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ON VAGUE CONVERGENCE OF STOCHASTIC PROCESSES

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Suppose Y, Y_n are stochastic processes in $C[0, 1]$ and the finite-dimensional distributions of Y_n converge vaguely to those of Y. Then a necessary and sufficient condition for the vague convergence of the distributions of Y_n to that of Y is an approximate equicontinuity of the sequence $\langle Y_n \rangle$. Dudley (1966) generalized this standard result. We generalize Dudley's result to the case when the values of X_n are in an arbitrary metric space and extend the result also to the case of the Skorohod metric. In our situation vague compactness does not imply tightness and thus a different proof than Dudley's (1966) must be used. The proof we use is simple and of interest even when other proofs are available.

1. Introduction. Let Y, Y_n be stochastic processes with a parameter space T and values in a metric space $\langle Z, d \rangle$ and let the finite dimensional distributions of Y_n converge to those of Y. Let $\mathscr{U}$ be the topology of uniform convergence in Z^T, the space of all functions on T into Z. We shall say that the distributions p_n of Y_n converge vaguely to the distribution p of Y, in symbols $p_n \to p$, if

$$\liminf \underline{p}_n O \geqq \bar{p} O \qquad \text{for every} \quad O \in \mathscr{U} \tag{1}$$

where $\underline{p}_n$ is the inner measure induced by p_n and $\bar{p}$ is the outer measure induced by p.

For Z the real line, $\langle T, v \rangle$ a compact metric space, p tight, and under some measurability conditions, it follows from Dudley's (1966) Theorem 1 and Proposition 2 that $p_n \to p$ if and only if $\langle Y_n \rangle$ is approximately equicontinuous, i.e. if for every positive ε there is a positive δ such that, eventually,

$$(p_n)_-\{x; \sup_{v(s,t)<\delta} |x_s - x_t| < \varepsilon\} > 1 - \varepsilon \,. \tag{2}$$

The importance of Dudley's result is that it relaxes the previously used requirement that the domains $\mathscr{D}p_n$ and $\mathscr{D}p$ of p_n and p should contain $\mathscr{U}$. This relaxation makes it possible to obtain stronger properties by simpler means in most applications when the Skorohod topology has been used (see Remark 2.12).

We shall generalize the result by deleting the assumption that Z is the real line and by weakening the assumption that p is tight to the requirement that (2) holds for $n = 0$ with $p_0 = p$. We shall talk about nets instead of sequences, and we shall relax some other assumptions as well (see Remark 2.11). Dudley's

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proof consists of two steps: by his Proposition 2, if $\langle Y_n \rangle$ is approximately equicontinuous then $\langle p_n \rangle$ is tight, and, by his Theorem 1, precompact. We shall see in Remark 2.10 that such a proof is no longer applicable in our situation since the approximate equicontinuity does not imply tightness.

The direct proof we use is similar to that used by Hájek and Šidák (1967, Theorem V.3.2) to establish the result for Y, Y_n in $C[0, 1]$. The proof is simple and short, and is barely affected by the generalization which makes it impossible to use the tightness argument. Thus the proof may be of interest even in cases where other proofs are available. In such special cases it can be still simplified and shortened, firstly because lesser generality requires fewer introductory explanations and conventions and secondly because we separated the argument into a part common for the uniform and Skorohod topology and other parts specific to these two topologies.

2. Conditions for the vague convergence of Y_n to Y.

2.1. *Preliminaries.* Throughout the paper X is a given set. A probability μ means a probability on a σ-algebra $\mathscr{D}\mu$ such that $\bigcup \mathscr{D}\mu = X$. If μ is a probability then $\bar{\mu}$ and $\underline{\mu}$ denote the outer and inner measure generated by μ. If f is a real function on X then $\bar{\int} f\, d\mu$ is the infimum of $\int h\, d\mu$ taken over all h for which $\int h\, d\mu$ is defined and $h \geqq f$. Similarly we define $\underline{\int} f\, d\mu$. It is easy to see that $\bar{\mu}A = \bar{\int} \chi_A\, d\mu$, $\underline{\mu}A = \underline{\int} \chi_A\, d\mu$ for every $A \subset X$.

