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ON THE PATH ABSOLUTE CONTINUITY OF SECOND ORDER PROCESSES[1]

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Necessary and sufficient conditions are given for almost all paths of a Gaussian process to be absolutely continuous (with derivative in L_p). Also sufficient conditions for almost sure path absolute continuity of second order processes are derived, slightly generalizing those previously known.

Sufficient conditions for a stochastic process to have absolutely continuous paths with probability one were first derived for weakly stationary processes in [6, pages 536–537] and were later generalized to second order processes in [7, pages 186–187]. For a Gaussian process it is known that its paths are absolutely continuous with probability zero or one [4], but necessary and sufficient conditions for the two alternatives were known only for stationary processes [2, 10], and thus also for processes with stationary increments, and for harmonizable processes [2]. Here we give several equivalent necessary and sufficient conditions for the paths of a Gaussian process to be absolutely continuous with probability one. These conditions, which are slightly more general than those of [7], are shown to be sufficient for almost sure path absolute continuity of a second order process. The results extend to the case where almost all paths have $n-1$ continuous derivatives with the $(n-1)$th derivative absolutely continuous.

In the following $T=[a, b]$ is a finite interval and $\xi=\{\xi(t, \omega), t \in T\}$ a real stochastic process of second order on the probability space $(\Omega, \mathscr{F}, P)$ with correlation function $R(t, s)=E(\xi_t \xi_s)$. $\xi(\cdot, \omega)$ denotes the path of the process ξ corresponding to $\omega \in \Omega$ and ξ_t denotes the random variable of the process corresponding to $t \in T$. $\mathscr{R}(R)$ denotes the reproducing kernel Hilbert space of the nonnegative definite function R. We will relate the absolute continuity of the paths of ξ on T with the absolute continuity of its correlation function R on $T \times T$. Recall that R is absolutely continuous on $T \times T$ if and only if there is a Lebesgue integrable function r on $T \times T$ such that for all $t_1, t_2, s_1, s_2 \in T$, $\Delta_{t_1}^{t_2} \Delta_{s_1}^{s_2} R=\int_{t_1}^{t_2} \int_{s_1}^{s_2} r(u, v)\, du\, dv$, where $\Delta_{t_1}^{t_2} \Delta_{s_1}^{s_2} R=R(t_2, s_2)-R(t_2, s_1)-R(t_1, s_2)+R(t_1, s_1)$ [9, Section 493]. Also the map $T \rightarrow L_2(\Omega)=L_2(\Omega, \mathscr{F}, P)$ defined by $t \rightarrow \xi_t$ is absolutely continuous if and only if it belongs to $W_{1,1}[T, L_2(\Omega)]$ [1, Appendice]. $W_{1,p}[T, L_2(\Omega)]$, $1 \leqq p<\infty$, is the set of all functions $f: T \rightarrow L_2(\Omega)$

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of the form $f(t) = f(a) + \int_a^t g(u)\,du$ for some $g \in L_p[T, L_2(\Omega)]$, i.e., $g: T \to L_2(\Omega)$ is measurable and $\int_T ||g(u)||^p_{L_2(\Omega)}\,du < \infty$.

THEOREM. *If $\xi = \{\xi(t, \omega),\ t \in T = [a, b]\}$ is a real separable stochastic process of second order on the probability space $(\Omega, \mathscr{F}, P)$ with correlation function $R(t, s) = E(\xi_t, \xi_s)$, the following are equivalent.*

(i) *With probability one the paths of ξ are absolutely continuous and there is a measurable second order process $\eta = \{\eta(t, \omega),\ t \in T\}$ such that*

$$\int_b^a \{E(\eta_t^2)\}^{\frac{1}{2}}\,dt < \infty \tag{1}$$

and

$$\xi(t, \omega) = \xi(a, \omega) + \int_a^t \eta(u, \omega)\,du \qquad \text{for all} \quad t \in T \tag{2}$$

with probability one.

