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CHARACTERIZATIONS OF SOME STOCHASTIC PROCESSES[1]

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In this paper, we extend known characterizations of normal and other distributions. Let $X(t)$, $t \geqq 0$, be a continuous (in probability) homogeneous process, with independent increments. Let $g(s, t)$ and $h(s)$ be continuous functions on $[a, b]^2$ and $[a, b]$, $0 \leqq a < b < \infty$. Define stochastic integrals $Y_1 = \int_a^b h(s)X(ds)$ and $Y_2 = \int_a^b \int_a^b g(s, t)X(ds)X(dt)$. It is known that Y_1 exists in the sense of convergence in probability. It is shown here that Y_2 exists at least in the sense of convergence in L_2, under the additional assumption that X is of second-order. The main results of this paper are to obtain, under additional appropriate assumptions on g and h, characterizations of a class of stochastic processes which include the Brownian motion, Poisson, negative binomial and gamma processes, based on the linear regression of Y_2 on Y_1.

1. Introduction. One of the most well-known theorems on the characterization of probability distributions involves the independence of the sample mean and sample variance. Historically, it is the first theorem on characterization. In 1925, Fisher [2] proved that if the population distribution is normal, then the sample mean and sample variance must be independent. The important Student's t distribution was derived using this result. In 1936, Geary [3] proved that the independence of the sample mean and sample variance is a sufficient condition for the population distribution to be normal under the assumption that it has moments of all order. This theorem was later improved upon by many authors, and closely related characterization theorems were obtained for other distributions such as the Poisson, binomial, negative binomial and gamma distributions. For a review of the literature in this area, the readers are referred to [4], [8] and [9].

The purpose of this paper is to extend the above results to stochastic processes. We prove two characterization theorems for stochastic processes based on the linear regressions of double stochastic integrals on simple stochastic integrals. Our results include Brownian motion, Poisson, gamma and negative binomial processes. For recent characterization theorems of Brownian motion and Poisson processes, see [10].

2. Preliminaries. Let $X(t)$, $t \geqq 0$, be a stochastic process. Fix $[a, b]$, $0 \leqq a < b < \infty$. Let $g(x_1, \cdots, x_k)$ be a real-valued function defined on $[a, b]^k$

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and

(2.1) $$P_i = \{(t_{i0}, t_{i1}, \cdots, t_{in_i}) : a = t_{i0} < t_{i1} < \cdots < t_{in_i} = b\} , \quad i = 1, 2, \cdots ,$$

be finite sets of points defining partitions of $[a, b]$. Let $S(n_1, \cdots, n_k)$, $k \geqq 1$, denote the approximating sum corresponding to the partitions $P_1, \cdots, P_k$:

(2.2) $$S(n_1, \cdots, n_k) = \sum_{j_1=1}^{n_1} \cdots \sum_{j_k=1}^{n_k} g(t^*_{1j_1}, \cdots, t^*_{kj_k}) X(\Delta t_{1j_1}, \cdots, t_{kj_k}) ,$$

where $X(\Delta t_{1j_1}, \cdots, \Delta t_{kj_k}) = \prod_{i=1}^k [X(t_{ij_i}) - X(t_{i,j_i-1})]$ and $t_{i,j_i-1} \leqq t^*_{ij_i} \leqq t_{ij_i}$, $j_i = 1, \cdots, n_i$, $i = 1, \cdots, k$. Evidently, $S(n_1, \cdots, n_k)$ is a sequence of random variables determined by the partitions $P_1, \cdots, P_k$ and the intermediate points t^*_{ij}'s. Now let $n_1 \to \infty, \cdots, n_k \to \infty$ such that $\max_{j_1,\cdots,j_k} \{\prod_{i=1}^k (t_{ij_i} - t_{i,j_i-1})\} \to 0$. If $S(n_1, \cdots, n_k)$ converges uniquely in probability or in L_2 to a limiting random variable, and if the limit is independent of the partitions $P_1, \cdots, P_k$ and the choices of the intermediate points t^*_{ij}'s, then the limiting random variable is called a stochastic integral which we write as

(2.3) $$Y_k = \int_a^b \cdots \int_a^b g(t_1, \cdots, t_k) X(dt_1) \cdots X(dt_k) .$$

We say that the function $\gamma(t_1, \cdots, t_l)$, $l \geqq 1$, is a function of bounded variation in $[a, b]$ if there exists a constant C, $0 < C < \infty$, independent of the partitions $P_1, \cdots, P_l$ but possibly depending on the interval $[a, b]$, for which

