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UNIFORM INEQUALITIES FOR CONDITIONAL p-MEANS GIVEN σ-LATTICES

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Let $(X, \mathscr{F}, P)$ be a probability space and $\mathscr{A}$ a sub-σ-lattice of $\mathscr{F}$: $\mathscr{A}$ is closed under countable union and countable intersection and contains X and $\emptyset$. Let $P^{\mathscr{A}}$ denote conditional expectation given $\mathscr{A}$ (Barlow and coworkers, 1972), and for fixed $p \geqq 1$ let $M^{\mathscr{A}}$ denote conditional p-mean given $\mathscr{A}$ (Brunk and Johansen, 1970). Rogge showed (1974) that for σ-fields $\mathscr{A}$ and $\mathscr{B}$, $\sup\{\|P^{\mathscr{A}}f - P^{\mathscr{B}}f\|_2 : 0 \leqq f \leqq 1\} \leqq 2\{\delta(\mathscr{A}, \mathscr{B})[1 - \delta(\mathscr{A}, \mathscr{B})]\}^{\frac{1}{2}}$, where $\delta(\mathscr{A}, \mathscr{B}) \equiv \max\{\sup_{A \in \mathscr{A}} \inf_{B \in \mathscr{B}} P(A \triangle B), \sup_{B \in \mathscr{B}} \inf_{A \in \mathscr{A}} P(A \triangle B)\}$: and that for $p \geqq 1$ the convergence to 0 of $\|P^{\mathscr{F}_n}f - P^{\mathscr{F}_\infty}f\|_p$ is uniform for $|f| \leqq 1$ if $\delta(\mathscr{F}_n, \mathscr{F}_\infty) \to 0$. In the present paper an inequality for conditional p-means given σ-lattices similar to Rogge's is obtained for $p \geqq 2$ and is applied to obtain uniformity of convergence to 0 of $\|M^{\mathscr{A}_n}f - M^{\mathscr{A}_\infty}f\|_p$ when $\delta(\mathscr{A}_n, \mathscr{A}_\infty) \to 0$.

1. Introduction. Let $(X, \mathscr{F}, P)$ be a probability space, and let $\mathscr{A}$ and $\mathscr{B}$ be σ-sublattices of $\mathscr{F}$: each is closed under countable union and countable intersection, and contains X and the empty set, $\emptyset$. A pseudo-metric on the set of all σ-sublattices of $\mathscr{F}$ is defined by $\delta(\mathscr{A}, \mathscr{B}) \equiv \max\{\sup_{A \in \mathscr{A}} \inf_{B \in \mathscr{B}} P(A \triangle B), \sup_{B \in \mathscr{B}} \inf_{A \in \mathscr{A}} P(A \triangle B)\}$. The restriction of this pseudo-metric to the set of σ-subfields of $\mathscr{F}$ was introduced by Rogge (1974). Boylan (1971) used a similar pseudo-metric in obtaining conditions sufficient for the convergence in L_1 of a sequence of conditional expectations $P(f \mid \mathscr{F}_n)$ to be uniform in the class of $\mathscr{F}$-measurable real valued functions f bounded in absolute value by 1; see also Neveu (1972). Rogge sharpened and extended Neveu's (loc. cit.) uniform inequality for conditional expectations and showed that $\|P^{\mathscr{F}_n}f - P^{\mathscr{F}_\infty}f\|_1 \to 0$ as $n \to \infty$ uniformly in the class f: $|f| \leqq 1$ if and only if $\delta(\mathscr{F}_n, \mathscr{F}_\infty) \to 0$; here $\|\cdot\|_1$ denotes the usual norm in $L_1(X, \mathscr{F}, P)$. Further, for $p \geqq 1$, $\delta(\mathscr{F}_n, \mathscr{F}_\infty) \to 0$ implies $\|P^{\mathscr{F}_n}f - P^{\mathscr{F}_\infty}f\|_p \to 0$ uniformly in the same class of functions f, where $\|\cdot\|_p$ denotes the norm in $L_p(X, \mathscr{F}, P)$. These results follow from his fundamental inequality,

$$\sup\{\|P^{\mathscr{A}}f - P^{\mathscr{B}}f\|_2 : 0 \leqq f \leqq 1\} \leqq [2\delta(\mathscr{A}, \mathscr{B})(1 - \delta(\mathscr{A}, \mathscr{B})]^{\frac{1}{2}} .$$

