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# THE CLASS OF SUBEXPONENTIAL DISTRIBUTIONS

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The class $\mathscr{S}$ of subexponential distributions is characterized by $F(0) = 0$, $1 - F^{(2)}(x) \sim 2\{1 - F(x)\}$ as $x \to \infty$. New properties of the class $\mathscr{S}$ are derived as well as for the more general case where $1 - F^{(2)}(x) \sim \beta\{1 - F(x)\}$. An application to transient renewal theory illustrates these results as does an adaptation of a result of Greenwood on randomly stopped sums of subexponentially distributed random variables.

**1. Introduction.** Let $F$ be a distribution function on $(0, \infty)$ for which $F(0+) = 0$, $F(x) < 1$ for all $x > 0$, $F(\infty) = 1$. Let the Laplace–Stieltjes transform of $F$ be denoted by $\phi$.

The distribution $F$ is said to belong to the *subexponential class* $\mathscr{S}$ if

$$\lim_{x\to\infty} \frac{1 - F^{(2)}(x)}{1 - F(x)} = 2 .$$

In spite of the simplicity of their definition, subexponential distributions have not been studied in much detail. The first paper dealing with $\mathscr{S}$ is due to Chistyakov [3], where an application to branching processes is given. Since then essential results in the latter field have been obtained using properties of $\mathscr{S}$ [1]. Especially the papers by Chover, Ney and Wainger [4, 5] are important in this connection. See also the papers by Pakes [10] and W. L. Smith [11].

In this paper we give a few new properties of $\mathscr{S}$ together with an application to renewal theory. Among others we will give proofs of the results, stated without proof in [12]. An adaptation of a recent result of Greenwood [8] is included as a further illustration.

**2. Properties of $\mathscr{S}$.** An elegant and basic property of $\mathscr{S}$ was found by Chistyakov [3]; for a proof see also [1].

LEMMA 1. *$F \in \mathscr{S}$ iff for every $n = 2, 3, \cdots$ $\lim_{x\to\infty} (1 - F^{(n)}(x))/(1 - F(x)) = n$.*

As a complement we prove in Section 3:

THEOREM 1. (i) *$F \in \mathscr{S}$ if and only if $(1 - F^{(2)}(x))/(1 - F(x)) \to \beta \in (0, \infty)$ and $\phi(\lambda)$ is not analytic at $\lambda = 0$;*

(ii) *$(1 - F^{(2)}(x))/(1 - F(x)) \to \beta > 2$ if and only if $F(x) = \beta/2 \int_0^x e^{-\gamma y} G(dy)$ where $G \in \mathscr{S}$ and $\phi(-\gamma) = \beta/2$, $\gamma > 0$.*

Theorem 1 (ii) provides us with a characterization of the class $\mathscr{S}(d)$ of distributions, allied to the subexponential class, and treated in [5] in connection with

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branching processes. For simplicity we say that $F \in \mathscr{S}_\gamma$ iff $F$ is of the form given in (ii) or iff $1/\phi(-\gamma) \int_0^x e^{\gamma y} F(dy) \in \mathscr{S} \equiv \mathscr{S}_0$. It then follows that $\mathscr{S}(d)$ coincides with $\mathscr{S}_\gamma$ where $d = \beta/2$.

The above result proves useful in deriving comparison theorems in transient renewal theory as illustrated in Theorem 4. Another elegant result is due to Kesten–Athreya–Ney [1].

LEMMA 2. *If $F \in \mathscr{S}$ then for any $\varepsilon > 0$ there exists a $K < \infty$ independent of $n$ such that for all $x \geqq 0$ and $n = 2, 3, \cdots$*

$$\frac{1 - F^{(n)}(x)}{1 - F(x)} \leqq K(1 + \varepsilon)^n .$$

LEMMA 3 [1, 3]. *If $F \in \mathscr{S}$, then $1 - F(\log x)$ is slowly varying as $x \to \infty$,* i.e. *as $x \to \infty$, uniformly on compact $y$-sets*

$$\frac{1 - F(x + y)}{1 - F(x)} \to 1 .$$

This result plays a fundamental role in the study of $\mathscr{S}$; unfortunately it does not characterize $\mathscr{S}$.

Since $\mathscr{S}$ is defined by a property of $F^{(2)}$ it seems intuitively difficult to characterize $\mathscr{S}$ solely in terms of $F$. The next result contains a nontrivial sharpening of a result of Chistyakov [3]. We put $\psi(x) = -\log\{1 - F(x)\}$.