(2.4) $$\sum_{i_1=1}^{n_1} \cdots \sum_{i_l=1}^{n_l} \Delta\gamma(t_{i_1}, \cdots, t_{i_l}) < C ,$$

for all partitions $P_1, \cdots, P_l$, where

(2.5) $$\Delta\gamma(t_{i_1}, \cdots, t_{i_l}) = \prod |\gamma(t_{i_1}, \cdots, t_{i_l}) - (t_{i_1'}, \cdots, t_{i_l'})|$$

and the product is taken for all $(i_1', \cdots, i_l')$ such that exactly one coordinate of $(i_1', \cdots, i_l')$ equals the corresponding one of $(i_1 - 1, \cdots, i_l - 1)$ and the rest $(l - 1)$ are equal to those of $(i_1, \cdots, i_l)$. (This definition of "bounded variation" is a generalization of the one given by Loève on page 473 in [5].)

If $k = 1$, it is known (c.f. [6], pages 103–7) that: a) If the covariance function $\gamma(t, s) = E[X(t)X(s)]$ is a function of bounded variation in $[a, b]$ and g is a continuous function in $[a, b]$, then the stochastic integral Y_1 defined by (2.3) exists in the sense of convergence in L_2; b) If X is a continuous (in probability) homogeneous process with independent increments and if g is a continuous function in $[a, b]$, then the stochastic integral Y_1 defined by (2.3) exists in the sense of convergence in probability. Since for a continuous, second-order, homogeneous process with independent increments, the covariance function $\gamma(t, s) = E[X(t)X(s)]$ is a function of bounded variation in $[a, b]$, if in b) we require that X be of second order, then the stochastic integral Y_1 defined by (2.3) exists also in the sense of convergence in L_2.

If $k \geqq 2$, it is an open question whether the stochastic integral Y_k defined by (2.3) exists in the sense of convergence in probability, even under the assumption that X is a continuous and homogeneous process with independent increments, and g is a continuous function in $[a, b]^k$. We shall prove the following theorem, which gives the existence of the stochastic integral Y_k defined by (2.3) in the sense of convergence in L_2.

THEOREM 2.1. *Suppose that g is a continuous function in $[a, b]^k$ and the covariance function $\gamma(t_1, \cdots, t_k, s_1, \cdots, s_k) = E[X(t_1) \cdots X(t_k) X(s_1) \cdots X(s_k)]$ is a function of bounded variation in $[a, b]$. Then the stochastic integral Y_k defined by* (2.3) *exists in the sense of convergence in L_2.*

PROOF. We show that the approximating sum defined by (2.2) is a Cauchy sequence in L_2, for which it suffices to show that

$$\lim_{n_i, n_i' \to \infty, 1 \leq i \leq k} E[S(n_1, \cdots, n_k)S(n_1', \cdots, n_k')] \tag{2.6}$$

exists and is independent of the manner in which n_i and n_i' tend to infinity.

It follows from (2.2) that

$$\begin{aligned} E[S(n_1, \cdots, n_k)S(n_1', \cdots, n_k')] \\ = \sum_{i_1=1}^{n_1} \cdots \sum_{j_k=1}^{n_k'} g(t_{i_1}^*, \cdots, t_{i_k}^*)g(s_{j_1}^*, \cdots, s_{j_k}^*)\gamma(\Delta t_{i_1}, \cdots, \Delta s_{j_k}) \end{aligned} \tag{2.7}$$

with

$$\begin{aligned} \gamma(\Delta t_{i_1}, \cdots, \Delta t_{i_k}, \Delta s_{j_1}, \cdots, \Delta s_{j_k}) \\ = \sum_{m=0}^{2k} (-1)^m \sum_m \gamma(t_{i_1}, \cdots, t_{i_k}, s_j, \cdots, s_{j_k}) \end{aligned} \tag{2.8}$$

and $\sum_m$ denotes the summation of all possible $\gamma(t_{i_1}, \cdots, t_{i_k}, s_{j_1}, \cdots, s_{j_k})$ such that exactly m $(0 \leq m \leq 2k)$ of $i_1, \cdots, i_k, j_1, \cdots, j_k$ equal $i_1 - 1, \cdots, i_k - 1, j_1 - 1, \cdots, j_k - 1$ and the rest $(2k - m)$ are equal to $i_1, \cdots, i_k, j_1, \cdots, j_k$.

Since by our assumption the covariance function γ is a function of bounded variation in $[a, b]$, the right-hand side of (2.8) converges absolutely, therefore the limit (2.6) exists.