Applications of conditional expectation given a σ-lattice are described by Barlow and coauthors (1972). Conditional generalized means were studied by Brunk and Johansen (1970). In the present paper, inequalities similar to Rogge's are obtained for conditional p-means given σ-lattices, when $p \geqq 2$. Whether or not similar inequalities obtain for other generalized means remains an open

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question. Also, the specialization of the inequality of the present paper to the case of conditional expectations and σ-fields is inferior to Rogge's inequality, and it is an open question whether inequalities for the more general situation which reduce to Rogge's can be obtained.

2. Terminology, notation, and preliminary results. As the reader has noticed above, some elements of the notation proposed by de Finetti are used. Thus P denotes not only the probability measure but also expectation according to the probability measure, and $P^{\mathscr{A}}$ denotes conditional expectation given the σ-lattice $\mathscr{A}$. Also A denotes the indicator of a set A in $\mathscr{F}$ as well as the set A itself; A is the function $X \to R$ taking the value 1 at points in A, the value 0 at points in A^c. Thus $A \triangle B = B(1 - A) + A(1 - B) = A \vee B - A \wedge B$, where $\vee$ denotes maximum and $\wedge$ denotes minimum. If $\mathscr{A}$ is a σ-sublattice of $\mathscr{F}$, a function $f\colon X \to R$ is $\mathscr{A}$-measurable if for each real a, $\{f > a\} \in \mathscr{A}$. One may think of the class of all $\mathscr{A}$-measurable functions as an extension of the class of indicators of sets in $\mathscr{A}$, and the class of $\mathscr{A}$-measurable real-valued functions will be denoted by $\mathscr{A}^*$. Fix $p > 0$. A function $f \in \mathscr{F}^*$ such that $P(|f|^p) < \infty$ determines an equivalence class of functions in $\mathscr{F}^*$ differing from f on sets of probability 0; such an equivalence class is an element of $L_p \equiv L_p(X, \mathscr{F}, P)$. The notation will not distinguish between a function and the equivalence class it represents. Throughout the remainder of the paper, p represents a fixed number, $p \geqq 2$, and $||f||$ denotes $P(|f|^p)$; it is the usual norm in L_p, whereas the norm in L_2 will be denoted by $||\cdot||_2$. We set $q = [1 - (1/p)]^{-1}$, so that $(1/p) + (1/q) = 1$.

It follows in the usual way from Clarkson's inequalities (1936, page 400) that if C is a closed convex cone in L_p, and if $f \in L_p$, there is a unique closest point of C to f, to be called the *projection of f on C* and denoted by $\pi_C f$. One can show also, as do Ando and Amemiya (1965) for the case of a subspace, that π_C is a continuous operator, not generally linear.

If $\mathscr{A}$ is a σ-sublattice of $\mathscr{F}$, $L_p(\mathscr{A})$ will denote the closed convex cone of $\mathscr{A}$-measurable functions in L_p. If $f \in L_p$, the projection of f on $L_p(\mathscr{A})$ is the *conditional p-mean of f* given $\mathscr{A}$, and is denoted by $M^{\mathscr{A}}f$. For $p = 2$ it is the conditional expectation of f given $\mathscr{A}$, denoted by $P^{\mathscr{A}}f$. And if $\mathscr{A}$ is a σ-subfield of $\mathscr{F}$, $P^{\mathscr{A}}f$ coincides with the usual conditional expectation given $\mathscr{A}$.