THEOREM 2. *If the following conditions are satisfied, then $F \in \mathscr{S}$:*

(i) *$\psi(x)$ is asymptotically concave;*
(ii) *there exists a function $g$ such that $0 < g(x) \to \infty$, $x - g(x) \to \infty$ for which*

$$\lim_{x\to\infty} \frac{1 - F[x - g(x)]}{1 - F(x)} = 1 ;$$

(iii) *$\psi(x) \exp\{-\psi[g(x)]\} \to 0$ as $x \to \infty$.*

Theorem 2 or the method used in its proof provides us with the following examples of members of $\mathscr{S}$.

COROLLARY 1. *If $\psi'$ is nonincreasing and*

$$\lim_{\varepsilon\to 0} \lim_{x\to\infty} \psi(x) \exp\left\{ -\psi\left[\frac{\varepsilon x}{\psi(x)}\right]\right\} ,$$

*then $F \in \mathscr{S}$.*

COROLLARY 2. (i) *If $1 - F(x) \sim x^{-\alpha}L(x)$; $\alpha \geqq 0$ and $L$ is slowly varying, then $F \in \mathscr{S}$;*

(ii) *if $\psi(x) \sim x^\alpha L(x)$, $0 < \alpha < 1$ and $L$ is slowly varying, then $F \in \mathscr{S}$.*

Part (i) has been derived by Feller [7], Chistyakov [3]. Part (ii) extends a result in [1, 3]. We can similarly show that if $\psi(x) = x(\log x)^{-\beta}$, $\beta > 0$, then $F \in \mathscr{S}$ iff $\beta > 1$; this shows that (iii) in Theorem 2 cannot be dropped entirely.

The next result shows that $\mathscr{S}$ is closed under asymptotic equality:

THEOREM 3. *If* $F \in \mathscr{S}$ *and* $1 - F(x) \sim 1 - G(x)$ *then* $G \in \mathscr{S}$.

## 3. Proofs.

3a. *Proof of Theorem* 1(i). Assume $(1 - F^{(2)}(x))/(1 - F(x)) \to \beta$, $\phi(\lambda)$ singular at 0.

Since $1 - F^{(2)}(x) = 1 - F(x) + \int_0^x [1 - F(x-y)]F(dy)$ we might as well assume that

$$I(x) \equiv \int_0^x \frac{1 - F(x-y)}{1 - F(x)} F(dy) \to \beta - 1 . \tag{1}$$

Since $F$ is nondecreasing we have for every $A$ such that $0 \leqq A \leqq x$

$$\begin{aligned} I(x) &\geqq \int_A^x \frac{1 - F(x-y)}{1 - F(x)} F(dy) + F(A) \\ &\geqq \frac{1 - F(x-A)}{1 - F(x)} \{F(x) - F(A)\} + F(A) . \end{aligned} \tag{2}$$

Hence by (1)

$$1 \leqq \frac{1 - F(x-A)}{1 - F(x)} \leqq \frac{I(x) - F(A)}{F(x) - F(A)} \to \frac{\beta - 1 - F(A)}{1 - F(A)} \equiv M(A) . \tag{3}$$

Incidentally, putting $A = 0$, we obtain $\beta \geqq 2$. For every $0 \leqq y$, finite

$$1 \leqq \liminf_{x\to\infty} \frac{1 - F(x-y)}{1 - F(x)} \leqq \limsup_{x\to\infty} \frac{1 - F(x-y)}{1 - F(x)} \leqq M(y) .$$

Let $\{x_n; n = 1, 2, \cdots\}$ be a nonnegative sequence of real numbers, increasing to $\infty$ with $n$. Put

$$t_n(y) = \frac{1 - F(x_n - y)}{1 - F(x_n)} .$$

Then $\{t_n(y), n = 1, 2, \cdots\}$ is a sequence of functions, uniformly bounded on finite intervals, by the selection principle [7, 14] there exists a subsequence $\tau_n \equiv x_{m_n} \uparrow \infty$ such that everywhere on $(0, \infty)$

$$\mu_n(y) \equiv \frac{1 - F(\tau_n - y)}{1 - F(\tau_n)} \to \mu(y) .$$

The limit $\mu(y)$ satisfies $\mu(0) = 1$, $\mu(y + s) = \mu(y)\mu(s)$; further $\mu(y)$ is nondecreasing in $y$ and is finite. But then $\mu(y) = e^{\gamma y}$ where $\gamma \geqq 0$.

But then for any finite $A$ by Fatou's lemma

$$\begin{aligned} \int_0^A e^{\gamma y} F(dy) &= \int_0^A \lim_{n\to\infty} \frac{1 - F(\tau_n - y)}{1 - F(\tau_n)} F(dy) \\ &\leqq \lim_{n\to\infty} \int_0^{\tau_n} \frac{1 - F(\tau_n - y)}{1 - F(\tau_n)} F(dy) = \beta - 1 . \end{aligned}$$

Letting $A \to \infty$ we see that $\phi(-\gamma) \leqq \beta - 1$, where $\gamma \geqq 0$. By the Vivanti–Pringsheim theorem the abscissa of convergence of $\phi(\lambda)$ is a singularity of $\phi(\lambda)$, hence $\gamma = 0$ for otherwise $\phi(\lambda)$ would be analytic.

