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A GENERALIZED SHANNON-MCMILLAN THEOREM FOR THE ACTION OF AN AMENABLE GROUP ON A PROBABILITY SPACE

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A generalization of the Shannon-McMillan Theorem (L^1 version) is obtained for the action of an amenable group on a probability space, thereby settling a conjecture of Pickel and Stepin. Interesting properties of the limit function are derived. The entropy of an action of an amenable group is defined.

Let G be a group and let $(\Omega, \mathscr{M}, \lambda)$ be a probability space. Let $T: G \times \Omega \to \Omega$ be an action of G on Ω; that is, for each $g \in G, T(g, \cdot): \Omega \to \Omega$ is measurable and measure-preserving and $T(g_1 g_2, \omega) = T(g_1, T(g_2, \omega))$, $g_1, g_2 \in G$, $\omega \in \Omega$. (From now on, we will write T^g for the map $T(g, \cdot)$.) We suppose from now on that G is amenable; that is, there exists an invariant mean on $B(G)$, the space of bounded real-valued functions defined on G. (The amenable groups have not been completely characterized as yet; finite groups and abelian groups are amenable, and any extension of an amenable group by an amenable group is amenable. Thus, for example, solvable groups are amenable.)

If E is a set, let $|E|$ denote the cardinality of E. Let K be a subgroup of G. Then K is amenable and there exists a net $\{A_\alpha\}$ of finite nonempty subsets of K such that

$$(1) \qquad \lim_\alpha |A_\alpha|^{-1}|gA_\alpha \cap A_\alpha| = 1\,, \quad g \in K\,.$$

(See [4]; conversely K is amenable if a net $\{A_\alpha\}$ of finite nonempty subsets of K exists satisfying (1).) From now on, we say that a net $\{A_\alpha\}$ satisfies property $\mathscr{P}$ with respect to a subgroup K if $\{A_\alpha\}$ is a net of finite nonempty subsets of K satisfying (1).

If Q is a partition of Ω, and $\omega \in \Omega$, let $Q(\omega)$ be the element of the partition Q which contains ω. Let P be a fixed countable measurable partition of Ω with finite entropy H. If E is a finite nonempty subset of G, let $h(E) \in L^1(\Omega)$ be the measurable function such that

$$h(E) = -\log \lambda[\{\bigvee_{g \in E} (T^g)^{-1}P\}(\omega)]\,, \quad \omega \in \Omega\,.$$

The Shannon-McMillan theorem states that if $G = Z$, the group of integers, and $A_n = \{0, 1, \cdots, n\}$, $n = 1, 2, \cdots$, then $|A_n|^{-1}h(A_n)$ converges in $L^1(\Omega)$ as $n \to \infty$ [1]. A generalization of the Shannon-McMillan theorem for the group

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Z^k (where k is a positive integer) has appeared which states that if $A_n = \{(x_1, x_2, \cdots, x_k) \in Z^k : 0 \leqq x_i \leqq n, i = 1, 2, \cdots, k\}$, $n = 1, 2, \cdots,$ then $|A_n|^{-1}h(A_n)$ converges in $L^1(\Omega)$. (See [5], [6], [12].) Pickel and Stepin [10] have shown that if G is the group of dyadic rationals modulo one and if A_n is the cyclic subgroup of G generated by 2^{-n}, then $|A_n|^{-1}h(A_n)$ converges in $L^1(\Omega)$ as $n \to \infty$. Pickel and Stepin conjectured that a result of this type holds for every countable abelian group G. The main result of this paper is the following theorem which includes these results as special cases, and settles the conjecture of Pickel and Stepin. (In the following, we say that a function $f \in L^1(\Omega)$ is invariant with respect to a subgroup K if for each $g \in K$, $f \cdot T^g = f$ a.e. $[\lambda]$.)