We shall consider a probability p and a net $\langle p_\alpha \rangle$ of probabilities. If $\mathscr{O}$ is a class of subsets of X we write

$$(1) \qquad p_\alpha \xrightarrow{\geq} p \quad \text{on} \quad \mathscr{O}$$

as an abbreviation for

$$(2) \qquad \liminf\, (p_\alpha)_{-} O \geqq \bar{p}O \quad \text{for every} \quad O \quad \text{in} \quad \mathscr{O}$$

and we say in this case that $\langle p_\alpha \rangle$ converges vaguely to p on $\mathscr{O}$.

We start with the simple Lemma 2.2, which gives conditions under which the class $\mathscr{O}$ for which $p_\alpha \xrightarrow{\geq} p$ on $\mathscr{O}$ can be enlarged. This is followed by Theorem 2.3 which shows that our concept of vague convergence agrees with that of Dudley (1966). Lemma 2.6 will be used in considering both the uniform and Skorohod topologies. In Definition 2.7 a concept of an almost simple probability p is introduced; it is a generalization of the property that p is concentrated on a union of equicontinuous families. Theorem 2.8 is the main result concerning the uniform topology and is followed by several remarks.

2.2. LEMMA. *Suppose*

$$p_\alpha \xrightarrow{\geq} p \quad \textit{on} \quad \mathscr{O},$$

$G \subset X$ *and suppose that for every* $\varepsilon > 0$ *there is an* O *in* $\mathscr{O}$ *such that*

$$(1) \qquad \bar{p}(G - O) < \varepsilon, \qquad \bar{p}_\alpha(O - G) < \varepsilon \quad \textit{eventually}.$$

Then

(2) $$p_\alpha \overset{\geq}{\to} p \quad \text{on} \quad \mathscr{O} \cup \{G\}.$$

PROOF. From Halmos (1950, Theorem 14.G) it follows that

$$(p_\alpha)_-G \geqq (p_\alpha)_-O - \bar{p}_\alpha(O - G),$$

the subadditivity of $\bar{p}$ yields

$$\bar{p}G \leqq \bar{p}O + \bar{p}(G - O)$$

and (2) follows easily from (1).

2.3. THEOREM. *Suppose τ is a topology for X, $C(X)$ is the class of all real bounded continuous functions on $\langle X, \tau\rangle$ and $\mathscr{O}$ is the class of all exactly open sets in τ*, i.e. $\mathscr{O} = \{f^{-1}[(0, +\infty)]; f \in C(X)\}$. *Then*

(1) $$p_\alpha \overset{\geq}{\to} p \quad \text{on} \quad \mathscr{O}$$

if and only if

(2) $$\operatorname{diam}\{\bar{\textstyle\int} f\,dp_\alpha, \underline{\textstyle\int} f\,dp_\alpha, \bar{\textstyle\int} f\,dp, \underline{\textstyle\int} f\,dp\} \to 0$$

for every $f \in C(X)$.

If τ is metrizable then $\mathscr{O} = \tau$.

PROOF. Suppose (2) holds, $O \in \mathscr{O}$, $\varepsilon > 0$. There is an f in $C(X)$ such that $O = f^{-1}[0, +\infty)]$ and then $f_n = (nf_+) \wedge 1$ are in $C(X)$, $f_n \uparrow \chi_O$, $\bar{\int} f_n\,dp > \bar{p}O - \varepsilon$ for some n, since $\bar{\int} \cdot\, dp$ is continuous from below for nonnegative integrands (see, e.g., Hewitt and Stromberg, 1969, the proof of Theorem III.9.17). This and (2) imply (1).

