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A NOTE ON THE PROOF OF THE ZERO-ONE LAW OF BLUM AND PATHAK

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Let $\{(\Omega, \mathscr{A}, \mu_n), n \geqq 1\}$ be a sequence of probability spaces. Blum and Pathak [*Ann. Math. Statist.* **43** (1972) 1008–1009] proved a zero-one law for permutation invariant sets $A \in \mathscr{A}^\infty$; which includes the zero-one laws of Hewitt and Savage [*Trans. Amer. Math. Soc.* **80** (1955) 470–501] and Horn and Schach [*Ann. Math. Statist.* **41** (1970) 2130–2131] as special cases. The proper reason for this is shown to be the fact that the set of measures admitting the zero-one law of Blum and Pathak coincides with the set of all strong limit points of measures admitting the zero-one law of Horn and Schach.

Consider the product-probability space $(\Omega^\infty, \mathscr{A}^\infty \bigotimes_{n=1}^\infty \mu_n \equiv \mu)$ of the probability spaces $(\Omega, \mathscr{A}, \mu_n)$, $n = 1, 2, \cdots$. In [1] the following proposition is proved: Let $A \in \mathscr{A}^\infty$ be a set which is invariant under all permutations of finitely many coordinates (π-invariant set); then $\mu(A) = 0$ or 1, provided the following condition is satisfied:

(B) for each $\varepsilon > 0$, $k \geqq 1$ and $m \geqq 1$ there is an $n \geqq m$ such that $\|\mu_k - \mu_n\| < \varepsilon$.

The purpose of this paper is to show that this result of [1] is the consequence of a result (Theorem 2.1) whose proof is trivial. The point which needs some consideration is to establish the equivalence between condition (B) and a special case of the conditions of Theorem 2.1 (Theorem 2.2). Some remarks on the role played by the property of recurrence (see Definition 2.2 and [2]) are added.

To establish some notation we start with

DEFINITION 2.1. Let $(R, \mathscr{F})$ be an arbitrary measurable space, $\mathscr{D} \subset \mathscr{F}$ an arbitrary subfamily of measurable sets. A probability measure μ on $(R, \mathscr{F})$ is said to be zero-one on $\mathscr{D}$, if $\mu(A) = 0$ or 1 for each $A \in \mathscr{D}$. A family Z of probability measures is said to be zero-one on $\mathscr{D}$ (or also $\mathscr{D}$-zero-one) if each $\mu \in Z$ is zero-one on $\mathscr{D}$.

THEOREM 2.1. *Let Z be a $\mathscr{D}$-zero-one family of probability measures on $(R, \mathscr{F})$; then the norm-topology-closure $\bar{Z}$ of Z is also $\mathscr{D}$-zero-one.*

PROOF. For $\mu \in \bar{Z}$ there exists a sequence $\mu^{(n)} \in Z$ with $\mu = s\lim_{n\to\infty} \mu^{(n)}$, where $s\lim$ denotes the limit in the norm topology; especially we have $\mu^{(n)}(A) \to \mu(A)$ for each $A \in \mathscr{D}$, whence $\mu(A) = 0$ or 1.

The following considerations serve to show that Theorem (1.1) of [1] is included in Theorem 2.1 and that the simple device of Theorem 2.1 provides a

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slightly more general class of probability measures being zero-one on the π-invariant sets than the Blum-Pathak zero-one law (Theorem 2.4 and Remark 2.2). From now on we restrict ourselves to the space $(\Omega^\infty, \mathscr{A}^\infty)$.

LEMMA 2.1. *Let* $\{\mu_k, \nu_k, k \geqq 1\}$ *be probability measures and suppose for some real numbers* ε_k, $k \geqq 1$, *that* $\|\mu_k - \nu_k\| < \varepsilon_k$. *Then* $\|\bigotimes_{k=1}^n \mu_k - \bigotimes_{k=1}^n \nu_k\| < \sum_{k=1}^n \varepsilon_k$.