This implies that for any finite $A$

$$\lim_{n\to\infty} \frac{1 - F(\tau_n - A)}{1 - F(\tau_n)} = 1 ,$$

and since the same argument can be repeated for any subsequence we conclude that

$$\lim_{t\to\infty} \frac{1 - F(t - A)}{1 - F(t)} = 1 \tag{4}$$

for any finite $A$.

It remains to show that (4) together with (1) implies that $\beta = 2$. Substitute $\tau_n$ for $x$ in (1); then for any given $\varepsilon > 0$ we can by (1) choose $B$ so large that

$$\int_B^{\tau_n} \frac{1 - F(\tau_n - y)}{1 - F(\tau_n)} F(dy) < \varepsilon$$

for $n \geqq n_0(\varepsilon)$. But then by bounded convergence and (4)

$$\int_0^B \frac{1 - F(\tau_n - y)}{1 - F(\tau_n)} F(dy) \to F(B) \leqq 1 .$$

Hence for $n_0$ large enough

$$\lim_{n\to\infty} \int_0^{\tau_n} \frac{1 - F(\tau_n - y)}{1 - F(\tau_n)} F(dy) \leqq 1 + \varepsilon$$

and henceforth $\beta = 2$. This proves one implication.

Conversely, if $F \in \mathscr{S}$ then Chistyakov has shown that $\phi(\lambda)$ is not analytic. A proof following Chistyakov's ideas is contained in the proof of part (ii).

3b. *Proof of Theorem* 1 (ii). Assume $\lim_{x\to\infty} (1 - F^{(2)}(x))/(1 - F(x)) = \beta > 2$. Again we can show that $\phi(-\gamma) \leqq \beta - 1$ as before, where now $\gamma > 0$ since otherwise $\beta = 2$ by part (i). Then $\phi(\lambda) = \int_0^\infty e^{-\lambda x} F(dx)$ exists for $\operatorname{Re} \lambda \geqq -\gamma$ by well-known properties of the Laplace–Stieltjes transform.

We show that $\lambda = -\gamma$ is actually a singularity of $\phi(\lambda)$. This follows partially from the Vivanti–Pringsheim theorem [14]. Indeed, pick any $\varepsilon > 0$. Then for all $x > 0$

$$\int_0^\infty e^{(\gamma+\varepsilon)y} F(dy) > e^{(\gamma+\varepsilon)x}\{1 - F(x)\} .$$

By the above construction $\psi(\tau_n) - \psi(\tau_n - y) \to \gamma y$, or

$$[\psi(\tau_n) - \gamma\tau_n] - [\psi(\tau_n - y) - \gamma(\tau_n - y)] \to 0 .$$

By a classical result of Cauchy or by the fact that for any slowly varying function $L(x)$, $\log L(x)/\log x \to 0$ as $x \to \infty$ [7, 9] we obtain that since $\tau_n \uparrow \infty$

$$\tau_n^{-1}\{\psi(\tau_n) - \gamma\tau_n\} \to 0$$

or

$$\phi(-\gamma-\varepsilon) > \exp\{(\gamma+\varepsilon)\tau_n - \psi(\tau_n)\} = \exp\tau_n \left\{\gamma+\varepsilon-\frac{\psi(\tau_n)}{\tau_n}\right\}.$$

Choose $n$ so large that $|\gamma - \psi(\tau_n)/\tau_n| < \varepsilon/2$, then since $\varepsilon > 0$

$$\phi(-\gamma-\varepsilon) > \exp\left\{\frac{\varepsilon}{2}\tau_n\right\} \to \infty .$$

Hence $\lambda = -\gamma$ is a singularity of $\phi(\lambda)$.

Now put

$$G(x) = \frac{1}{\phi(-\gamma)} \int_0^x e^{\gamma y} F(dy)$$

then $G(x)$ is a distribution function with Laplace–Stieltjes transforms $\phi_\gamma(\lambda) = \phi(\lambda-\gamma)/\phi(-\gamma)$. Using some elementary algebra one easily shows that $(1-F^{(2)}(x))/(1-F(x)) \to \beta$ implies

$$\lim_{x\to\infty} \frac{1-G^{(2)}(x)}{1-G(x)} = \frac{\beta}{\phi(-\gamma)} \equiv \beta_\gamma . \tag{5}$$

By the above considerations $\phi_\gamma(\lambda)$ is singular at $\lambda = 0$; but then by (i) of this theorem $\beta_\gamma = 2$ or $\beta = 2\phi(-\gamma)$. Hence from the definition of $G(x)$

$$F(x) = \frac{\beta}{2} \int_0^x e^{-\gamma y} G(dy)$$

where $G(x) \in \mathscr{S}$.