Suppose (1) holds, $f \in C(X)$. We may assume that f is into (0, 1). Let k be a positive integer. Take $F_i = \{x; f(x) \geqq i/k\}$ to obtain

$$\frac{1}{k} \sum_{i=1}^k \chi_{F_i} \leqq f \leqq \frac{1}{k} + \frac{1}{k} \sum_{i=1}^k \chi_{F_i}.$$

The subadditivity of upper integrals and superadditivity of lower integrals yield

$$\bar{\textstyle\int} f\,dp_\alpha \leqq \frac{1}{k} + \frac{1}{k} \sum_{i=1}^k \bar{p}_\alpha(F_i), \qquad \frac{1}{k} \sum_{i=1}^k \underline{p}(F_i) \leqq \underline{\textstyle\int} f\,dp.$$

Since $X - F_i$ are in $\mathscr{O}$ we obtain $\limsup \bar{p}_\alpha F_i \leqq \underline{p}F_i$ and

$$\limsup \bar{\textstyle\int} f\,dp_\alpha \leqq \underline{\textstyle\int} f\,dp.$$

This, applied to f and $1 - f$, implies (2).

The last assertion, that $\tau = \mathscr{O}$ if τ is metrizable, is easy.

The following will be assumed from now on:

2.4. ASSUMPTION. $\langle Z, d\rangle$ is a metric space, T is a set, $X \subset Z^T$.

2.5. NOTATION. If $T_0 \subset T$ then u_{T_0} denotes the pseudometric for X defined by

$$u_{T_0}(x, y) = \sup\{d(x_t, y_t); t \in T_0\}$$

and π_{T_0} denotes the topology induced by u_{T_0} if T_0 is finite. For T_0 infinite,

$$\pi_{T_0} = \bigcup \{\pi_{T_1}; T_1 \subset T_0, T_1 \text{ finite}\} .$$

Note that the vague convergence on π_T is the vague convergence of the finite dimensional distributions (cf. Billingsley (1968, Section 5)).

If ρ is a pseudometric and $\varepsilon > 0$ then $S_\rho(x, \varepsilon)$ denotes the open ε-sphere around x, with respect to ρ. We write S_{T_0} to mean $S_{u_{T_0}}$.

The topology $\mathscr{U}$ associated with u_T is the topology of uniform convergence. If T is given a topology, the family of all continuous functions on T into $\langle Z, d\rangle$ will be denoted by C.

2.6. LEMMA. *Suppose that ρ is a pseudometric for X, τ the topology induced by ρ. Suppose that for every $\varepsilon > 0$, T_ε is a finite subset of T, and C_ε is a subset of X satisfying*

$$S_{T_\varepsilon}(x, \eta_\varepsilon) \cap C_\varepsilon \subset S_\rho(x, \varepsilon) \tag{1}$$

for an $\eta_\varepsilon > 0$ and all $x \in C_\varepsilon$. Set $T_* = \bigcup \{T_\varepsilon; \varepsilon > 0\}$. Suppose that $\mathscr{D}p \supset \pi_{T*}$, $p_\alpha \overset{>}{\to} p$ on π_{T*} and

$$\underline{p}C_\varepsilon > 1 - \varepsilon \tag{2}$$

for every $\varepsilon > 0$.

Then a sufficient condition for

$$p_\alpha \overset{>}{\to} p \quad \text{on} \quad \tau \tag{3}$$

is that, for every $\varepsilon > 0$

$$(p_\alpha)_- C_\varepsilon > 1 - \varepsilon \quad \textit{eventually}. \tag{4}$$

PROOF. Let $G \in \tau$, $\delta > 0$. Set

$$G_0 = \{x; S_\rho(x, \varepsilon) \subset G\} \tag{5}$$

and select ε such that $0 < \varepsilon < \delta$ and

$$\bar{p}G_0 > \bar{p}G - \delta . \tag{6}$$

The set

$$O = \bigcup \{S_{T_\varepsilon}(x, \eta_\varepsilon); x \in G_0 \cap C_\varepsilon\} \tag{7}$$

is in π_{T*} and satisfies $C_\varepsilon \cap O \subset G$ by (1) and (5). Thus

$$G_0 \cap C_\varepsilon \subset O \subset G \cup (X - C_\varepsilon) . \tag{8}$$