It follows from (2.8) that if we let $n_i \to \infty$, $n_i' \to \infty$ for all i, the limit of (2.6) is

$$\int_a^b \cdots \int_a^b g(t_1, \cdots, t_k)g(s_1, \cdots, s_k)\gamma(dt_1, \cdots, ds_k) . \tag{2.9}$$

This fact gives us the following theorem. A stochastic process $X(t)$ is said to be a lth order process if $E|X(t)|^l < \infty$ for all t.

THEOREM 2.2. *Suppose that X is a* $2k$th *order process with covariance function $\gamma(t_1, \cdots, t_k, s_1, \cdots, s_k) = E[X(t_1) \cdots X(t_k) X(s_1) \cdots X(s_k)]$ and let g be a continuous function in $[a, b]^k$. Then the stochastic integral* (2.3) *exists if and only if the Riemann–Stieltjes integral* (2.9) *exists.*

From now on we shall assume that X is a continuous (in probability) homogeneous process with independent increments. Also to make sure that the double integrals Y_2' and Y_2'' defined by (2.11) below exist, at least in the sense of convergence in L_2, we shall assume that X is a process of second order.

Let $g_1(t, s)$ and $g_2(t, s)$ be two continuous real-valued functions defined on $[a, b]^2$ such that

$$\begin{aligned} &\int_a^b \int_a^b g_1(t, s)\, dt\, ds = 0 , \qquad \int_a^b \int_a^b g_2(t, s)\, dt\, ds = c \neq 0 \qquad \text{and} \\ &\int_a^b g_1(t, t)\, dt = \int_a^b g_2(t, t)\, dt = 1 . \end{aligned} \tag{2.10}$$

(The "1" in (2.10) is taken for convenience. In the proofs in Section 3, we only require that the corresponding integrals do not vanish.)

We define two double stochastic integrals Y_2' and Y_2'' by

$$Y_2' = \int_a^b \int_a^b g_1(t, s)X(dt)X(ds) \quad \text{and} \quad Y_2'' = \int_a^b \int_a^b g_2(t, s)X(dt)X(ds)\,. \tag{2.11}$$

The following results are well known, and their proofs can be found in many standard textbooks on probability theory, such as Chung [1] and Lòeve [5].

LEMMA 2.1. *Suppose X is a continuous (in probability) homogeneous process with independent increments, then:*

a) *X is infinitely divisible and hence its characteristic function $\phi(u, t) = E[e^{iuX(t)}]$ never vanishes for all real u and $t \geqq 0$.*

b) *$\phi(u, t) = \phi^t(u)$, for all $t \geqq 0$, where $\phi(u) = \phi(u, 1)$.*

If, in addition, we assume that X is of second order, then:

c) $E[X(t)] = t\mu$, $\operatorname{Var}[X(t)] = t\sigma^2$ *and* $\operatorname{Cov}[X(t), X(s)] = \sigma^2 \min(t, s)$ *for all* $t, s \geqq 0$, *where* $\mu = E[X(1)]$ *and* $\sigma^2 = \operatorname{Var}[X(1)]$.

d) *For the stochastic integrals defined by* (2.11), *we have* $E(Y_2') = \sigma^2$ *and* $E(Y_2'') = \sigma^2 + \mu^2$. *If in* (2.3) *we require that* $\int_a^b g(t)\,dt = 1$, *then* $E(Y_1) = \mu$.

Let $a \leqq a_i < b_i \leqq b$, $i = 1, 2$, be such that $[a_1, b_1]^2$ and $[a_2, b_2]^2$ are two smallest squares inside $[a, b]^2$ satisfying $g_i(t, s) = 0$ for $(t, s) \notin [a_i, b_i]^2$, $i = 1, 2$. Also let I_i, $i = 1, 2$, be any intervals satisfying $[a_i, b_i] \subseteqq I_i \subseteqq [a, b]$, $i = 1, 2$. Define

$$h_i(t) = 1/(\text{length of } I_i)\,, \quad \text{if} \quad t \in I_i \quad \text{and} \quad = 0 \quad \text{elsewhere,} \quad i = 1, 2\,. \tag{2.12}$$

Suppose $a = a_i$, $b = b_i$, $i = 1, 2$, then $I_i = [a_i, b_i] = [a, b]$ and $h_i(t) = (b - a)^{-1}$ for $t \in [a, b]$ and $= 0$ elsewhere, $i = 1, 2$.