THEOREM 1. *Let K be any subgroup of the amenable group G. There exists a function $\not{h}(K) \in L^1(\Omega)$, invariant with respect to K, such that for every net $\{A_\alpha\}$ satisfying property $\mathscr{P}$ with respect to K, $\lim_\alpha |A_\alpha|^{-1}h(A_\alpha) = \not{h}(K)$ in $L^1(\Omega)$.*

It turns out that Theorem 1 will follow from Theorem 3, which is a special case of Theorem 1 for countable subgroups. We therefore have to defer our proof of Theorem 1 until after Theorem 3 is proved.

DEFINITIONS. If $g \in G$ and A is a nonempty subset of G, let $h(g \mid A) \in L^1(\Omega)$ be the function such that

$$h(g \mid A)(\omega) = -\log \lambda[\{(T^g)^{-1}P\}(\omega) \mid \mathscr{M}_A](\omega)\,, \quad \omega \in \Omega\,,$$

where $\mathscr{M}_A$ is the smallest subsigma-field of $\mathscr{M}$ containing $\bigcup_{g \in A} (T^g)^{-1}P$. Let $h(g \mid \phi) = h(\{g\})$. Let $H(g \mid A) = \int h(g \mid A)\, d\lambda$, $H(E) = \int h(E)\, d\lambda$. Let e be the identity of G. If K is a subgroup of G, let $\mathscr{C}(K) = \{M \in \mathscr{M}$: for each $g \in G$, $\lambda[T^g(M) \triangle M] = 0\}$. We remark that a function $f \in L^1(\Omega)$ is invariant with respect to K if and only if f is measurable with respect to $\mathscr{C}(K)$.

The following lemma is easily proved and is left to the reader.

LEMMA 1. *If $\{A_\alpha\}$ satisfies property $\mathscr{P}$ with respect to the subgroup K, and if E is a finite subset of K, $\lim_\alpha |A_\alpha|^{-1}|\bigcap_{g \in E} gA_\alpha| = 1$.*

LEMMA 2. a) *If $\{E_n\}$ is a sequence of subsets of G and $E_n \uparrow E$, then $h(e \mid E_n) \to h(e \mid E)$ in $L^1(\Omega)$ and* a.e. $[\lambda]$. *Furthermore, $\int \sup_n h(e \mid E_n)\, d\lambda \leqq H + 1$.*

b) $h(e \mid E)T^g = h(g \mid Eg)$ a.e. $[\lambda]$, $g \in G$.

c) $H(e \mid E) \leqq H(e \mid F)$, $E \supset F$.

d) *If $E \supset F$ and $H(e \mid E) = H(e \mid F)$, then $h(e \mid E) = h(e \mid F)$* a.e. $[\lambda]$.

PROOF. Parts a) and c) may be found in [9], Chapters 1 and 2. Part b) is easily shown from the definition. Part d) follows because if the conditional information $H(e \mid F) - H(e \mid E)$ is zero, then the conditional information density $h(e \mid F) - h(e \mid E)$ is zero almost everywhere ([11], page 35).

A total order of the set W is a transitive relation on W such that if $x, y \in W$ exactly one of the following holds: $x < y$, $y < x$, $x = y$.

Let K_1 be a fixed countable subgroup of G. Corresponding to K_1, construct a probability space $(S, \mathscr{S}, \mu)$, an action U of K_1 on S, and a total order $<$ of S

such that the following two properties hold:

(2.1) For each $s \in S$, if g_1 and g_2 are distinct elements of K_1, then $U^{g_1}(s)$ and $U^{g_2}(s)$ are distinct elements of S.

(2.2) For each $g \in K_1$, $\{s \in S : U^g(s) < s\} \in \mathscr{S}$.

To see that one can do this, if K_1 is finite take $S = K_1$ with each point of S measurable with measure $|S|^{-1}$, and let $<$ be an arbitrary total order of S. Let K_1 act on S by left multiplication.