The properties of $M^{\mathscr{A}}$ listed below are easily established by standard techniques; see also Nikolskii (1963). Define the function h by

$$h(a) \equiv |a|^{p-1} \operatorname{sgn} a$$

for real a where $\operatorname{sgn} a = 1$ if $a > 0$, $\operatorname{sgn} a = -1$ if $a < 0$. If $f \in L_p$, if C is a closed convex cone in L_p, and if $g = \pi_C f$, then

(1) $$P[gh(f - g)] = 0\,,$$

and for all $z \in C$,

(2) $$P[zh(f - g)] \leqq 0\,.$$

In particular, if $f \in L_2$, if $\mathscr{A}$ is a σ-sublattice of $\mathscr{F}$, and if $g = P^{\mathscr{A}}f$, then

$$(1)' \qquad P[g(f - g)] = 0,$$

and if $z \in L_2(\mathscr{A})$ then

$$(2)' \qquad P[z(f - g)] \leqq 0.$$

Setting $z = 1$ and $z = -1$ in (2)′, we find that

$$(3) \qquad Pf = Pg.$$

In fact it can be shown (Barlow and coworkers, 1972, page 343) that if ψ is a Borel function such that $\psi(g) \in L_2$, then

$$(4) \qquad P[f\psi(g)] = P[g\psi(g)].$$

If $f \in L_p$, if $a \leqq f \leqq b$, and if $\mathscr{A}$ is a σ-sublattice of $\mathscr{F}$, then

$$(5) \qquad a \leqq M^{\mathscr{A}}f \leqq b,$$

and

$$(6) \qquad ||f - M^{\mathscr{A}}f|| \leqq ||f||.$$

Further, if $a \geqq 0$ and b is real,

$$(7) \qquad M^{\mathscr{A}}(af + b) = aM^{\mathscr{A}}f + b.$$

3. The theorem.

THEOREM. *Let $f \in L_p$, $p \geqq 2$. Let $\mathscr{A}$, $\mathscr{B}$ be σ-sublattices of $\mathscr{F}$, at distance $\delta(\mathscr{A}, \mathscr{B}) \leqq \frac{1}{2}$. If $0 \leqq f \leqq 1$, then*

$$(8) \qquad ||M^{\mathscr{A}}f - M^{\mathscr{B}}f||^p \leqq 2^{p-1}\{\delta(\mathscr{A}, \mathscr{B})[1 - \delta(\mathscr{A}, \mathscr{B})]\}^{1/p}.$$

If $|f| \leqq 1$ then

$$(8)' \qquad ||M^{\mathscr{A}}f - M^{\mathscr{B}}f||^p \leqq 2^p\{\delta(\mathscr{A}, \mathscr{B})[1 - \delta(\mathscr{A}, \mathscr{B})]\}^{1/p}.$$

COROLLARY. *Let $\mathscr{A}_n$ be a σ-sublattice of $\mathscr{F}$, $n = 1, 2, \cdots$, and $\mathscr{A}$ a σ-sublattice of $\mathscr{F}$. If $\delta(\mathscr{A}_n, \mathscr{A}_m) \to 0$ as $n, m \to \infty$, then to each $f \in L_p$ corresponds a function $f_\infty \in L_p$ such that $M^{\mathscr{A}_n}f \to f_\infty$, and the convergence is uniform for $|f| \leqq 1$. If $\delta(\mathscr{A}_n, \mathscr{A}) \to 0$ then $||M^{\mathscr{A}_n}f - M^{\mathscr{A}}f|| \to 0$ for each $f \in L_p$, and the convergence is uniform for $|f| \leqq 1$.*

This corollary generalizes Rogge's improvement of Boylan's martingale uniform convergence theorem to conditional p-means given σ-sublattices.

The theorem is proved in a sequence of lemmas.