PROOF (cf. [1], Lemma (1.3)). For $n = 1$ it is clear; suppose the lemma is true for some $n \geqq 1$; let f_k, g_k, $k \geqq 1$ be the densities of μ_k, ν_k, $k \geqq 1$ with respect to a dominating measure λ, then use Fubini's Theorem and the formula

$$\prod_{k=1}^{n+1} f_k(\omega_k) - \prod_{k=1}^{n+1} g_k(\omega_k) = \left(\prod_{k=1}^{n} f_k(\omega_k)\right)(f_{n+1}(\omega_{n+1}) - g_{n+1}(\omega_{n+1})) + g_{n+1}(\omega_{n+1})\left(\prod_{k=1}^{n} f_k(\omega_k) - \prod_{k=1}^{n} g_k(\omega_k)\right)$$

to obtain:

$$\begin{aligned}
&\|\textstyle\bigotimes_{k=1}^{n+1} \mu_k - \bigotimes_{k=1}^{n+1} \nu_k\| \\
&\quad = \tfrac{1}{2} \int_\Omega \cdots \int_\Omega \left|\prod_{k=1}^{n+1} f_k(\omega_k) - \prod_{k=1}^{n+1} g_k(\omega_k)\right| d\lambda(\omega_1) \cdots d\lambda(\omega_{n+1}) \\
&\quad \leqq \tfrac{1}{2} \int_\Omega |f_{n+1}(\omega_{n+1}) - g_{n+1}(\omega_{n+1})|\, d\lambda(\omega_{n+1}) \\
&\qquad + \tfrac{1}{2} \int_\Omega \cdots \int_\Omega \left|\prod_{k=1}^{n} f_k(\omega_k) - \prod_{k=1}^{n} g_k(\omega_k)\right| d\lambda(\omega_1) \cdots d\lambda(\omega_n) \\
&\quad = \|\mu_{n+1} - \nu_{n+1}\| + \|\textstyle\bigotimes_{k=1}^{n} \mu_k - \bigotimes_{k=1}^{n} \nu_k\| < \varepsilon_{n+1} + \sum_{k=1}^{n} \varepsilon_k\,.
\end{aligned}$$

DEFINITION 2.2. A product probability measure $\mu = \bigotimes_{i=1}^\infty \mu_i$ is called recurring, if for each $i \geqq 1$ there is some $j > i$ with $\mu_i = \mu_j$.

THEOREM 2.2. μ *is in the norm closure of a family of recurring probability measures* iff *it satisfies condition* (B).

PROOF. 1) Suppose (B) holds and let $\delta > 0$, $\varepsilon_k > 0$, $k = 1, 2, \cdots$, be real numbers with $\sum_{k=1}^\infty \varepsilon_k \leqq \delta$. We choose a sequence $\mathscr{N}_1 \equiv \{n_j(1), j \geqq 1\}$ of natural numbers with the properties $n_1(1) = 1$, $n_{j+1}(1) > n_j(1)$, $\|\mu_1 - \mu_{n_j(1)}\| < \varepsilon_1/2^j$, $j \geqq 1$. Due to (B) such a choice is possible.

Defining $\mathscr{R}_1 \equiv \{n : \mu_n = \mu_1\}$ and $\mathscr{M}_1 \equiv \mathscr{N}_1 \cup \mathscr{R}_1$ we have either $\mathscr{M}_1 = \mathbb{N}$ (the set of all natural numbers), or $\mathbb{N} - \mathscr{M}_1$ contains infinitely many elements. Indeed, assume $\mathbb{N} - \mathscr{M}_1$ has only finitely many elements and let $r \in \mathbb{N} - \mathscr{M}_1$; from the definition of $\mathscr{M}_1$ we have $\|\mu_r - \mu_1\| > 0$, and the assumption implies for arbitrary $\eta > 0$ the existence of an $N(\eta)$ so that $\|\mu_1 - \mu_m\| < \eta$ for all $m \geqq N(\eta)$, whence taking $\eta = \frac{1}{2}\|\mu_r - \mu_1\|$ we obtain

$$\|\mu_r - \mu_m\| \geqq |\|\mu_r - \mu_1\| - \|\mu_1 - \mu_m\|| \geqq \tfrac{1}{2}\|\mu_r - \mu_1\| > 0 \quad \forall\, m \geqq N(\eta) \tag{2.1}$$

which is impossible in view of (B).