In particular, $O - G \subset X - C_\varepsilon$ and

$$\bar{p}(O - G) < \delta , \qquad \bar{p}_\alpha(O - G) < \delta , \quad \text{eventually.} \tag{9}$$

On the other hand, by (2) and (6),

$$\bar{p}(G) - 2\delta \leqq \bar{p}(G_0 \cap C_\varepsilon) \leqq pO \leqq p_-(O \cap G) + \bar{p}(O - G)$$

and (9) implies that

$$p_{-}(O \cap G) \geqq \bar{p}G - 3\delta .$$

Thus

(10) $$\bar{p}(G - O) \leqq \bar{p}(G) - p_{-}(O \cap G) \leqq 3\delta .$$

Lemma 2.2 applies now and yields (3).

2.7. DEFINITION. p (resp. $\langle p_\alpha \rangle$, resp. $\langle p_\alpha \rangle, p$) is called *almost simple* if for every $\varepsilon > 0$ there is a finite cover $\mathscr{T}_\varepsilon$ of T such that, with

(1) $$D_\varepsilon = \{x; x \in X, \operatorname{diam} x[A] < \varepsilon \text{ for every } A \in \mathscr{T}_\varepsilon\} ,$$

$p_{-}D_\varepsilon > 1 - \varepsilon$ (resp. $(p_\alpha)_{-}D_\varepsilon > 1 - \varepsilon$ eventually, resp. $p_{-}D_\varepsilon > 1 - \varepsilon$ and $(p_\alpha)_{-}D_\varepsilon > 1 - \varepsilon$ eventually). In addition, if T_0 is a subset of T which intersects every A in $\bigcup \{\mathscr{T}_\varepsilon; \varepsilon > 0\}$ then we say that p (resp. $\langle p_\alpha \rangle$, resp. $p, \langle p_\alpha \rangle$) is *almost simple* with base T_0.

2.8. THEOREM. *Suppose p is almost simple with base T_0,*

(1) $$\mathscr{D}p \supset \pi_{T_0}, \qquad p_\alpha \underset{\rightarrow}{\gtrsim} p \quad on \quad \pi_{T_0} .$$

Then a necessary and sufficient condition for

(2) $$p_\alpha \underset{\rightarrow}{\gtrsim} p \qquad on \quad \mathscr{U}$$

is that $p, \langle p_\alpha \rangle$ be almost simple with base T_0.

PROOF. Apply Lemma 2.6 with $\tau = \mathscr{U}$, $C_\varepsilon = D_{\varepsilon/3}$, $\eta_\varepsilon = \varepsilon/3$, T_ε a finite subset of T_0 which intersects every A in $\mathscr{T}_{\varepsilon/3}$. We have $T_0 \supset T_*$ and thus $\mathscr{D}p \supset \pi_{T_*}$, $p_\alpha \underset{\rightarrow}{\gtrsim} p$ on π_{T_*} and the sufficiency follows from the lemma. The necessity follows since D_ε is in $\mathscr{U}$ and so if $p_{-}D_\varepsilon > 1 - \varepsilon$ then $\liminf (p_\alpha)_{-}D_\varepsilon \geqq \bar{p}D_\varepsilon > 1 - \varepsilon$.

2.9. LEMMA.

(i) *If T is a compact metric space, $\mathscr{D}p \supset \mathscr{U}$, $pC = 1$ then p is almost simple.*

(ii) *If $p_\alpha \underset{\rightarrow}{\gtrsim} p$ on π_T, $\langle p_\alpha \rangle$ is almost simple and p is tight* (i.e. *to every $\varepsilon > 0$ there is a compact subset C_ε of $\langle X, \mathscr{U} \rangle$ such that $p_{-}C_\varepsilon > 1 - \varepsilon$*) *then p is almost simple.*

(iii) *If p is defined on the σ-algebra generated by π_T, $\langle p_\alpha \rangle$ is almost simple and $p_\alpha \underset{\rightarrow}{\gtrsim} p$ on π_T then p can be extended to a probability $\bar{p}$ such that $\bar{p}, \langle p_\alpha \rangle$ is almost simple.*

PROOF. Part (i): Let v be a compact metric for T. Every x in C is uniformly continuous and C is the union of p-measurable sets A_k in $\mathscr{U}$ of all x in C for which $\sup \{d(x_t, x_s); v(s, t) < 1/k\} < \varepsilon$. From the compactness of T it follows easily that p is almost simple.