Conversely if $F(x)$ is of the above form the same algebra as used to derive (5) proves that $(1-F^{(2)}(x))/(1-F(x)) \to \beta > 2$ since $\gamma > 0$. This proves the theorem.

Corollary 3. *If $F \in \mathscr{S}_\gamma$, then $\phi(\lambda)$ has a singularity at $\lambda = -\gamma \leqq 0$.*

Corollary 4. *If $F \in \mathscr{S}_\gamma$, then $x^{-1}\psi(x) \to \gamma$.*

We remark that putting $A = -y$ and $\beta = 2$ in (3) we proved also Lemma 3.

3c. *Proof of Theorem* 2. Similarly to the Feller inequalities [7] we can show that for any $t < x/2$

$$2F(t) + J(x,t) \leqq \frac{1-F^{(2)}(x)}{1-F(x)} \leqq 2\,\frac{1-F(x-t)}{1-F(x)} + J(x,t) \tag{6}$$

where

$$J(x,t) = \int_t^{x-t} \frac{1-F(x-u)}{1-F(x)} F(du) .$$

Using $\psi(x)$ as defined above we obtain

$$J(x,t) = \int_t^{x-t} e^{\psi(x)-\psi(x-u)-\psi(u)} \psi(du) .$$

We show that under the three given conditions $J(x, g(x)) \to 0$ as $x \to \infty$. By the asymptotic concavity, and $0 < t \leqq u \leqq x/2 \leqq x-u \leqq x-t < x$,

$$\psi(x-u) + \psi(u) \geqq \psi(x-t) + \psi(t) .$$

Now $F \in \mathscr{S}$ iff $J(x, g(x)) \to 0$ as $x \to \infty$. But

$$J(x, g(x)) \leqq \frac{1 - F[x - g(x)]}{1 - F(x)} a(x)$$

where

$$a(x) = \{1 - F[g(x)]\} \log \{1 - F[g(x)]\} - \{1 - F[g(x)]\} \log \left\{\frac{1 - F[x - g(x)]}{1 - F(x)}\right\}$$
$$- \{1 - F[g(x)]\} \log \{1 - F(x)\} .$$

The first term tends to zero since $\lim_{u \to 0} u \log u = 0$; the second clearly tends to zero, as does the third by condition (iii) of the theorem. This finishes the proof.

3d. *Proof of Corollary* 1. Assume $\psi'(x) \downarrow$. Then $x\psi'(x) \leqq \int_0^x \psi'(u)\,du = \psi(x)$. Moreover $\psi'(u) \geqq \psi'(t - u)$ for $u \leqq t/2$; by integration we obtain

$$\psi(u) + \psi(x - u) \geqq \psi(t) + \psi(x - t) , \qquad t \leqq u \leqq x - t$$

which implies (i).

To prove (ii) take any $\varepsilon > 0$ and put $g_\varepsilon(x) = \varepsilon x/\psi(x)$. By the mean value theorem there exists a $\theta(x) \in [0, 1]$ such that

$$0 \leqq \log \frac{1 - F[x - g_\varepsilon(x)]}{1 - F(x)} = \psi(x) - \psi\left\{x - \varepsilon \frac{x}{\psi(x)}\right\} = \frac{\varepsilon x}{\psi(x)} \psi'\left\{x - \frac{\theta \varepsilon x}{\psi(x)}\right\} .$$

Now $\psi' \downarrow$ and hence this expression is bounded by

$$\frac{\varepsilon x}{\psi(x)} \psi'\left\{x - \frac{\varepsilon x}{\psi(x)}\right\}$$
$$= \varepsilon \frac{\left[x - \frac{\varepsilon x}{\psi(x)}\right] \psi'\left[x - \frac{\varepsilon x}{\psi(x)}\right]}{\psi\left[x - \frac{\varepsilon x}{\psi(x)}\right]} \cdot \frac{1}{1 - \varepsilon/\psi(x)} \cdot \frac{\psi[x - \varepsilon x/\psi(x)]}{\psi(x)}$$
$$\leqq \varepsilon \cdot 1 \cdot \left\{1 - \frac{\varepsilon}{\psi(x)}\right\}^{-1} \cdot 1 \leqq 2\varepsilon \qquad \text{for } x \text{ large enough.}$$

This shows that

$$\lim_{\varepsilon \downarrow 0} \lim_{x \to \infty} \frac{1 - F[x - g_\varepsilon(x)]}{1 - F(x)} = 1 .$$

The additional condition in the statement of the corollary replaces (iii) of Theorem 2.

3e. *Proof of Corollary* 2.

(i) The proof is simple by using $g(x) = \varepsilon x$ and letting $\varepsilon \to 0$.