If $\mathbb{N} - \mathscr{M}_1$ has infinitely many elements, choose a sequence $\mathscr{N}_2 \equiv \{n_j(2), j \geqq 1\}$ with the properties $\mathscr{N}_2 \subset \mathbb{N} - \mathscr{M}_1$, $n_1(2) = \min(\mathbb{N} - \mathscr{M}_1)$, $n_{j+1}(2) > n_j(2)$, $\|\mu_{n_1(2)} - \mu_{n_j(2)}\| < \varepsilon_2/2^j$, $j \geqq 1$.

Using again an argument analogous to (2.1) such a choice is possible due to (B).

Defining $\mathscr{R}_2 \equiv \{n : n \notin \mathscr{M}_1, \mu_n = \mu_{n_1(2)}\}$ and $\mathscr{M}_2 \equiv \mathscr{N}_2 \cup \mathscr{R}_2$, we have $\mathscr{M}_1 \cap \mathscr{M}_2 = \emptyset$, and, due to the same reasons as before, either $\mathscr{M}_1 \cup \mathscr{M}_2 = \mathbb{N}$ or $\mathbb{N} - \mathscr{M}_1 \cup \mathscr{M}_2$ consists of infinitely many elements.

We proceed now by induction: let $\mathscr{M}_k$ be defined and suppose $\mathbb{N} - \bigcup_{i=1}^k \mathscr{M}_i$ consists of infinitely many elements. By the same argument as before it is, due to (B), possible to choose $\mathscr{N}_{k+1} \equiv \{n_j(k+1), j \geqq 1\}$ with

$$\mathscr{N}_{k+1} \subset \mathbb{N} - \bigcup_{i=1}^k \mathscr{M}_i , \qquad n_1(k+1) = \min (\mathbb{N} - \bigcup_{i=1}^k \mathscr{M}_i) ,$$

$$n_{j+1}(k+1) > n_j(k+1) , \qquad ||\mu_{n_1(k+1)} - \mu_{n_j(k+1)}|| < \frac{\varepsilon_{k+1}}{2^j} , \qquad j \geqq 1$$

so that, defining $\mathscr{R}_{k+1} \equiv \{n : \mu_n = \mu_{n_1(k+1)}, n \notin \bigcup_{i=1}^k \mathscr{M}_i\}$ and $\mathscr{M}_{k+1} \equiv \mathscr{N}_{k+1} \cup \mathscr{R}_{k+1}$, we obtain $\mathscr{M}_{k+1} \cap (\bigcup_{i=1}^k \mathscr{M}_i) = \emptyset$. We thus have constructed a partition of $\mathbb{N}$ into (finitely or countable infinitely many) sets $\mathscr{M}_k$. Therefore, for each $n \in \mathbb{N}$ there is exactly one $k(n)$ with $n \in \mathscr{M}_{k(n)}$.

Define now $\nu_n \equiv \mu_{n_1(k(n))}$, $n = 1, 2, \cdots$, then $\nu = \bigotimes_{n=1}^\infty \nu_n$ is a recurring probability measure with $||\nu - \mu|| < 2\delta$. To show this, let $A \in \mathscr{S}^\infty$ be arbitrary fixed; there is a sequence $A_n \in \mathscr{S}^n$ of cylinder-sets with $\lambda(A \triangle A_n) \to 0$, where λ dominates μ and ν, therefore $\mu(A \triangle A_n) \to 0$, $\nu(A \triangle A_n) \to 0$ (with the *same* sequence $\{A_n\}$), whence for $\delta > 0$ there is an $N(\delta)$ such that

$$\max (|\mu(A) - \mu(A_n)|, |\nu(A) - \nu(A_n)|) < \delta/2 \qquad \text{for} \quad n \geqq N(\delta) .$$