LEMMA 1. *Let $\mathscr{A}$, $\mathscr{B}$ be σ-sublattices of $\mathscr{F}$, For $B \in \mathscr{B}$,*

$$P(P^{\mathscr{A}}B \wedge [1 - P^{\mathscr{A}}B]) \leqq \delta(\mathscr{A}, \mathscr{B}).$$

PROOF. Set $g = P^{\mathscr{A}}B$, and let $A \in \mathscr{A}$. We have

$$P(A \triangle B) = P[A(1 - B) + B(1 - A)].$$

It follows from (2)′ and (3) that

$$\begin{aligned} P(A \triangle B) &\geqq P[A(1-g) + g(1-A)] \\ &\geqq P\{A[g \wedge (1-g)] + (1-A)[g \wedge (1-g)]\} \\ &= P[g \wedge (1-g)]\,. \end{aligned}$$

Hence

$$\begin{aligned} P[g \wedge (1-g)] &\leqq \inf_{A \in \mathscr{A}} P(A \triangle B) \leqq \sup_{B \in \mathscr{B}} \inf_{A \in \mathscr{A}} P(A \triangle B) \\ &\leqq \delta(\mathscr{A}, \mathscr{B})\,. \end{aligned}$$ □

LEMMA 2. *Let* $f \in \mathscr{B}^*$, $0 \leqq f \leqq 1$, *If* $\delta(\mathscr{A}, \mathscr{B}) \leqq \frac{1}{2}$, *then*

$$\|f - M^{\mathscr{A}} f\|^p \leqq \delta(\mathscr{A}, \mathscr{B})[1 - \delta(\mathscr{A}, \mathscr{B})]\,. \tag{9}$$

PROOF. Case 1: $f = B \in \mathscr{B}^*$. Set $g = P^{\mathscr{A}} f$. Then

$$\|f - g\|^p = P(|f-g|^p) = P[B(1-g)^p + (1-B)g^p]\,.$$

But from (4) we have

$$P[B(1-g)^p] = P[g(1-g)^p]$$

and

$$P[(1-B)g^p] = P(g^p) - P(g^{p+1}) = P[g^p(1-g)]\,.$$

Thus

$$\|f - g\|^p = P\{g(1-g)[g^{p-1} + (1-g)^{p-1}]\} \leqq P[g(1-g)]\,.$$

Set $k = g \wedge (1-g)$. Then

$$\begin{aligned} P[g(1-g)] &= P[k(1-k)] = P(k) - P(k^2) \leqq P(k) - [P(k)]^2 \\ &= P(k)[1 - P(k)] \leqq \delta(\mathscr{A}, \mathscr{B})[1 - \delta(\mathscr{A}, \mathscr{B})] \end{aligned}$$

by Lemma 1. A fortiori, (9) holds.

The general case: $f \in \mathscr{B}^*$, $0 \leqq f \leqq 1$. Obviously every constant function f satisfies (9), for then the left hand member is 0. Now suppose f_0 is simple,

$$f_0 = \textstyle\sum_{i=0}^k b_i B_i\,,$$

where $0 = b_0 < b_1 < \cdots < b_k = 1$, $B_i \cap B_j = \emptyset$ for $i \neq j$, $i, j = 1, 2, \cdots, k$. Then

$$f_0 = \textstyle\sum_{j=1}^k (b_j - b_{j-1}) \bigvee_{i=j}^k B_i\,.$$

Set

$$\mathscr{C} \equiv \{f \in \mathscr{B}^* : 0 \leqq f \leqq 1, \|f - M^{\mathscr{A}} f\|^p \leqq \delta(\mathscr{A}, \mathscr{B})[1 - \delta(\mathscr{A}, \mathscr{B})]\}\,.$$