(ii) We use some results from the theory of regular variation [6, 7, 9]. It is well-known that since $\alpha > 0$ we can replace $\psi(x) = x^\alpha L(x)$ by $\psi_1(x) = x^\alpha L_1(x)$ where $\psi_1(x)$ is differentiable and $xL_1'(x)/L_1(x) \to 0$. But then $\psi_1'(x) \sim \alpha x^{\alpha-1} L_1(x)$ which is asymptotically equal to a nonincreasing function since $\alpha - 1 < 0$. Hence $\psi(x) \sim x^\alpha L_2(x)$ where $L(x) \sim L_2(x)$ and $x^\alpha L_2(x)$ is asymptotically concave.

Put

$$\psi(x) = x^\alpha L_2(x) L_3(x)$$

where $L_3(x) \to 1$.

Pick any $\varepsilon \leqq \frac{1}{2}$ and choose $x$ so large that $|L_3(x-u) - 1| < \varepsilon$ for $t \leqq u \leqq x - t$. Hence for this range

$$\begin{aligned}\psi(x-u) + \psi(u) &= (x-u)^\alpha L_2(x-u) L_3(x-u) + u^\alpha L_2(u) L_3(u) \\ &\geqq (1-\varepsilon)\{(x-u)^\alpha L_2(x-u) + u^\alpha L_2(u)\} \\ &\geqq \frac{1-\varepsilon}{1+\varepsilon}\{\psi(x-t) + \psi(t)\}\end{aligned}$$

since $x^\alpha L_2(x)$ is concave. For this fixed $\varepsilon$ the proof of Theorem 2 can be repeated using in (ii) $g(x) = x^\gamma$ where $\gamma = (1-\alpha)/2 > 0$. For (iii) choose $k \geqq 1 + [\gamma^{-1}]$; then for $x$ large enough $e^{-x} < Cx^{-k}$, hence

$$e^{-\psi[g(x)]}\psi(x) \leqq C(1+\varepsilon)^2 x^{\alpha(1-\gamma k)} L(x) L^{-k}(x^\gamma) \to 0$$

by another property of slowly varying functions, since $\alpha(1-\gamma k) < 0$.

3f. *Proof of Theorem* 3. As indicated in the proof of Theorem 1 it suffices to prove that

$$\lim_{x\to\infty} \int_0^x \frac{1-G(x-y)}{1-G(x)} G(dy) \leqq 1 . \tag{7}$$

Let $A$ be an arbitrary positive quantity. Then for $x > A$

$$I_1 \equiv \int_{x-A}^x \frac{1-G(x-y)}{1-G(x)} G(dy) \leqq \frac{G(x) - G(x-A)}{1-G(x)} = \frac{1-G(x-A)}{1-G(x)} - 1 .$$

Since $1 - F(x) \sim 1 - G(x)$, $I_1 \to 0$ as $x \to \infty$ for all finite $A$ by Lemma 3.

Choose now $\varepsilon > 0$ and $y_0(\varepsilon)$ such that $1 - \varepsilon \leqq \{1 - G(y)\}/\{1 - F(y)\} \leqq 1 + \varepsilon$ for all $y \geqq y_0$. Take $A \geqq y_0$ and $x \geqq y_0$. Then

$$\begin{aligned}I_2 &= \int_0^{x-A} \frac{1-G(x-y)}{1-G(x)} G(dy) \leqq \frac{1+\varepsilon}{1-\varepsilon} \int_0^{x-A} \frac{1-F(x-y)}{1-F(x)} G(dy) \\ &= \frac{1+\varepsilon}{1-\varepsilon}\{1-F(x)\}^{-1}\{G(x) - G * F(x) - \int_{x-A}^x [1-F(x-y)]G(dy)\} \\ &\leqq \frac{1+\varepsilon}{1-\varepsilon}\left\{1 - \frac{1-G(x)}{1-F(x)} + \int_0^x \frac{1-G(x-y)}{1-F(x)} F(dy)\right\} .\end{aligned}$$

But the integral is dominated by

$$(1+\varepsilon)\int_0^{x-A} \frac{1-F(x-y)}{1-F(x)} F(dy) + [1-G(A)] \int_{x-A}^x \frac{F(dy)}{1-F(x)} \leqq (1+\varepsilon)^2 + \varepsilon$$

since $F \in \mathscr{S}$ and by Lemma 3 on $F$ again. Putting everything together we obtain

$$\lim_{x\to\infty} \int_0^x \frac{1-G(x-y)}{1-G(x)} G(dy) \leqq \lim_{x\to\infty} I_1 + \lim_{x\to\infty} I_2 \leqq 0 + \frac{(1+2\varepsilon)^2}{1-\varepsilon} .$$

Letting $\varepsilon \to 0$ we see that $G$ satisfies (7). This finishes the proof.