If K_1 is countably infinite, one can assign to $\{0, 1\}^{K_1}$ (the set of all maps from K_1 to $\{0, 1\}$, or equivalently, the Cartesian product of a countable number of copies of $\{0, 1\}$, indexed by K_1) the usual product sigma-field, and then an appropriate product measure. (A measure which induces in each factor $\{0, 1\}$ of the product space $\{0, 1\}^{K_1}$ the measure p such that $p(\{0\}) = p(\{1\}) = \frac{1}{2}$, will do.) Let the action of K_1 on $\{0, 1\}^{K_1}$ be left translation. It is not hard to show that there is a measurable subset S of $\{0, 1\}^{K_1}$ of measure one, invariant under the action of K_1, such that (2.1) holds. One then restricts the action of K_1 to the subspace S and orders S by a lexicographical order. Then (2.2) will also hold.

DEFINITIONS. For each $s \in S$ let $<_s$ be the total order of K_1 such that if $g_1, g_2 \in K_1$, $g_1 <_s g_2$ if and only if $U^{g_1}(s) < U^{g_2}(s)$. Let $V_g(s) = \{g' \in K_1 : g' <_s g\}$, $g \in K_1$, $s \in S$. Let $(S \times \Omega, \mathscr{S} \times \mathscr{M}, \mu \times \lambda)$ be the probability space which is the Cartesian product of $(S, \mathscr{S}, \mu)$ and $(\Omega, \mathscr{M}, \lambda)$.

LEMMA 3. *For each $E \subset K_1$ there exists a jointly measurable function $\phi_E \in L^1(S \times \Omega)$ such that*

a) *For almost every s with respect to μ, $\phi_E(s, \cdot) = h(e \mid E \cap V_e(s))(\cdot)$, a.e.* $[\lambda]$;

b) *If $\{E_n\}$ is a sequence of subsets of K_1 and $E_n \uparrow E$, then $\phi_{E_n} \to \phi_E$ in $L^1(S \times \Omega)$;*

c) *If K_2 is a subgroup of K_1 and $\mathscr{F}$ is the family of all finite subsets of K_2 directed by inclusion ($\supset$), then* $\lim_{F \in \mathscr{F}} \phi_F = \phi_{K_2}$ *in* $L^1(S \times \Omega)$.

PROOF. Using condition (2.2) above, for each finite subset E of K_1, ϕ_E can be defined as a jointly measurable function satisfying a). If $F \subset K_1$ is not finite, choose a sequence $\{F_n\}$ of finite sets such that $F_n \uparrow F$. Then by a) and Lemma 2a, ϕ_{F_n} converges almost everywhere to a jointly measurable function we will call ϕ_F which satisfies a). Now if $E_n \uparrow E$, then $\phi_{E_n} \to \phi_E$ a.e. $[\mu \times \lambda]$ by Lemma 2a and a). Also by Lemma 2a, $\sup_n \phi_{E_n} \in L^1(S \times \Omega)$ and so by the dominated convergence theorem, $\phi_{E_n} \to \phi_E$ in $L^1(S \times \Omega)$. For the proof of c), if $\lim_{F \in \mathscr{F}} \phi_F \neq \phi_{K_2}$ then there exists a sequence $\{F_n\}$ from $\mathscr{F}$ such that $F_n \uparrow K_2$ and $\lim_n \phi_{F_n} \neq \phi_{K_2}$, which is a contradiction of b).

We will make essential use of the following generalized ergodic theorem due to Chatard [2].

THEOREM 2. *Let K be a subgroup of G. Let the sequence $\{A_n\}$ satisfy property $\mathscr{P}$ with respect to K. Then if $f \in L^1(\Omega)$* $\lim_n |A_n|^{-1} \sum_{g \in A_n} f \cdot T^g = E(f \mid \mathscr{C}(K))$ *in* $L^1(\Omega)$.

