)\, dP . \end{aligned}$$

It follows now that outside a P-null set, $\Psi_n(t, \omega) \to f(t)$ $(n \to \infty)$ for m-almost all $t \in R$, and thus the convergence holds for all t (see e.g. page 190 of Loève (1960)). Consequently the first assertion of the lemma is proved, and the other is proved likewise.

PROOF OF THE THEOREM. For $m < k < r$ observe that

$$\begin{aligned} S^m(\sigma_1 \cdots \sigma_r) &\geqq S^m(\sigma_1 \cdots \sigma_k) \geqq S^k(\sigma_1 \cdots \sigma_k) \\ &\geqq S^r(\sigma_1 \cdots \sigma_k) \geqq S^r(\sigma_1 \cdots \sigma_m) , \end{aligned}$$

so

$$\beta(j, \omega; m, r) \leqq \eta_k(j, \omega) \leqq \beta(j, \omega; r, m) .$$

Putting $m = n^2$ and $r = (n + 1)^2$ so that $n^2 \leqq k \leqq (n + 1)^2$, we apply Lemma 2, and the theorem is proved.

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REFERENCES

[1] JOFFE, A. and MONCAYO, A. R. (1972). On sums of independent random variables defined on a binary tree. *Bol. Soc. Mat. Mexicana* **18** 1.

[2] JOFFE, A. and MONCAYO, A. R. (1973). Random variables, trees, and branching random walks. *Advances in Math.* **10** 401-416.

[3] LOEVE, M. (1960). *Probability Theory*. Van Nostrand, Princeton.

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