We define two simple integrals Y_1' and Y_1'' by

$$Y_1' = \int_a^b h_1(t)X(dt) \quad \text{and} \quad Y_1'' = \int_a^b h_2(t)X(dt)\,. \tag{2.13}$$

It follows from Lemma 2.1-d) that $E(Y_1') = E(Y_2'') = \mu = E[X(1)]$.

3. The results. The first theorem we show is a characterization of a class of stochastic processes which includes the Brownian motion process with linear mean value function, and the Poisson process based on the property that the regression of Y_2' and Y_1' is a.e. linear. A homogeneous process X with independent increments is said to be 1) a Brownian motion process (also Wiener process or Wiener–Lévy process) with linear mean value function if the increments $X(t + \tau) - X(t)$ have normal distributions with means $\mu\tau$ and variances $\sigma^2\tau$, for all $\tau \geqq 0$, 2) a Poisson process if the increments $X(t + \tau) - X(t)$ have Poisson distributions with parameters $\lambda\tau$, for all $\tau \geqq 0$. It follows from Lemma 2.1-d) that the two stochastic integrals Y_1' and Y_2' as defined by (2.13) and (2.11) are unbiased for μ and σ^2, the location and scale parameters of the process X, and therefore they are the stochastic process versions of the sample mean and variance. Since the Brownian motion with linear mean value function is a stochastic process version of the normal random variable, we can expect to obtain a characterization theorem of it based on the property $E(Y_2' \mid Y_1') =$ constant a.e. Similarly, for the Poisson process, the location and scale parameters are equal, and we look forward to characterizing the Poisson process with the property $E(Y_2' \mid Y_1') = Y_1'$ a.e.

For convenience and without loss of generality we shall assume that $[a, b] = [a_i, b_i] = I_i = [0, 1]$, $i = 1, 2$.

LEMMA 3.1. *Suppose that $X(t)$, $t \geqq 0$, is a continuous (in probability) homogeneous second-order process with independent increments and let $\phi(u, t) = E[e^{iuX(t)}]$ be the characteristic function of $X(t)$ and $\phi(u) = \phi(u, 1)$. Then*

a) *$E[(X(t_j) - X(t_{j-1}))e^{iuX(1)}] = (1/i)(t_j - t_{j-1})\phi'(u)$, for all $0 \leqq t_{j-1} \leqq t_j \leqq 1$ and all real u.*

b) *If $(s_{i-1}, s_i) \cap (t_{j-1}, t_j) = \emptyset$, then $E[(X(s_i) - X(s_{i-1}))(X(t_j) - X(t_{j-1}))e^{iuX(1)}] = -(s_i - s_{i-1})(t_j - t_{j-1})[\phi'(u)]^2\phi^{-1}(u)$, for all $0 \leqq s_{i-1} \leqq s_i \leqq 1$, $0 \leqq t_{j-1} \leqq t_j \leqq 1$ and all real u.*

c) *If $(s_{i-1}, s_i) = (t_{j-1}, t_j)$, then $E[(X(s_i) - X(s_{i-1}))(X(t_j) - X(t_{j-1}))e^{iuX(1)}] = -(s_i - s_{i-1})^2[\phi'(u)]^2\phi^{-1}(u) + (s_i - s_{i-1})\{[\phi'(u)]^2\phi^{-1}(u) - \phi''(u)\}$, for all $0 \leqq s_{i-1} \leqq s_i \leqq 1$ and all real u.*

PROOF. Note first that by Lemma 2.1-a) $\phi^{-1}(u)$ in b) and c) above are well defined for all real u.

a)
$$\begin{aligned} &E[(X(t_j) - X(t_{j-1}))e^{iuX(1)}] \\ &\quad = E[(X(t_j) - X(t_{j-1}))e^{iu(X(t_j)-X(t_{j-1}))}]\phi(u, 1 - t_j)\phi(u, t_{j-1}) \\ &\quad = \frac{1}{i}\frac{\partial}{\partial u}\phi(u, t_j - t_{j-1})\phi(u, 1 - t_j + t_{j-1}) = \frac{1}{i}(t_j - t_{j-1})\phi'(u) . \end{aligned}$$