Part (ii): Suppose $\varepsilon > 0$, $\mathscr{T}_\varepsilon$, D_ε are as in the definition of almost simplicity of $\langle p_\alpha \rangle$. Form $D_\varepsilon' = \{x; \operatorname{diam} x[A] < 2\varepsilon$ for every $A \in \mathscr{T}_\varepsilon\}$. Then $K_\varepsilon = C_\varepsilon - D_\varepsilon'$ is compact in $\langle X, \mathscr{U} \rangle$. The sets $B_{t_1,t_2} = \{x; d(x_{t_1}, x_{t_2}) > \varepsilon\}$, with $\{t_1, t_2\} \subset A \in \mathscr{T}_\varepsilon$, are open and cover K_ε. Thus there is a union B of a finite number of such sets for which $B \supset K_\varepsilon$. Since $B \subset X - D_\varepsilon$, $B \in \pi_T$, we obtain $\bar{p}B \leqq \liminf (p_\alpha)_- B < \varepsilon$ and $p_- D_\varepsilon' > 1 - 2\varepsilon$. Thus p is almost simple with cover $\mathscr{T}_\varepsilon' = \mathscr{T}_{\varepsilon/2}$.

Part (iii): Since $\langle p_\alpha \rangle$ is almost simple and $p_\alpha \overset{\sim}{\to} p$ on π_T we obtain that there are finite covers $\mathscr{T}_\varepsilon$ of T satisfying

$$(1) \qquad p\{x; \operatorname{diam} x[A \cap T_1] < \tfrac{1}{2}\varepsilon \text{ for every } A \in \mathscr{T}_\varepsilon\} \geqq 1 - \varepsilon/2$$

for every finite subset T_1 of T. It follows that (1) holds for every countable subset of T.

The rest of the proof consists of extending p to describe a separable stochastic process. To facilitate the notation let Y be a stochastic process with the probability distribution p. If A is in $\mathscr{T} = \bigcup \{\mathscr{T}_\varepsilon; \varepsilon^{-1} = 1, 2, \cdots\}$, select a $t_0 \in A$. The family of all random variables $d(Y_t, Y_{t_0})$, $t \in A$ has a countable subfamily with the same essential supremum, i.e., there is a countable subset T_A of A such that, for each t,

$$(2) \qquad d(Y_t, Y_{t_0}) \leqq \sup \{d(Y_s, Y_{t_0}); s \in T_A\} \quad \text{a.e.}$$

(This should be well known and is easy to establish by taking $T_A = \{t_1, t_2, \cdots\}$ for which $E \arctan (\bigvee_{i=1}^{\infty} d(Y_{t_i}, Y_{t_0}))$ is maximized.)

For every t in A change the definition of Y_t on a null set to obtain (2) everywhere, instead of almost everywhere, for every A in $\mathscr{T}$. Define $\bar{p}$ by $\bar{p}M = P(Y^{-1}[M])$ for every M such that $Y^{-1}[M]$ is P-measurable. Then $\bar{p}$ extends p since the change did not affect the finite dimensional distributions of Y and, for every $\varepsilon^{-1} = 1, 2, \cdots$,

$$\bar{p}_-\{x, \operatorname{diam} x[A] < \varepsilon \text{ for every } A \in \mathscr{T}_\varepsilon\} > 1 - \varepsilon$$

which shows that $\bar{p}$ is almost simple. Since the family of covers $\mathscr{T}_\varepsilon$ can be taken the same for p as for p_α, we obtain the desired result.