(ii) *The map $T \to L_2(\Omega)$ defined by $t \to \xi_t$ is absolutely continuous.*

(iii) *There is a measurable nonnegative definite function r on $T \times T$ with $\mathscr{R}(r)$ separable and*

$$\int_a^b \{r(u, u)\}^{\frac{1}{2}}\,du < \infty \tag{3}$$

such that for all $t_1, t_2, s_1, s_2 \in T$,

$$\Delta_{t_1}^{t_2} \Delta_{s_1}^{s_2} R = \int_{t_1}^{t_2} \int_{s_1}^{s_2} r(u, v)\,du\,dv\,. \tag{4}$$

(iv) *$R(t, \cdot)$ is absolutely continuous on T for every fixed $t \in T$, (R is absolutely continuous on $T \times T$) and there is a measurable subset T_0 of T with Lebesgue measure zero such that $\partial^2 R(t, s)/\partial t\,\partial s$ exists for all $t, s \in T - T_0$ and satisfies*

$$\int_a^b \left(\frac{\partial^2 R(t, s)}{\partial t\,\partial s}\bigg|_{t=s=u} \right)^{\frac{1}{2}} du < \infty\,. \tag{5}$$

When ξ is Gaussian each of (ii), (iii) *and* (iv) *is necessary and sufficient for almost sure path absolute continuity.*

In general, (i) is stronger than the absolute continuity of the paths of ξ with probability one (the Gaussian case being an exception), and (iii) and (iv) are stronger than the absolute continuity of R on $T \times T$ (the stationary case is an exception here). Condition (iv) may in some cases be easier to verify than condition (iii). Trivial examples show that the presence of the zero Lebesgue measure set T_0 in (iv) is necessary. Also (iv) implies that for each $t \in T - T_0$, $\partial R(t, \cdot)/\partial t$ is absolutely continuous on T.

When ξ is harmonizable or weakly stationary with (two- and one-dimensional respectively) spectral distribution F, then condition (iv) is equivalent to $\iint_{-\infty}^{\infty} |\lambda\mu|\,dF(\lambda, \mu) < \infty$ and $\int_{-\infty}^{\infty} \lambda^2\,dF(\lambda) < \infty$ respectively; that (iv) implies (i) was shown in [6, pages 536–537; 7, pages 186–187] and the equivalence of almost sure path absolute continuity to (iv) in the Gaussian case was shown in [2, 10]. In the stationary case (iv) is also equivalent to the absolute continuity of R as a function of two variables on $T \times T$.

The relationship between the almost sure path absolute continuity of ξ and the absolute continuity of the map $t \to \xi_t$ has also been considered in [11] where, under certain conditions (not applicable here), it is shown that the former implies the latter.

When ξ is Gaussian and $1 \leqq p < \infty$, the statement "almost all paths of ξ are absolutely continuous with derivative in L_p" is equivalent to (i) to (iv) with $p/2$ replacing $\frac{1}{2}$ is (1), (3) and (5) and the map in (ii) belonging to $W_{1,p}[T, L_2(\Omega)]$. Also all results extend in a straightforward way to give necessary and sufficient conditions for almost all paths of ξ to have $(n-1)$ continuous derivatives with $\xi^{(n-1)}$ satisfying (i).

PROOF. (iv) *implies* (iii). If we define r by $r(t, s) = \partial^2 R(t, s)/\partial t\, \partial s$ for $t, s \in T - T_0$ and $r(t, s) = 0$ elsewhere, everything in (iii) is obvious except perhaps that $\mathscr{R}(r)$ is separable, which is shown as follows. Since $\partial^2 R(t, s)/\partial t\, \partial s$ exists for $t = s \in T - T_0$, the mean square derivative $\dot{\xi}$ of ξ exists on $T - T_0$. Define the process ζ by $\zeta_t = \dot{\xi}_t$, $t \in T - T_0$, and $\zeta_t = 0$, $t \in T_0$. Then $E(\zeta_t \zeta_s) = r(t, s)$, $t, s \in T$ and $H(\zeta) \subset H(\xi)$ where $H(\zeta)$ is the closure in $L_2(\Omega)$ of the linear space generated by $\{\zeta_t, t \in T\}$ and similarly for $H(\xi)$. Now $\mathscr{R}(R)$ and $H(\xi)$ are isomorphic, and so are $\mathscr{R}(r)$ and $H(\zeta)$, and thus the separability of $\mathscr{R}(R)$ implies that of $\mathscr{R}(r)$.