COROLLARY 5. *If $F \in \mathscr{S}_\gamma$ and $1 - F \sim 1 - G$ then $G \in \mathscr{S}_\gamma$.*

PROOF. Since $1 - F \sim 1 - G$ we have successively

$$\begin{aligned} 1 - F_\gamma(x) &\equiv \frac{1}{\phi(-\gamma)} \int_x^\infty e^{\gamma y} F(dy) \\ &= \frac{1}{\phi(-\gamma)} \{e^{\gamma x}[1 - F(x)] + \gamma \int_x^\infty e^{\gamma y}[1 - F(y)]\, dy\} \\ &\sim \frac{1}{\phi(-\gamma)} \int_x^\infty e^{\gamma y} G(dy) \equiv 1 - G_\gamma(x)\,. \end{aligned}$$

But $F_\gamma \in \mathscr{S}$. So from the above theorem $G_\gamma \in \mathscr{S}$ and hence $G \in \mathscr{S}_\gamma$.

Actually one can assume $1 - F \sim c(1 - G)$ for $c > 0$ and still reach the same conclusion.

**4. An application to renewal theory.** Let $F$ be a defective distribution on $(0, \infty)$ such that $F(0+) = 0$, $F(x) < F(\infty) = \alpha$ for all $x \geqq 0$ where $0 < \alpha < 1$. The renewal function associated with $F$ is given by

$$U(x) = \sum_{n=1}^\infty F^{(n)}(x)$$

and satisfies the renewal integral equation

$$U(x) = F(x) + F * U(x)\,. \tag{8}$$

As is obvious $U(\infty) = \alpha/(1 - \alpha)$. We are interested in the common tail behavior of $F(\infty) - F(x)$ and $U(\infty) - U(x)$. The importance of $\mathscr{S}$ is fully illustrated in the next theorem which generalizes a result of Callaert-Cohen [2].

THEOREM 4. *The following statements are equivalent.*

(i) $F^{-1}(\infty)F(x) \in \mathscr{S}$;

(ii) $U^{-1}(\infty)U(x) \in \mathscr{S}$;

(iii) $\lim_{x\to\infty} \dfrac{1 - U^{-1}(\infty)U(x)}{1 - F^{-1}(\infty)F(x)} = \dfrac{1}{1 - \alpha}$.

PROOF OF (i) $\to$ (ii). A short and elegant proof can be based on Lemma 2 and can be found in [1]. Another proof in the spirit of [2] can be derived by the method used in the second part of this proof.

(ii) $\to$ (i). Let us use some abbreviations:

$$K^{-1}(x) = G(x) = \frac{U(\infty) - U(x)}{F(\infty) - F(x)} \qquad \text{and} \qquad S(x, y) = \frac{U(\infty) - U(x - y)}{U(\infty) - U(x)}\,.$$

The integral equation (8) can be rewritten in the form

$$1 - K(x)[1 + U(\infty)] = \int_0^x S(x, y)F(dy)\,. \tag{9}$$

Since $S(x, y) \geqq 1$, we obtain that

$$\bar{K} \equiv \limsup_{x\to\infty} K(x) \leqq (1 - \alpha)^2 \equiv c_0\,. \tag{10}$$

To obtain an inequality in the other direction, take $A$ such that $0 < A < x$ and write

$$1 - K(x)[1 + U(\infty)] = \{\int_0^A + \int_A^{x-A} + \int_{x-A}^x\} S(x, y)F(dy) = \mathrm{I} + \mathrm{II} + \mathrm{III}\,.$$

The integrals I and III can be easily estimated; the second requires first an integration by parts.

$$\mathrm{I} \leqq S(x, A)F(A)$$
$$\mathrm{II} \leqq \{F(\infty) - F(A)\}S(x, A) - \{U(\infty) - U(A)\}S(x, A)K(x - A) + \int_A^{x-A} K(x-t)S(x, t)U(dt)$$
$$\mathrm{III} \leqq U(\infty)K(x - A)S(x, A) - U(\infty)K(x)\,.$$

Combination of these estimates leads to

$$1 - K(x) \leqq U(A)K(x - A)S(x, A) + F(\infty)S(x, A) + \int_A^{x-A} K(x-t)S(x, t)U(dt)\,. \tag{11}$$

Pick an $\varepsilon > 0$. Choose $A$ so that for large enough $x$, $1 \leqq S(x, A) \leqq 1 + \varepsilon$ and $\int_{[A,x-A]} S(x, t)U(dt) \leqq \varepsilon$ which can be done since by assumption $U(x)/U(\infty) \in \mathscr{S}$. But then for $x$ large enough

$$1 - K(x) < U(\infty)(\bar{K} + \varepsilon)(1 + \varepsilon) + F(\infty)(1 + \varepsilon) + (\bar{K} + \varepsilon)\varepsilon\,.$$