From Lemma 2.1 and the definition of ν we have $||\bigotimes_{i=1}^n \mu_i - \bigotimes_{i=1}^n \nu_i|| < \delta/2$, so that from $\mu(A_n) = (\bigotimes_{i=1}^n \mu_i)(\hat{A}_n)$, $\nu(A_n) = (\bigotimes_{i=1}^n \nu_i)(\hat{A}_n)$ (where $\hat{A}_n$ is the 'basis' of A_n) we obtain

$$|\mu(A) - \nu(A)| \leqq |\mu(A) - \mu(A_n)| + |\mu(A_n) - \nu(A_n)| + |\nu(A_n) - \nu(A)| < 3\delta/2 ,$$

whence $||\mu - \nu|| < 2\delta$. (ν depends on δ, of course). Since δ was arbitrary we have thus proved that μ is a cluster point of recurring probability measures.

2) Let $\nu^{(n)} = \bigotimes_{i=1}^\infty \nu_i^{(n)}$ be a sequence of recurring probability measures with $\mu = s \lim_{n\to\infty} \nu^{(n)}$. Let $\varepsilon > 0$ be given, let n be fixed and so large that $||\mu - \nu^{(n)}|| < \varepsilon/2$. Consider an arbitrary index k; due to recurrence there is for each $m \geqq 1$, a $p \geqq m$ with $\nu_k^{(n)} = \nu_p^{(n)}$. But since $||\mu_j - \nu_j^{(n)}|| \leqq ||\mu - \nu^{(n)}||$, $\forall j \geqq 1$, this implies $||\mu_j - \mu_p|| \leqq ||\mu_j - \nu_j^{(n)}|| + ||\nu_p^{(n)} - \mu_p|| < \varepsilon$, whence (B) is established for μ.

With the aid of Theorem 2.2 the proof of Theorem (1.1) of [1] is now simple:

THEOREM 2.3. (*Theorem* (1.1) *of* [1]). *Suppose the product probability measure μ satisfies condition* (B); *then μ is zero-one on the π-invariant sets.*

PROOF. Due to Theorem 2.2, μ is element of the strong closure of the family of recurring measures, the latter being zero-one on the π-invariant sets due to the Horn–Schach zero-one law [2].

REMARK 2.1. The arguments leading to Theorem 2.2 are of course used implicitly (in a slightly weaker form) in the proof of Theorem (1.1) of [1], so that from that point of view the present method does not provide a real simplification of that proof; however, it seems to give a better insight into the structure of the problem.

THEOREM 2.4. *Let R be the class of all product probability measures which are dominated by a recurring measure; then the strong closure $\bar{R}$ is zero-one on the π-invariant sets.*

PROOF. If μ is $\mathscr{D}$-zero-one for some family $\mathscr{D}$, then $\nu \ll \mu$ implies that ν is also $\mathscr{D}$-zero-one.

REMARK 2.2. The zero-one family obtained by Theorem 2.4 is strictly wider than the class characterized by condition (B). Indeed it is easy to see that μ can be dominated by a recurring ν without satisfying (B). Moreover, R also contains elements which are not product probability measures.

REMARK 2.3. The question arises if some kind of "nearby-recurrence" in the sense of Theorem 2.4 could provide a necessary condition for a measure to be zero-one on the π-invariant sets. The answer is in general negative as shown by the following (however trivial) example: let $\Omega = [0, 1]$, $\mathscr{A}$ the Borel-sets of $[0, 1]$, $\mu_n = \delta_{1/n}$ (Dirac measure on $1/n$); then $\mu = \bigotimes_{n=1}^{\infty} \mu_n$ is concentrated at the point $(1, \frac{1}{2}, \frac{1}{3}, \cdots)$ and is therefore trivially zero-one on $\mathscr{A}^{\infty}$; on the other hand, μ is roughly spoken of "as nonrecurrent as possible."

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