(iii) *implies* (ii). There exists a separable Hilbert space H and a function $f: T \to H$ such that $\langle f(u), f(v)\rangle_H = r(u, v)$ and $\{f(t), t \in T\}$ is complete in H; for instance take $H = \mathscr{R}(r)$ and $f(t) = r(t, \cdot)$. Every $h \in H$ is the limit of a sequence of linear combinations from $\{f(t), t \in T\}$: $h = \lim_n h_n$, $h_n = \sum_{i=1}^{N_n} a_{n,i} f(t_{n,i})$. Thus for all $t \in T$, $\langle h, f(t)\rangle_H = \lim_n \sum_{i=1}^{N_n} a_{n,i} r(t_{n,i}, t)$ and since r is measurable, f is weakly measurable and also measurable since H is separable. Then (3) implies that $f \in L_1[T, L_2(\Omega)]$ and from (4) we have that for all $t, s \in T$,

$$\langle \xi_t - \xi_a, \xi_s - \xi_a \rangle_{L_2(\Omega)} = \langle \textstyle\int_a^t f(u)\, du, \int_a^s f(v)\, dv\rangle_H .$$

It follows that there is an isomorphism A between the closure in H of the linear space generated by $\{\int_a^t f(u)\, du, t \in T\}$ and the closure in $L_2(\Omega)$ of the linear space generated by $\{\xi_t - \xi_a, t \in T\}$, such that

$$\xi_t - \xi_a = A \textstyle\int_a^t f(u)\, du \qquad \text{for all} \quad t \in T .$$

Define $g: T \to L_2(\Omega)$ by $g(t) = Af(t)$. Then $g \in L_1[T, L_2(\Omega)]$ and $A \int_a^t f(u)\, du = \int_a^t g(u)\, du$ [8, page 83]. It follows that $\xi_t = \xi_a + \int_a^t g(u)\, du$, $t \in T$, and thus (ii) is satisfied.

(ii) *implies* (i). Since the function $t \to \xi_t$ is absolutely continuous, there is a $g \in L_1[T, L_2(\Omega)]$ such that

$$\xi_t = \xi_a + \textstyle\int_a^t g(u)\, du \qquad \text{for all} \quad t \in T$$

the equality being in $L_2(\Omega)$. Since g is measurable, there is a measurable subset N of T with Lebesgue measure zero such that $g(T - N)$ is separable. Now define the process ζ by $\zeta_t = g(t)$ for $t \in T - N$ and $\zeta_t = 0$ for $t \in N$. Then ζ is of second order, $\{\zeta_t, t \in T\}$ is separable as a subset of $L_2(\Omega)$, and hence so is $H(\zeta)$, the closure of the linear space generated by it. Let $Q(t, s) = E[g(t)g(s)]$ and $K(t, s) = E(\zeta_t \zeta_s)$, $t, s \in T$. Since g is measurable, it is weakly measurable and thus for all fixed $t \in T$, $Q(t, \cdot)$ is measurable. Since for fixed $t \in T$, $K(t, \cdot)$ and $Q(t, \cdot)$ agree on $T - N$, it follows that $K(t, \cdot)$ is also measurable. An inspection of the proof of Theorem 1 of [3] shows that $H(\zeta)$ separable and $K(t, \cdot)$ measurable for all $t \in T$ imply that ζ has a measurable modification, denoted by η. Now for all $t \in T - N$, $E(\eta_t^2) = E(\zeta_t^2) = E[g^2(t)]$ and thus $g \in L_1[T, L_2(\Omega)]$ implies (1). This in turn implies $E \int_a^b |\eta(t, \omega)| \, dt < \infty$, and thus $\eta(\cdot, \omega) \in L_1[T]$ with probability one, i.e., for all $\omega \in \Omega - \Omega_0$ with $P(\Omega_0) = 0$. Now define the process X by

$$\begin{aligned} X(t, \omega) &= \xi(a, \omega) + \int_a^t \eta(u, \omega) \, du && \text{for} \quad t \in T, \quad \omega \in \Omega - \Omega_0 \\ &= 0 && \text{for} \quad t \in T, \quad \omega \in \Omega_0 . \end{aligned}$$

It is easily seen that each of $E(\xi_t - \xi_a)^2$, $E(X_t - \xi_a)^2$ and $E[(\xi_t - \xi_a)(X_t - \xi_a)]$ equals $\int_a^t \int_a^t E[g(u)g(v)] \, du \, dv$. Hence for all fixed $t \in T$, $E(\xi_t - X_t)^2 = 0$ and thus $P\{\omega \in \Omega : \xi(t, \omega) = X(t, \omega)\} = 1$. If S is a countable dense subset of T which is a separating set for ξ, we have $P\{\omega \in \Omega : \xi(t, \omega) = X(t, \omega), t \in S\} = 1$ and since X has continuous paths with probability one it follows that $P\{\omega \in \Omega : \xi(t, \omega) = X(t, \omega), t \in T\} = 1$ and thus (i) is satisfied.

(i) *implies* (iii). This is obvious, with $r(u, v) = E(\eta_u \eta_v)$, when we note that the measurability of the second order process η implies that r is measurable and $\mathscr{R}(r)$ is separable [3, Theorem 1].