THEOREM 3. *Let K_1 be a countable subgroup of G. Let K_2 be any subgroup of K_1. Let the sequence $\{A_n\}$ satisfy condition $\mathscr{P}$ with respect to K_2. Then $|A_n|^{-1}h(A_n) \to E(\int_S \phi_{K_2}\, d\mu \mid \mathscr{C}(K_2))$ in $L^1(\Omega)$.*

PROOF. From the definition of conditional probability, it easily follows that $h(A_n)(\omega) = \sum_{g \in A_n} h(g \mid A_n \cap V_g(s))(\omega)$, a.e. $[\mu \times \lambda]$. Using Lemma 2b, we see that $h(A_n)(\omega) = \sum_{g \in A_n} h(e \mid A_n g^{-1} \cap V_g(s)g^{-1})(T^g\omega)$. Since $V_g(s)g^{-1} = V_e(U^g s)$, $h(A_n)(\omega) = \sum_{g \in A_n} \phi_{A_n g^{-1}}(U^g s, T^g\omega)$. Integrating with respect to μ, $h(A_n) = \sum_{g \in A_n} [\int \phi_{A_n g^{-1}}\, d\mu] \cdot T^g$, a.e. $[\lambda]$. Now from Theorem 2, we need only show that $\lim_{n\to\infty} \{|A_n|^{-1}h(A_n) - |A_n|^{-1} \sum_{g \in A_n} [\int \phi_{K_2}\, d\mu] \cdot T^g\} = 0$, in $L^1(\Omega)$. We have $||h(A_n) - \sum_{g \in A_n} [\int \phi_{K_2}\, d\mu] \cdot T^g|| \leqq \sum_{g \in A_n} ||\phi_{A_n g^{-1}} - \phi_{K_2}||'$, where $||\cdot||$ denotes the $L^1(\Omega)$ norm, and $||\cdot||'$ denotes the $L^1(S \times \Omega)$ norm. It suffices then to show that $\lim_n |A_n|^{-1} \sum_{g \in A_n} ||\phi_{A_n g^{-1}} - \phi_{K_2}||' = 0$. Fix $\varepsilon > 0$. Using Lemma 3c, choose a finite subset B of K_2 such that $||\phi_A - \phi_{K_2}||' < \varepsilon$ for each finite subset A of K_2 such that $A \supset B$. Let $Z_n = \bigcap_{b \in B} b^{-1}A_n$, $n = 1, 2, \cdots$. Let Z_n^c be the complement of Z_n in G. Now if $g \in Z_n$, then $A_n g^{-1} \supset B$. Thus $\sum_{g \in A_n} ||\phi_{A_n g^{-1}} - \phi_{K_2}||' = \sum_{g \in A_n \cap Z_n} ||\phi_{A_n g^{-1}} - \phi_{K_2}||' + \sum_{g \in A_n \cap Z_n^c} ||\phi_{A_n g^{-1}} - \phi_{K_2}||' \leqq \varepsilon|A_n \cap Z_n| + 2H(A_n \cap Z_n^c)$. By Lemma 1, $\lim_n |A_n|^{-1}|A_n \cap Z_n| = 1$ and $\lim_n |A_n|^{-1}|A_n \cap Z_n^c| = 0$. Thus $\limsup_n |A_n|^{-1} \sum_{g \in A_n} ||\phi_{A_n g^{-1}} - \phi_{K_2}||' \leqq \varepsilon$. This completes the proof.

PROOF OF THEOREM 1.

PART I. Let $\{A_\alpha\}$ be a net satisfying property $\mathscr{P}$ with respect to K. Suppose $\lim_\alpha |A_\alpha|^{-1}h(A_\alpha)$ does not exist in $L^1(\Omega)$. Then for some $\varepsilon > 0$, we may define inductively a sequence $\{E_n\}_0^\infty$ of finite subsets of K and a sequence $\{C_n\}_0^\infty$ extracted from $\{A_\alpha\}$ such that

a) C_0 is any A_α and $E_0 = \{e\}$;
b) $|||C_n|^{-1}h(C_n) - |C_{n-1}|^{-1}h(C_{n-1})|| \geqq \varepsilon$, $n \geqq 1$;
c) $|C_n|^{-1}|\bigcap_{g \in E_{n-1}} gC_n| \geqq 1 - n^{-1}$, $n \geqq 1$;
d) $E_n = [E_{n-1} \cup C_n \cup C_n^{-1}]^n$, $n \geqq 1$.