b)
$$\begin{aligned} &E[(X(s_i) - X(s_{i-1}))(X(t_j) - X(t_{j-1}))e^{iuX(1)}] \\ &\quad = E[(X(s_i) - X(s_{i-1}))e^{iu(X(s_i)-X(s_{i-1}))}] \\ &\quad\quad \times E[(X(t_j) - X(t_{j-1}))e^{iu(X(t_j)-X(t_{j-1}))}] \\ &\quad\quad \times \phi(u, 1 - t_j + t_{j-1} - s_i + s_{i-1}) \\ &\quad = \left[\frac{1}{i}\frac{\partial}{\partial u}\phi(u, s_i - s_{i-1})\right]\left[\frac{1}{i}\frac{\partial}{\partial u}\phi(u, t_j - t_{j-1})\right] \\ &\quad\quad \times \phi(u, 1 - t_j + t_{j-1} - s_i + s_{i-1}) \\ &\quad = -(s_i - s_{i-1})(t_j - t_{j-1})[\phi'(u)]^2\phi^{-1}(u) . \end{aligned}$$

The proof of c) proceeds similarly and is omitted.

THEOREM 3.1. *Suppose $X(t)$, $t \geqq 0$, is a continuous (in probability) homogeneous second-order process with independent increments, and let Y_1' and Y_2' be defined as in (2.13) and (2.11), respectively. Then the conditional expectation*

$$E(Y_2' \mid Y_1') = \alpha Y_1' + \beta \tag{3.1}$$

holds a.e., *where α and β are real constants, if and only if*

(1) $\alpha = 0$, *and X is a Brownian motion process with linear mean value function.*
(2) $\alpha \neq 0$, *and* $X(t) = \alpha Y(t) + (\beta/\alpha)t$, *where Y is a Poisson process.*

PROOF. As we have mentioned earlier, it is assumed, without loss of generality, that $[a, b] = [a_i, b_i] = I_i = [0, 1]$, $i = 1, 2$. It follows from Theorem 6.1.1 in Lukacs and Laha [7] that the condition (3.1) is equivalent to

$$E[Y_2' e^{iuX(1)}] = \alpha E[X(1) e^{iuX(1)}] + \beta E[e^{iuX(1)}], \qquad \text{for all real } u. \tag{3.2}$$

The left-hand side of (3.2) can be expressed as

$$\begin{aligned} E[Y_2' e^{iuX(1)}] = \lim \textstyle\sum_{j=1}^n \sum_{k=1}^m g_1(t_j^*, s_k^*) \\ \times E[(X(t_j) - X(t_{j-1}))(X(s_k) - X(s_{k-1})) e^{iuX(1)}], \end{aligned} \tag{3.3}$$

where $0 = t_0 < t_1 < \cdots < t_n = 1$, $0 = s_0 < s_1 < \cdots < s_m = 1$, $t_{j-1} \leqq t_j^* \leqq t_j$, $s_{k-1} \leqq s_k^* \leqq s_k$ and the limit is taken in such a way that $\max_{j,k} [t_j - t_{j-1}][s_k - s_{k-1}] \to 0$, as $n, m \to \infty$. To simplify our calculation, we shall assume, without loss of generality, that $n = m$ and $s_i = t_i$ for all $i = 0, 1, \cdots, n$. (Otherwise, a finer partition which includes t_i's and s_j's may be taken.) Then, using Lemma 3.1-b) and c), we have

$$\begin{aligned} &\textstyle\sum_{j=1}^n \sum_{k=1}^n g_1(t_j^*, s_k^*) E[(X(t_j) - X(t_{j-1}))(X(s_k) - X(s_{k-1})) e^{iuX(1)}] \\ &\quad = \textstyle\sum_{j=1}^n g_1(t_j^*, s_j^*) E[(X(t_j) - X(t_{j-1}))^2 e^{iuX(1)}] \\ &\qquad + \textstyle\sum_{j=1}^n \sum_{k \neq j} g_1(t_j^*, s_k^*) E[(X(t_j) - X(t_{j-1}))(X(s_k) - X(s_{k-1})) e^{iuX(1)}] \\ &\quad = -\textstyle\sum_{j=1}^n \sum_{k=1}^n g_1(t_j^*, s_k^*)(t_j - t_{j-1})(s_k - s_{k-1})[\phi'(u)]^2 \phi^{-1}(u) \\ &\qquad + \textstyle\sum_{j=1}^n g_1(t_j^*, s_j^*)(t_j - t_{j-1})\{[\phi'(u)]^2 \phi^{-1}(u) - \phi''(u)\}. \end{aligned} \tag{3.4}$$

Letting $n \to \infty$ with $\max_{i,j} (t_j - t_{j-1})(s_i - s_{i-1}) \to 0$, the first summation in (3.4) converges to

$$-\textstyle\int_0^1 \int_0^1 g_1(t, s)\, dt\, ds [\phi'(u)]^2 \phi^{-1}(u) = 0. \tag{3.5}$$