Since $f_0 \in \mathscr{B}^*$, $\bigvee_{i=j}^k B_i \in \mathscr{B}^*$ for $j = 1, 2, \cdots, k$. By Case 1, $\bigvee_{i=j}^k B_i \in \mathscr{C}$. But $\mathscr{C}$ is convex, $b_j - b_{j-1} \geqq 0$ for $j = 1, 2, \cdots, k$, and $\sum_{j=1}^k (b_j - b_{j-1}) = 1$. Hence $f_0 \in \mathscr{C}$. The set $\mathscr{C}$ is also closed under increasing limits; since every function f in $\mathscr{B}^*$ bounded by 0 and 1 is the limit of simple functions in $\mathscr{C}$, also $f \in \mathscr{C}$. □

LEMMA 3. *For real a and b, and for* $p \geqq 2$,

$$|a - b|^p \leqq 2^{p-2}(a - b)[h(a) - h(b)] , \tag{10}$$

where

$$h(x) \equiv (\operatorname{sgn} x)|x|^{p-1} . \tag{11}$$

PROOF. If $\operatorname{sgn} a = \operatorname{sgn} b$, (10) is a consequence of the inequality $(1 + x)^{p-1} \geqq 1 + x^{p-1}$, $x \geqq 0$. If $\operatorname{sgn} a \neq \operatorname{sgn} b$, (10) follows from the inequality $(1 + x)^{p-1} \leqq 2^{p-2}(1 + x^{p-1})$, $x \geqq 0$.

LEMMA 4. *Let* $f_0 \in L_p$, $x^* \in L_q$, $A \equiv \{f \in L_p : P(x^*f) \leqq 0\}$, $M \equiv \{f \in L_p : P(x^*f) = 0\}$, $g_A \equiv \pi_A f_0$, $g_M \equiv \pi_M f_0$. *If* $f_0 \notin A$, *then* $g_A = g_M \in M$, *so that* $P(x^*g_A) = 0$.

PROOF. Since $A \supseteqq M$, $||f_0 - g_A|| \leqq ||f_0 - g_M||$. If $g_A \in M$ then $g_A = g_M \in M$. Suppose then the contrary, $g_A \notin M$, i.e., $P(x^*g_A) < 0$. Since $f_0 \notin A$, $P(x^*f_0) > 0$. Set $\varphi(t) \equiv P\{x^*[tf_0 + (1 - t)g_A]\}$, $t \in [0, 1]$. Then φ is continuous, $\varphi(0) < 0$, $\varphi(1) > 0$. Hence $\exists\, t_0 \in (0, 1)$ such that $\varphi(t_0) = 0$. Set $g_1 \equiv t_0 f_0 + (1 - t_0)g_A$. Then $g_1 \in M \subseteqq A$, $f_0 - g_1 = (1 - t_0)(f_0 - g_A)$, $||f_0 - g_1|| = (1 - t_0)||f_0 - g_A|| < ||f_0 - g_A||$, a contradiction. Hence $g_A \in M$ and $g_M = g_A$. □

LEMMA 5. *Let* $f_0 \in L_p$. *Let* C_i *be a closed convex cone in* L_p, $i = 1, 2$. *Set* $g_i \equiv \pi_{C_i} f_0$, $i = 1, 2$, $g_{21} \equiv \pi_{C_2} g_1$, $g_{12} \equiv \pi_{C_1} g_2$. *Then*

$$||g_1 - g_2||^p \leqq 2^{p-2}[||f_0 - g_2||^{p/q}||g_1 - g_{21}|| + ||f_0 - g_1||^{p/q}||g_2 - g_{12}||] . \tag{12}$$

PROOF. By Lemma 3,

$$\begin{aligned} ||g_1 - g_2||^p &= ||(f_0 - g_2) - (f_0 - g_1)||^p \\ &\leqq 2^{p-2}P\{(g_1 - g_2)[h(f_0 - g_2) - h(f_0 - g_1)]\} . \end{aligned}$$