Letting $x$ tend to infinity and then $\varepsilon \downarrow 0$ we see that $\bar{K} \geqq c_0$. Hence $\bar{K} = c_0$. To prove that also

$$\underline{K} \equiv \liminf_{x\to\infty} K(x) = c_0$$

we start again from (11). For $x$ large and $A$ as above we rewrite (11) in the form

$$K(x) \geqq 1 - U(A)(\bar{K} + \varepsilon)S(x, A) - F(\infty)S(x, A) - \varepsilon(\bar{K} + \varepsilon)\,.$$

Letting $x$ tend to infinity we obtain

$$\underline{K} \geqq 1 - U(A)(\bar{K} + \varepsilon) - F(\infty) - \varepsilon(\bar{K} + \varepsilon)\,.$$

Now let $\varepsilon \downarrow 0$ and then $A \uparrow \infty$ to obtain that $\underline{K} \geqq c_0$. This then shows that

$$\lim_{x\to\infty} K(x) = c_0 = (1 - \alpha)^2\,.$$

But then also $F(x)/F(\infty)$ belongs to $\mathscr{S}$ by virtue of Theorem 3.

(iii) $\to$ (i). We write (8) in the form

$$G(x) = 1 + U(\infty) + \int_0^x G(x - y)\frac{F(\infty) - F(x - y)}{F(\infty) - F(x)}F(dy) \tag{12}$$

where $G(x) \to c \in (0, \infty)$. Since $G(x)$ is bounded away from zero and infinity it is clear that

$$\limsup_{x\to\infty} \int_0^x \frac{F(\infty) - F(x - y)}{F(\infty) - F(x)}F(dy) < \infty\,.$$

Following the same argument as in the proof of Theorem 1 (i), we can find a sequence $\tau_n \uparrow \infty$ such that

$$\frac{F(\infty) - F(\tau_n - y)}{F(\infty) - F(\tau_n)} \to e^{\gamma y}$$

for $\gamma \geqq 0$. Using this sequence in (12) we obtain by Lebesgue's theorem

$$c = 1 + U(\infty) + c\int_0^\infty e^{\gamma y}F(dy)\,.$$

Since however $F(\infty) = \alpha$ and $c = (1-\alpha)^{-2}$, necessarily $\gamma = 0$. But then

$$T(x, y) \equiv \frac{F(\infty) - F(x-y)}{F(\infty) - F(x)} \to 1$$

as $x \to \infty$ for all $y$.

Given $\varepsilon > 0$, choose $A$ so large that $|G(y) - c| < \varepsilon$ for $y > A$. Then take $x$ so large that $T(x, A) - 1 \leqq \varepsilon$. Then

$$\begin{aligned} &|c \textstyle\int_0^x T(x, y)F(dy) - [G(x) - 1 - U(\infty)]| \\ &\quad \leqq \textstyle\int_0^{x-A} |G(x-y) - c| T(x, y)F(dy) + \int_{x-A}^x |G(x-y) - c| T(x, y)F(dy) \\ &\quad \leqq \varepsilon \textstyle\int_0^x T(x, y)F(dy) + \{\sup_{0 \leqq t \leqq A} G(t) + c\}F(\infty)\{T(x, A) - 1\} \leqq M\varepsilon \,. \end{aligned}$$

for some constant $M$.

Letting $x \to \infty$ and then $\varepsilon \to 0$ we see that

$$\lim_{x\to\infty} \int_0^x \frac{F(\infty) - F(x-y)}{F(\infty) - F(x)} F(dy) = \frac{c - 1 - U(\infty)}{c} \,.$$

Put $H(x) = F^{-1}(\infty)F(x)$. Then

$$\frac{1 - H^{(2)}(x)}{1 - H(x)} = \frac{1}{F(\infty)} \left\{ \frac{F^2(\infty) - F(\infty)F(x)}{F(\infty) - F(x)} + \int_0^x \frac{F(\infty) - F(x-y)}{F(\infty) - F(x)} F(dy) \right\}$$

and henceforth

$$\lim_{x\to\infty} \frac{1 - H^{(2)}(x)}{1 - H(x)} = 1 + \frac{1}{F(\infty)} \frac{c - 1 - U(\infty)}{c} \equiv 2 \,.$$

But then $H \in \mathscr{S}$; by using Theorem 3 also $U(\infty)U^{-1}(x) \in \mathscr{S}$. This finishes the proof.

We generalize the above theorem to the class $\mathscr{S}_\gamma$; part of the theorem can be found in [5].