2.10. Remark. Suppose p is almost simple, $p_\alpha \overset{\sim}{\to} p$ on π_T, $\mathscr{D}p \supset \pi_T$ and $\langle T, v \rangle$ is a compact metric space. Set $w_x(\delta) = \sup \{d(x_s, x_t); v(s, t) < \delta\}$. Then a sufficient (and necessary, if $pC = 1$) condition for

$$(1) \qquad p_\alpha \overset{\sim}{\to} p \quad \text{on} \quad \mathscr{U}$$

is that for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$(2) \qquad (p_\alpha)_-\{x; w_x(\delta) < \varepsilon\} > 1 - \varepsilon \quad \text{eventually.}$$

Indeed the sufficiency follows easily, since if the condition holds, one can find finite covers $\mathscr{T}_\varepsilon$ such that the corresponding sets D_ε contain the events in (2). We then obtain the sufficiency from Theorem 2.8. The necessity follows since the sets in (2) are in $\mathscr{U}$ and their union, as $\delta \to 0$, contains C.

The above result follows from Dudley's Theorem 1 and Proposition 2 under the additional assumptions that $\langle Z, d \rangle$ is the real line, X is the set of all bounded functions on T, p is tight, and

(3) $$S_T(x, \varepsilon) \in \mathscr{D}p \cap \mathscr{D}p_\alpha \qquad \text{for every} \quad \alpha, \varepsilon > 0, x \in X.$$

Our proof is not substantially affected by considering a metric space $\langle Z, d\rangle$ instead of the real line. However, the original proof, using a tightness argument, cannot be successful in this generality. Indeed, suppose T is a singleton, p a nontight probability, $\mathscr{D}p \supset \pi_T$, $p_\alpha = p$. Then $p_\alpha \to p$ on $\mathscr{U}$ (trivially), but $\langle p_\alpha\rangle$ is not tight. On the other hand conditions of Theorem 2.8 are satisfied with $\mathscr{T}_\varepsilon = \{T\}$, $D_\varepsilon = X$.

2.11. Remark. Suppose p is almost simple, $p_\alpha \gtrsim p$ on π_T, $\mathscr{D}p \supset \pi_T$, $\langle T, v\rangle$ is a compact metric space. Suppose (2.10.3) holds.

Then a sufficient (and necessary, if $pC = 1$) condition for

(1) $$p_\alpha \gtrsim p \qquad \text{on} \quad \mathscr{U}$$

is that for every $\varepsilon > 0$, there is a $\delta > 0$ such that

(2) $$\bar{p}_\alpha\{x;\ w_x(\delta) < \varepsilon\} > 1 - \varepsilon \quad \text{eventually.}$$

In view of Remark 2.10 it is enough to show that (2) is sufficient. To do so, repeat the argument in Remark 2.10, observing that the sets D_ε are in $\mathscr{D}p_\alpha \cap \mathscr{D}_p$ because of (2.10.3) and (2) thus implies that $p_\alpha D_\varepsilon = (p_\alpha)_- D_\varepsilon > 1 - \varepsilon$ eventually.

2.12. Remark. The standard definition of $p_\alpha \to p$ vaguely on $\mathscr{U}$ is stronger than ours in that it requires $\mathscr{D}p_\alpha \supset \mathscr{U}$, $\mathscr{D}p \supset \mathscr{U}$ in addition to $p_\alpha \gtrsim p$ on $\mathscr{U}$. Of course, with $\mathscr{D}p_\alpha \supset \mathscr{U}$, $\mathscr{D}p \supset \mathscr{U}$, the condition $p_\alpha \gtrsim p$ on $\mathscr{U}$ can be restated as

(1) $$\liminf p_\alpha O \geqq pO \qquad \text{for every} \quad O \in \mathscr{U}.$$

There are frequent situations in which $\mathscr{D}p_\alpha$ do not contain $\mathscr{U}$. The standard solution has been to find another topology, say $\mathscr{S}$, such that $\mathscr{D}p_\alpha \supset \mathscr{S}$, $\mathscr{D}p \supset \mathscr{S}$ and (1) holds with $\mathscr{U}$ replaced by $\mathscr{S}$. In many cases this program works with $\mathscr{S}$ the Skorohod topology (see Billingsley, 1968, Section 14). However, the following question occurs: why, in selecting a smaller family than $\mathscr{U}$, should we limit ourselves to topologies, and not, e.g., to all sets O in $\mathscr{U}$ which are measurable? Or, more generally, why not replace p_α, p in (1) by $(p_\alpha)_-$ and $\bar{p}$?