(To accomplish c) above, use Lemma 1.) Then $\{C_n\}$ satisfies property $\mathscr{P}$ with respect to the countable group $\bigcup_n E_n$. This implies by Theorem 3 that $|C_n|^{-1}h(C_n)$ converges in $L^1(\Omega)$, a contradiction of b). Thus $\lim_\alpha |A_\alpha|^{-1}h(A_\alpha)$ exists in $L^1(\Omega)$.

PART II. Let $\{A_\alpha\}$ and $\{B_\beta\}$ satisfy property $\mathscr{P}$ with respect to K. Let $\lim_\alpha |A_\alpha|^{-1}h(A_\alpha) = h_1$, $\lim_\beta |B_\beta|^{-1}h(B_\beta) = h_2$. We show that $h_1 = h_2$. Define inductively sequences $\{C_n\}_1^\infty$, $\{D_n\}_1^\infty$, $\{E_n\}_0^\infty$ such that

a) $E_0 = \{e\}$, $\{C_n\} \subset \{A_\alpha\}$, $\{D_n\} \subset \{B_\beta\}$;
b) $|||C_n|^{-1}h(C_n) - h_1|| < n^{-1}$, $|||D_n|^{-1}h(D_n) - h_2|| < n^{-1}$, $n \geqq 1$;
c) $|C_n|^{-1}|\bigcap_{g \in E_{n-1}} gC_n| \geqq 1 - n^{-1}$, $|D_n|^{-1}|\bigcap_{g \in E_{n-1}} gD_n| \geqq 1 - n^{-1}$, $n \geqq 1$;
d) $E_n = [E_n \cup C_n \cup C_n^{-1} \cup D_n \cup D_n^{-1}]^n$.

Then $\{C_n\}$ and $\{D_n\}$ both satisfy property $\mathscr{P}$ with respect to the countable group $\bigcup_n E_n$, and so by Theorem 3, $\lim_n |C_n|^{-1}h(C_n) = \lim_n |D_n|^{-1}h(D_n)$. Applying b), we see that $h_1 = h_2$.

All that remains to be shown is the invariance of the limit $\mathscr{h}(K)$ with respect to K, which is an immediate consequence of the following lemma and Theorem 3.

LEMMA 4. *Let K be a subgroup of G. Let L be a countable subgroup of K. Then there exists a countable subgroup L' of K such that $L' \supset L$ and $\mathscr{h}(L') = \mathscr{h}(K)$.*

PROOF. Let $\{A_n\}$ be a sequence of finite subsets of L such that $A_n \uparrow L$. Define inductively sequences $\{B_n\}_1^\infty$ and $\{E_n\}_0^\infty$ of finite subsets of K such that

a) $E_0 = \{e\}$;
b) $\| |B_n|^{-1}h(B_n) - \mathscr{h}(K)\| < n^{-1}$, $n \geqq 1$;
c) $|B_n|^{-1}|\bigcap_{g \in E_{n-1}} gB_n| \geqq 1 - n^{-1}$, $n \geqq 1$;
d) $E_n = [E_{n-1} \cup B_n \cup B_n^{-1} \cup A_n \cup A_n^{-1}]^n$, $n \geqq 1$.

We have that $\{B_n\}$ satisfies property $\mathscr{P}$ with respect to the countable group $L' = \bigcup_n E_n$; thus $|B_n|^{-1}h(B_n) \to \mathscr{h}(L')$. From b) above, we see that $|B_n|^{-1}h(B_n) \to \mathscr{h}(K)$. Thus $\mathscr{h}(L') = \mathscr{h}(K)$. Also, note that $L \subset L' \subset K$.