COROLLARY 6. *Assume that* $\phi(-\gamma) = \int_0^\infty e^{\gamma x} F(dx) < 1$ *where* $\gamma \geqq 0$. *Then the following statements are equivalent.*

(i) $F^{-1}(\infty)F(x) \in \mathscr{S}_\gamma$;
(ii) $U^{-1}(\infty)U(x) \in \mathscr{S}_\gamma$;
(iii) $\lim_{x\to\infty} \dfrac{1 - U^{-1}(\infty)U(x)}{1 - F^{-1}(\infty)F(x)} = \dfrac{1}{1 - \phi(-\gamma)}$.

PROOF. Put $F_\gamma(x) = \int_0^x e^{\gamma y} F(dy)$; then (i) is equivalent to

(i)′ $F_\gamma^{-1}(\infty)F_\gamma(x) \in \mathscr{S}$ in view of Theorem 1.

Put $U_\gamma(x) = \int_0^x e^{\gamma y} U(dy)$; then (ii) is equivalent to

(ii)′ $U_\gamma^{-1}(\infty)U_\gamma(x) \in \mathscr{S}$.

A simple calculation shows that $U_\gamma$ is the renewal function generated by $F_\gamma$; hence by Theorem 4, (i), (i)′, (ii) and (ii)′ are all equivalent to

(iii)′ $\lim_{x\to\infty} \dfrac{U_\gamma(\infty) - U_\gamma(x)}{F_\gamma(\infty) - F_\gamma(x)} = (1 - \phi(-\gamma))^{-2}$.

Performing similar operations as in the proof of Corollary 5, one can then show that (iii)′ is equivalent to (iii).

The above theorem shows clearly how $\mathscr{S}$ is the natural class for a comparison theorem of the given character. For if the limit in (iii) is replaced by any constant $c$, we naturally end up in a class $\mathscr{S}_\gamma$.

Applications to branching processes can be derived by consulting the framework of [1]. In [12] an application to the ruin problem is investigated.

COROLLARY 7. [2]. *With the above notations the following statements are equivalent*: (i) $F(\infty) - F(x) \sim x^{-\beta}L(x)$;

(ii) $U(\infty) - U(x) \sim (1-\alpha)^{-2}x^{-\beta}L(x)$.

**5. On a theorem of Greenwood.** In [8] Greenwood considers the following problem. Let $S_n$, $n = 1, 2, \cdots$ be a sequence of sums of independent, identically distributed random variables $X_i$, let $N$ be a stopping time for $S_n$ with finite mean. When is

$$\lim_{x\to\infty} \frac{P\{S_N > x\}}{P\{X_1 > x\}} = E(N)\,?$$

Greenwood assumes that $P\{X_1 > x\} \sim x^{-\alpha}L(x)$, for $x \to \infty$. She shows that if $N \wedge n = \min(N, n)$, then for $n = 1, 2, \cdots$

$$P\{S_{N\wedge n} > x\} \sim P\{X_1 > x\}E\{N \wedge n\}\,. \tag{13}$$

We show that if $F(x) \equiv P\{X_1 \leqq x\} \in \mathscr{S}$, then (13) is satisfied. The basic step in the proof is the verification of the relation for $x \to \infty$

$$P\{S_n > x, S_{n-1} \leqq x, N \geqq n\} - P\{S_{n-1} > x, S_n \leqq x, N \geqq n\}$$
$$\sim P\{X_1 > x\}P\{N \geqq n\}$$

Conditioning on $X_n$ and using the positivity of the random variables involved we have to show that

$$P\{x - X_n < S_{n-1} \leqq x, N \geqq n\} \sim P\{X_1 > x\}P\{N > n\}\,.$$

The left hand side is equal to

$$P\{0 < S_{n-1} \leqq x, N \geqq n\}P\{X_1 > x\} + P\{0 < x - X_n < S_{n-1} \leqq x, N \geqq n\}\,.$$

Hence the first term divided by $P\{X_1 > x\}$ tends to $P\{N \geqq n\}$; the second is bounded above by

$$\int_0^x \{F^{(n-1)}(x) - F^{(n-1)}(x-u)\}F(du)$$

which leads to

$$\int_0^x \frac{F^{(n-1)}(x) - F^{(n-1)}(x-u)}{1-F(x)}F(du)$$
$$= \frac{[1 - F^{(n)}(x)] - [1 - F^{(n-1)}(x)]F(x) - [1 - F(x)]}{1 - F(x)}$$
$$\to n - (n-1) - 1 = 0 \qquad \text{as} \quad x \to \infty\,.$$

The above result now allows an extension of Greenwood's results to a larger class of asymptotic decays, however for the case of positive random variables.

REMARK. This paper deals only with a few problems on $\mathscr{S}$. A sharpening of Theorem 2 would be helpful in studying the closure properties of $\mathscr{S}$ under convolution, convex combinations, convergence in distribution, etc. A characterization of $\mathscr{S}$ in terms of properties of $\phi(\lambda)$ would be highly interesting. An extension of $\mathscr{S}$ to $(-\infty, \infty)$ might also be considered.

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