It turns out that if $T = [0, 1]$, Z is the real line, $pC = 1$ and $p_\alpha \to p$ vaguely on $\mathscr{S}$ in the standard sense, then, indeed, a stronger result holds, namely $p_\alpha \gtrsim p$ on $\mathscr{U}$ (see Remark 3.3).

The results by Dudley, and our preceding theorems, give necessary and sufficient conditions to establish $p_\alpha \gtrsim p$ on $\mathscr{U}$. Thus they can be used to simplify and strengthen all results concerning vague convergence, with respect to the Skorohod topology, to a probability concentrated on C.

3. Extension to the Skorohod metric.

3.1. ASSUMPTION. We assume now that $T = [0, 1]$, D is the family of all functions on T into $\langle Z, d\rangle$ which are right-continuous and have left-hand limits. Define the Skorohod pseudometric σ for $X = Z^T$, and the function $w_x'(\delta)$ as in Billingsley (1968, Section 14), replacing the distance $|x_t - y_{\lambda t}|$ by $d(x_t, y_{\lambda t})$. Denote by $\mathscr{S}$ the topology induced by σ.

3.2. THEOREM. *Suppose Assumption* 3.1 *holds,* T_0 *is a dense subset of* $[0, 1]$, $1 \in T_0$, $p_\alpha \gtrsim\!\!\to p$ *on* π_{T_0}, $\mathscr{S} \subset \mathscr{D}p$, $p(D) = 1$. *For* $\varepsilon > 0$, $\delta > 0$ *let*

$$D_{\varepsilon,\delta} = \{x;\, x \in X,\, w_x'(\delta) < \varepsilon\}\,.$$

Then

$$(1) \qquad p_\alpha \gtrsim\!\!\to p \quad on \quad \mathscr{S}$$

if and only if to every $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that*

$$(2) \qquad (p_\alpha)_- D_{\varepsilon,\delta} > 1 - \varepsilon \quad eventually.$$

PROOF. Suppose $\varepsilon > 0$. Extending Lemma 1 in Billingsley (1968, Section 14) we obtain $D \cap D_{\varepsilon,\delta} \uparrow D$ as $\delta \to 0$. Also, $D_{\varepsilon,\delta} \in \mathscr{S}$. Select δ in $(0, \varepsilon)$ such that $pD_{\varepsilon/4,\delta} > 1 - \varepsilon$ and set $C_\varepsilon = D_{\varepsilon/4,\delta}$.

Take a finite subset T_ε of T_0 such that every point in T has distance $\delta/5$ or less from T_ε, $1 \in T_\varepsilon$. We want to prove that (2.6.1) holds for $\rho = \sigma$, $\eta_\varepsilon = \varepsilon/4$. Thus let

$$(3) \qquad x \in C_\varepsilon,\, y \in S_{T_\varepsilon}(x, \varepsilon/4) \cap C_\varepsilon\,.$$

There are finite decompositions H_x, H_y of $[0, 1)$ into intervals of form $[a, b)$ and length at least δ and such that

$$(4) \qquad \operatorname{diam} x[I] < \varepsilon/4\,, \qquad \operatorname{diam} y[J] < \varepsilon/4$$

for all $I \in H_x$, $J \in H_y$. Define H as the family of all nonempty intersections $I \cap J$ with $I \in H_x$, $J \in H_y$. Consider a U in H.