THEOREM 4. *Let K be a countable subgroup of G. Let $\{K_n\}$ be a sequence of subgroups of K such that $K_n \uparrow K$. Then $\mathscr{h}(K_n) \to \mathscr{h}(K)$ in $L^1(\Omega)$.*

PROOF. Let $\{B_n\}_1^\infty$ be a sequence of finite subsets of K such that $B_n \uparrow K$, $B_n \subset K_n$. Define inductively sequences $\{D_n\}_1^\infty$ and $\{E_n\}_0^\infty$ of finite subsets of K such that

a) $E_0 = \{e\}$, $D_n \subset K_n$, $n \geqq 1$;
b) $\| |D_n|^{-1}h(D_n) - \mathscr{h}(K_n)\| \leqq n^{-1}$, $n \geqq 1$;
c) $|D_n|^{-1}|\bigcap_{g \in E_{n-1}} gD_n| \geqq 1 - n^{-1}$, $n \geqq 1$;
d) $E_n = [E_{n-1} \cup B_n \cup B_n^{-1} \cup D_n \cup D_n^{-1}]^n$, $n \geqq 1$.

Then $K = \bigcup_n E_n$ and $\{D_n\}$ satisfies property $\mathscr{P}$ with respect to K; therefore, $|D_n|^{-1}h(D_n) \to \mathscr{h}(K)$. It follows from b) above that $\mathscr{h}(K_n) \to \mathscr{h}(K)$.

DEFINITION. If K is a subgroup of G, let $\mathscr{H}(K) = \int \mathscr{h}(K)\, d\lambda$.

THEOREM 5. *Let K_1, K_2 be subgroups of G with $K_2 \subset K_1$. Then $\mathscr{H}(K_1) \leqq \mathscr{H}(K_2)$.*

PROOF. If K_1 is countable then from Theorem 3 $\mathscr{H}(K_i) = \int_S H(e \mid K_i \cap V_e(s))\, d\mu(s)$, $i = 1, 2$, and the result follows from Lemma 2c. If K_1 is uncountable, choose a countable subgroup K_2' of K_2 such that $\mathscr{H}(K_2') = \mathscr{H}(K_2)$. Then choose a countable subgroup K_1' of K_1 such that $K_1' \supset K_2'$ and $\mathscr{H}(K_1') = \mathscr{H}(K_1)$. Then $\mathscr{H}(K_1') \leqq \mathscr{H}(K_2')$ and thus $\mathscr{H}(K_1) \leqq \mathscr{H}(K_2)$.

The following result gives an interesting martingale property.

THEOREM 6. *If K_1 and K_2 are subgroups of G such that $K_2 \subset K_1$ and $\mathscr{H}(K_1) = \mathscr{H}(K_2)$, then $E[\mathscr{h}(K_2) \mid \mathscr{C}(K_1)] = \mathscr{h}(K_1)$.*

PROOF. If K_1 is countable, then $\mathscr{H}(K_i) = \int H(e \mid K_i \cap V_e(s))\, d\mu(s)$, $i = 1, 2$. Thus $\mathscr{H}(K_1) = \mathscr{H}(K_2)$ implies by Lemma 2c that $H(e \mid K_1 \cap V_s(e)) = H(e \mid K_2 \cap V_s(e))$ a.e. $[\mu]$; this implies by Lemma 2d that $\phi_{K_1} = \phi_{K_2}$ a.e. $[\mu \times \lambda]$. Therefore,

$\int \phi_{K_1} d\mu = \int \phi_{K_2} d\mu$ a.e. $[\lambda]$. Now $h(K_i) = E[\int \phi_{K_i} d\mu \mid \mathscr{C}(K_i)]$, $i = 1, 2$. Thus

$$\begin{aligned} E[h(K_2) \mid \mathscr{C}(K_1)] &= E[E[\int \phi_{K_2} d\mu \mid \mathscr{C}(K_2)] \mid \mathscr{C}(K_1)] \\ &= E[E[\int \phi_{K_1} d\mu \mid \mathscr{C}(K_2)] \mid \mathscr{C}(K_1)] \\ &= E[\int \phi_{K_1} d\mu \mid \mathscr{C}(K_1)] = h(K_1) . \end{aligned}$$

If K_1 is uncountable, choose a countable subgroup K_2' of K_2 such that $h(K_2') = h(K_2)$. Then choose a countable subgroup K_1' of K_1 such that $K_1' \supset K_2'$ and $h(K_1') = h(K_1)$. We have $E[h(K_2') \mid \mathscr{C}(K_1')] = h(K_1')$ and thus $E[h(K_2) \mid \mathscr{C}(K_1')] = h(K_1)$. Taking the expectation of both sides with respect to $\mathscr{C}(K_1)$ gives $E[h(K_2) \mid \mathscr{C}(K_1)] = h(K_1)$.