(i) *implies* (iv). Since (i) is equivalent to (iii), and (iii) is a condition on the two-dimensional distributions of ξ, it suffices to prove that (i) implies (iv) when ξ is Gaussian. In fact we will show that if ξ is Gaussian and its paths are absolutely continuous with probability one then (i) and (iv) are satisfied, proving thus the last claim of the theorem as well.

Since ξ has with probability one continuous paths it is product measurable [12, page 122]. If $T_d(\omega)$ is the set of points in T where the path $\xi(\cdot, \omega)$ is differentiable, the almost sure path absolute continuity of ξ implies $\text{Leb}\{T - T_d(\omega)\} = 0$ a.s. Also, if T_d is the set of points in T where the paths of ξ are differentiable with probability one, it is shown in [2, Theorem 3 (ii)] that $\text{Leb}\{T_d(\omega) \triangle T_d\} = 0$ a.s. It follows that $\text{Leb}\{T - T_d\} = 0$ and thus we will take $T_0 = T - T_d$. For every $t \in T_d = T - T_0$ the paths of ξ are differentiable with probability one, hence ξ is mean square differentiable at t and thus $\partial^2 R(t, s)/\partial t \, \partial s$ exists for all $t, s \in T_d$. Now let $\Omega_0 \in \mathscr{F}$ with $P(\Omega_0) = 0$ be such that for all $\omega \in \Omega - \Omega_0$, $\xi(\cdot, \omega)$ is absolutely continuous and define ζ by

$$\begin{aligned} \zeta(t, \omega) &= \limsup_{n \to \infty} n \left[\xi\left(t + \frac{1}{n}, \omega\right) - \xi(t, \omega) \right] && \text{for} \quad t \in [a, b), \ \omega \in \Omega - \Omega_0 \\ &= 0 && \text{for} \quad t \in [a, b), \ \omega \in \Omega_0 \ \text{and} \ t = b, \ \omega \in \Omega . \end{aligned}$$

Then ζ is product measurable and for all $\omega \in \Omega - \Omega_0$, we have $\zeta(t, \omega) = \xi'(t, \omega)$ for $t \in T_d(\omega) - \{b\}$, where $\xi'(\cdot, \omega)$ denotes the path derivative of $\xi(\cdot, \omega)$. Also for all $t \in T_d - \{b\}$, $\zeta(t, \omega) = \xi'(t, \omega)$ a.s. Now define η by $\eta(t, \omega) = \zeta(t, \omega)$ for $t \in T_d$, $\omega \in \Omega$, and $\eta(t, \omega) = 0$ for $t \in T_0$, $\omega \in \Omega$. It is clear that η is product measurable and also Gaussian, since for all $t \in [a, b) - T_0$, $\eta_t = \lim_{n\to\infty} n(\xi_{t+1/n} - \xi_t)$ a.s. and ξ is Gaussian. Also for all $\omega \in \Omega - \Omega_0$, $\eta(\cdot, \omega) = \zeta(\cdot, \omega) = \xi'(\cdot, \omega)$ a.s. [Leb] on T and since $\xi(\cdot, \omega)$ is absolutely continuous, $\xi'(\cdot, \omega) \in L_1[T]$. It follows that with probability one $\eta(\cdot, \omega) \in L_1[T]$ and hence by a result in [13, page 391] (1) is satisfied. (5) follows if we note that for all $u \in T_d - \{b\}$, $E(\eta_u^2) = E(\zeta_u^2) = E(\xi_u'^2) = (\partial^2 R(t, s)/\partial t\, \partial s)_{t=s=u}$. We also have

$$\xi(t, \omega) = \xi(a, \omega) + \int_a^t \eta(u, \omega)\, du \qquad \text{for} \quad t \in [a, b]\,, \quad \omega \in \Omega - \Omega_0\,.$$

Now an application of Fubini's theorem justified by (1) gives

$$R(t, s) = R(t, a) + \int_a^s E(\xi_t \eta_u)\, du \qquad t, s \in T$$

$$\Delta_{t_1}^{t_2} \Delta_{s_1}^{s_2} R = \int_{t_1}^{t_2} \int_{s_1}^{s_2} E(\eta_u \eta_v)\, du\, dv \qquad t_1, t_2, s_1, s_2 \in T$$

and since by (1) the functions inside the integrals are Lebesgue integrable, it follows that $R(t, \cdot)$ is absolutely continuous on T for every fixed $t \in T$ and that R is absolutely continuous on $T \times T$. Thus (i) and (iv) are satisfied.

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