If the length of U is greater than $\delta/5$ then U intersects T_ε and from (3) and (4)

$$(5) \qquad u_U(x, y) < \tfrac{3}{4}\varepsilon\,.$$

Suppose the length of U is $\delta/5$ or less, $U = [b, c)$. Then there are $[a, c)$, $[b, d)$ in H with lengths at least $\frac{4}{5}\delta$ and such that $[a, c) \subset I \in H_x$, $[b, d) \subset J \in H_y$ (or the same holds with $I \in H_y$, $J \in H_x$; we will treat the first case only).

Take an increasing function λ on $[a, d]$ onto $[a, d)$ which maps $[a, c)$ onto $[a, b)$ and satisfies $\sup_t |\lambda t - t| \leqq \delta/5$, and $\lambda t = t$ on $[a, b - \delta/5)$, $[c + \delta/5, d)$. Since $d(x_{t_1}, y_{t_1}) < \varepsilon/4$ for a t_1 in $[a, b) \cap T_\varepsilon$ and because $\operatorname{diam} x[[a, c)] < \varepsilon/4$, $\operatorname{diam} y[[a, b)] < \varepsilon/4$, we obtain

$$(6) \qquad d(x_t, y_{\lambda t}) < \tfrac{3}{4}\varepsilon$$

for all $t \in [a, c)$. By a similar argument, we obtain that (6) holds for every t in $[c, d)$, thus for every t in $[a, d)$.

Extend the definition of λ by the same construction for all other U with the length $\delta/5$ or less, verify that this does not lead to any conflict, and complete the definition by setting $\lambda t = t$ for t where λ has not been yet defined. We obtain then from (5) that (6) holds for all t in [0, 1] and

$$\rho(x, y) \leqq u_T(x, y \circ \lambda) + \sup |\lambda t - t| < \varepsilon . \tag{7}$$

This shows that (2.6.1) is satisfied, if δ is sufficiently small.

If our condition holds, δ can be chosen so that (2.6.4) is satisfied and by Lemma 2.6 our condition is sufficient for (1). Its necessity follows since the sets $D_{\varepsilon,\delta}$ are in $\mathscr{S}$.

3.3. COROLLARY. *Suppose the assumptions of Theorem* 3.2 *hold,* $\langle Z, d\rangle$ *is the real line,* $\mathscr{U} \subset \mathscr{D}p$ *and* $p(C) = 1$.

Then

$$p_\alpha \overset{\geq}{\rightarrow} p \quad \text{on} \quad \mathscr{U} \tag{1}$$

if and only if for every $\varepsilon > 0$ *there is a* $\delta > 0$ *such that*

$$(p_\alpha)_- D_{\varepsilon,\delta} > 1 - \varepsilon \quad \text{eventually}. \tag{2}$$

PROOF. In view of Theorem 3.2 and since $\mathscr{U} \supset \mathscr{S}$, it is enough to show that under the present conditions $p_\alpha \overset{\geq}{\rightarrow} p$ on $\mathscr{S}$ implies (1).

The proof follows an argument which Billingsley (1968, Section 18) shows to almost prove that under our conditions, $p_\alpha \overset{\geq}{\rightarrow} p$ on $\mathscr{S}$ implies $\liminf p_\alpha O \geqq pO$ for every $O \in \mathscr{U}$. For our definition of vague convergence, the proof goes through without any difficulty. If F is a $\mathscr{U}$-closed subset of X then the closure $\bar{F}^{\mathscr{S}}$ of F in $\langle X, \mathscr{S}\rangle$ is a subset of $F \cup (X - C)$, since the restrictions of $\mathscr{S}$ and $\mathscr{U}$ to C concide (Billingsley 1968, Section 14). Thus

$$\limsup \bar{p}_\alpha F \leqq \underline{p}(F \cup (X - C)) = pF$$

and (1) holds.

3.4. REMARK. Under the assumptions of Corollary 3.3, both the conditions in this Corollary and the condition in Remark 2.10 are necessary and sufficient for $p_\alpha \overset{\geq}{\rightarrow} p$ on $\mathscr{U}$, but relation (3.3.2) is easier to verify (with $D_{\varepsilon,\delta}$ as in Theorem 3.2) than (2.10.1).

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