Then (1) implies

$$||g_1 - g_2||^p \leqq 2^{p-2}P\{g_1 h(f_0 - g_2) - g_2 h(f_0 - g_1)\} .$$

Note that $h(f_0 - g_i) \in L_q$, $i = 1, 2$. Set $A_i \equiv \{f \in L_p : P[fh(f_0 - g_i)] \leqq 0\}$, $i = 1, 2$, and $g'_{21} \equiv \pi_{A_2} g_1$, $g'_{12} \equiv \pi_{A_1} g_2$. By Lemma 4, with x^* set successively equal to $h(f_0 - g_2)$ and $h(f_0 - g_1)$, $P[g'_{21} h(f_0 - g_2)]$ is 0 if $g_1 \notin A_2$ and is nonpositive if $g_1 \in A_2$, in which case $g'_{21} = g_1$; also $P[g'_{12} h(f_0 - g_1)]$ is 0 if $g_2 \notin A_1$ and is nonpositive if $g_2 \in A_1$, in which case $g'_{12} = g_2$. In any case,

$$||g_1 - g_2||^p \leqq 2^{p-2}P\{(g_1 - g'_{21})h(f_0 - g_2) + (g_2 - g'_{12})h(f_0 - g_1)\} .$$

But for u, v, in L_p,

$$P[|uh(v)|] = P(|u|\,|v|^{p-1}) \leqq ||u||\,||v||^{p/q} .$$

Also, since by (2), $C_i \subseteqq A_i$, $i = 1, 2$, we have $||g_1 - g'_{21}|| \leqq ||g_1 - g_{21}||$ and $||g_2 - g'_{12}|| \leqq ||g_2 - g_{12}||$. Inequality (12) follows. □

PROOF OF THEOREM. Let $f \in L_p$, $0 \leqq f \leqq 1$. In Lemma 5, set $C_1 \equiv L_p(\mathscr{A})$,

$C_2 \equiv L_p(\mathscr{B})$, $f_0 = f$. We have $g_1 = M^{\mathscr{A}}f$, $g_2 = M^{\mathscr{B}}f$, $g_{21} = M^{\mathscr{B}}(M^{\mathscr{A}}f)$, $g_{12} = M^{\mathscr{A}}(M^{\mathscr{B}}f)$. By Lemma 5,

$$\|M^{\mathscr{A}}f - M^{\mathscr{B}}f\|^p \leqq 2^{p-2}[\|f - M^{\mathscr{B}}f\|^{p/q}\|M^{\mathscr{A}}f - M^{\mathscr{B}}(M^{\mathscr{A}}f)\| + \|f - M^{\mathscr{A}}f\|^{p/q}\|M^{\mathscr{B}}f - M^{\mathscr{A}}(M^{\mathscr{B}}f)\|] .$$

By Lemma 2 and (5), $\|M^{\mathscr{B}}f - M^{\mathscr{A}}(M^{\mathscr{B}}f)\| \leqq \{\delta(\mathscr{A}, \mathscr{B})[1 - \delta(\mathscr{A}, \mathscr{B})]\}^{1/p}$, and $\|M^{\mathscr{A}}f - M^{\mathscr{B}}(M^{\mathscr{A}}f)\|$ has the same bound. Also $\|f - M^{\mathscr{B}}f\| \leqq \|f\| \leqq 1$ and $\|f - M^{\mathscr{A}}f\| \leqq \|f\| \leqq 1$, so that (8) follows. If $|f| \leqq 1$, set $f_1 \equiv (1 + f)/2$; then $0 \leqq f_1 \leqq 1$. Also $\|M^{\mathscr{A}}f - M^{\mathscr{B}}(M^{\mathscr{A}}f)\| = 2\|M^{\mathscr{A}}f_1 - M^{\mathscr{B}}(M^{\mathscr{A}}f_1)\| \leqq 2\{\delta(\mathscr{A}, \mathscr{B})[1 - \delta(\mathscr{A}, \mathscr{B})]\}^{1/p}$ and $\|M^{\mathscr{B}}f - M^{\mathscr{A}}(M^{\mathscr{B}}f)\|$ has the same bound, yielding (8)′. □

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