The second summation in (3.4) converges to (note, we have $t_j = s_j$, for all j)

$$\textstyle\int_0^1 g_1(t, t)\, dt \{[\phi'(u)]^2 \phi^{-1}(u) - \phi''(u)\} = [\phi'(u)]^2 \phi^{-1}(u) - \phi''(u). \tag{3.6}$$

Therefore, from Lemma 3.1-a), and equations (3.5) and (3.6), the condition (3.1) is equivalent to

$$[\phi'(u)]^2 \phi^{-1}(u) - \phi''(u) = \frac{\alpha}{i} \phi'(u) + \beta\phi(u), \qquad \text{for all real } u. \tag{3.7}$$

or (for detail from here on see [8]),

$$\frac{d}{du}\left(\frac{d}{du}\phi(u)\Big/\phi(u)\right) = \alpha i \left(\frac{d}{du}\phi(u)\Big/\phi(u)\right) - \beta, \qquad \text{for all real } u. \tag{3.8}$$

From (3.8), we conclude that if $\alpha = 0$

$$\phi(u) = \exp\{i\mu u - \tfrac{1}{2}\sigma^2 u^2\}, \qquad \text{for all real } u, \tag{3.9}$$

where μ is a real number and $\sigma^2 > 0$, both independent of u. If $\alpha \neq 0$

$$\phi(u) = \exp\{-i(\beta/\alpha)u + \lambda(e^{i\alpha u} - 1)\}, \qquad \text{for all real } u \cdot \tag{3.10}$$

where λ is a nonnegative real constant independent of u. The proof is completed by Lemma 2.1-b).

REMARK. It follows from Theorem 3.1 that if $\alpha = 1$ and $\beta = 0$, that is $E(Y_2' \mid Y_1') = Y_1'$ a.e., then we have as expected, a characterization theorem of the Poisson process.

Next, we present a characterization theorem for the gamma and negative binomial processes. A homogeneous process $X(t)$ with independent increments is said to be 1) a gamma process, if the increments $X(t+\tau) - X(t)$ have gamma distributions with parameters $\alpha\tau$ ($\alpha > 0$) and $\beta > 0$, for all $\tau \geqq 0$, 2) a negative binomial process, if the increments $X(t+\tau) - X(t)$ have negative binomial distributions with parameters $r\tau$ ($r \geqq 0$) and p ($0 < p \leqq 1$), for all $\tau \geqq 0$. Since, by Lemma 2.1-d) the two stochastic integrals Y_1'' and Y_2'' as defined by (2.13) and (2.11) are unbiased for μ and $\sigma^2 + \mu^2$, they are two stochastic process versions of the sample mean and the quadratic statistic $S^{2\prime} = (1/n) \sum_{i=1}^n X_i^2$. The following theorem is a logical extension of Theorem 3.1 in [8].

THEOREM 3.2. *Suppose $X(t)$, $t \geqq 0$ is a continuous (in probability) homogeneous second-order process with independent increments, and let Y_1'' and Y_2'' be as defined by* (2.13) *and* (2.11), *respectively. Then the conditional expectation*

$$E(Y_2'' \mid Y_1'') = \alpha Y_1'' \tag{3.11}$$

holds a.e., *where α is a real number, if and only if*

(1) $\alpha = 0$, *and X or $-X$ is a gamma process,*
(2) $\alpha \neq 0$, $c < 0$, *and $X(t) = \alpha Y(t)$, where Y is a negative binomial process.*

PROOF. Following the steps in the proof of Theorem 3.1, we can rewrite the condition (3.11) as

$$\frac{d}{du}\left(\frac{d}{du}\,\phi^c(u)\right) = \alpha i \left(\frac{d}{du}\,\phi^c(u)\right), \qquad \text{for all real } u . \tag{3.12}$$

From (3.12) it follows that if $\alpha = 0$, then

$$\phi(u) = (1 - i\theta u)^{-\lambda}, \qquad \text{for all real } u . \tag{3.13}$$

where $\lambda = -c^{-1} > 0$ and θ is a real constant independent of u. If $\alpha \neq 0$ and $c < 0$, then

$$\phi(u) = \left(\frac{p}{1 - qe^{i\alpha u}}\right)^r, \qquad \text{for all real } u . \tag{3.14}$$

where $r = -c^{-1} > 0$, $0 < p \leqq 1$ and $q = 1 - p$. The proof is completed by Lemma 2.1-b).

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