LEMMA 5. *Let K, L be subgroups of G such that $K \subset L$ and $h(K) = h(L)$. If K' is a subgroup such that $K \subset K' \subset L$, then $h(K') = h(L)$.*

PROOF. By Theorem 5, $\mathscr{H}(K') = \mathscr{H}(K)$ and so by Theorem 6, $E[h(K) \mid \mathscr{C}(K')] = h(K')$. But $E[h(K) \mid \mathscr{C}(K')] = E[h(L) \mid \mathscr{C}(K')] = h(L)$.

THEOREM 7. *Let L be a subgroup of G. Let $\mathscr{L}$ be a directed family (directed by inclusion) of subgroups of L whose union is L. Then in $L^1(\Omega)$ $\lim_{K \in \mathscr{L}} h(K) = h(L)$, and $\lim_{K \in \mathscr{L}} \mathscr{H}(K) = \inf_{K \in \mathscr{L}} \mathscr{H}(K) = \mathscr{H}(L)$.*

PROOF. If $\lim_{K \in \mathscr{L}} h(K)$ does not exist, there exists an increasing sequence $\{K_n\}$ of subgroups in $\mathscr{L}$ such that $\{h(K_n)\}$ does not converge. Choose by Lemma 4 an increasing sequence $\{J_n\}$ of countable subgroups such that $h(J_n) = h(K_n)$. Then $\{h(J_n)\}$ must not converge, a contradiction of Theorem 4.

Thus $\lim_{K \in \mathscr{L}} h(K) = h$ for some function h. Let L' be a countable subgroup of L such that $h(L') = h(L)$. Choose a sequence $\{E_n\}$ of finite subsets such that $E_n \uparrow L'$. Choose an increasing sequence $\{K_n\}$ from $\mathscr{L}$ such that $h(K_n) \to h$ and $K_n \supset E_n$ for each n. Choose an increasing sequence $\{K_n'\}$ of countable subgroups such that $K_n' \subset K_n$ and $h(K_n') = h(K_n)$ for each n. Let J_n be the group generated by $E_n \cup K_n'$. Then $K_n' \subset J_n \subset K_n$ and so $h(J_n) = h(K_n)$ by Lemma 5. Therefore $h(J_n) \to h$. Now $\{J_n\}$ increases to a countable subgroup J. Thus $h(J_n) \to h(J)$, and hence $h(J) = h$. Also, $L \supset J \supset L'$ and so $h(J) = h(L)$ by Lemma 5. We conclude $h = h(L)$. Thus $\lim_{K \in \mathscr{L}} h(K) = h(L)$. Integrating, $\lim_{K \in \mathscr{L}} \mathscr{H}(K) = \mathscr{H}(L)$. But $\lim_{K \in \mathscr{L}} \mathscr{H}(K)$ must be $\inf_{K \in \mathscr{L}} \mathscr{H}(K)$ by Theorem 5.

FINAL REMARKS. Denote $\mathscr{H}(G)$ by $\mathscr{H}(P, T)$ to denote the dependence of $\mathscr{H}(G)$ on the action T and the partition P. Define the entropy $\mathscr{H}(T)$ of the action T to be $\mathscr{H}(T) = \sup_P \mathscr{H}(P, T)$, where the supremum is taken over all countable measurable partitions P of Ω with finite entropy. Conze [3] has defined the entropy of an action of the group Z^k. Our definition above extends his.

The entropy of an action of G is easily seen to be an invariant under isomorphism. Conversely, for actions of Z which are Bernoulli shifts, Ornstein [8] has shown that equality of entropy implies isomorphism. The concept of Bernoulli shift has been generalized to an action of an arbitrary group G by

Kirillov [7]. It would be interesting to know whether Ornstein's theorem generalizes to actions of an amenable group G which are generalized Bernoulli shifts.

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