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AN APPROXIMATION THEOREM FOR CONVOLUTIONS OF PROBABILITY MEASURES

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An extension of the usual problem of bounding the total variation of the difference of two probability measures is considered for certain convolutions of probability measures on a measurable Abelian group. The result is a fairly general approximation theorem which also yields an L_p approximation theorem and a large deviation result in some special cases. A limit theorem in equally general setting is proved as a consequence of the main theorem. As the convolutions of probability measures under consideration reduce to the Poisson binomial distribution as a special case, an alternative proof of the approximation theorem in this special case is discussed.

1. Introduction and notation. Let $(\mathscr{X}, \mathscr{A})$ be a measurable Abelian group; that is, $\mathscr{X}$ is an Abelian group and $\mathscr{A}$ is a σ-algebra of subsets of $\mathscr{X}$ such that the mapping from $\mathscr{X} \times \mathscr{X}$ to $\mathscr{X}$ defined by the group operation is $(\mathscr{A} \times \mathscr{A}, \mathscr{A})$ measurable. The class $\mathscr{M}$ of all finite signed measures on $\mathscr{A}$ with the usual operations of real scalar multiplication, addition and convolution, and the norm defined to be the total mass of total variation is a real commutative Banach algebra. We assume that $\mathscr{A}$ contains the singleton consisting of the identity of $\mathscr{X}$ so that $\mathscr{M}$ contains the identity measure I which is the probability measure concentrated at the identity of $\mathscr{X}$. Let μ and ν be two finite signed measures. We shall denote the convolution of μ and ν by $\mu\nu$, the total variation of μ by $|\mu|$ and the norm of μ by $||\mu||$. We also define $\mu \leqq \nu$ by $\mu(A) \leqq \nu(A)$ for all $A \in \mathscr{A}$.

Let p_{ni} be numbers between 0 and 1 and μ_{ni} probability measures on $\mathscr{A}$ where $i = 1, \cdots, n$ and let $\lambda_n = \sum_{i=1}^n p_{ni}$ and $\mu_n = \lambda_n^{-1} \sum_{i=1}^n p_{ni} \mu_{ni}$. Consider the probability measures

$$\tilde{Q}_n = \prod_{i=1}^n [(1 - p_{ni})I + p_{ni}\mu_{ni}] \tag{1.1}$$

and

$$Q_n = e^{\lambda_n(\mu_n - I)}\,. \tag{1.2}$$

It is well known that

$$||\tilde{Q}_n - Q_n|| \leqq 2 \sum_{i=1}^n p_{ni}^2\,. \tag{1.3}$$

See, for example, Le Cam (1960). In this paper, we consider the more general problem of bounding $\int h\, d|\tilde{Q}_n - Q_n|$ where h is a measurable nonnegative

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function defined on $\mathscr{X}$ such that $\int h\, dQ_n < \infty$. If $h = 1$, then $\int h\, d|\tilde{Q}_n - Q_n| = ||\tilde{Q}_n - Q_n||$. In the main theorem, $\mu_{n1}, \cdots, \mu_{nn}$ are assumed to be mutually absolutely continuous with uniformly bounded Radon–Nikodym derivatives. This condition is satisfied if all μ_{ni}'s are equal. Despite this restriction on the μ_{ni}'s, the main theorem is fairly general in the sense that it also yields an L_p approximation theorem of the type considered by Erickson (1973) and a large deviation result in cases where $\mathscr{X}$ is a subgroup of the additive group of real numbers.

A limit theorem in an equally general setting is proved as a consequence of the main theorem. This limit theorem generalizes a previous result of the author (1974) who proved that if $\mathscr{X}$ is the additive group of integers and the μ_{ni}'s are concentrated at 1, then $\lim_{n\to\infty} \int h\, d|\tilde{Q}_n - Q_n| = 0$ where h is any nonnegative function defined on $\mathscr{X}$ such that $\int h\, dQ_n < \infty$ for all n, provided λ_n remains constant for sufficiently large n and $\max_{1\leqq i\leqq n} p_{ni} \to 0$ as $n \to \infty$.

In the subsequent sections, all notations will be the same as in this section unless otherwise stated. We shall omit the subscript n for brevity but shall pick it up whenever we need it. All functions will be assumed to be real-valued, defined on $\mathscr{X}$ and measurable. The indicator of a set A will be denoted by $\chi(A)$. Finally, we shall adopt the convention that the sum $\sum_a^b$ is empty if $b < a$.

2. The main theorem.

We first state two simple lemmas without proof.

LEMMA 2.1. *If $\alpha > 0$ and m is a nonnegative integer, then* $(1 + \alpha)^m \leqq 1 + m\alpha(1 + \alpha)^{m-1}$.

LEMMA 2.2. *Let $p_1, \cdots, p_n$ be numbers between* 0 *and* 1 *and* $\lambda = \sum_{i=1}^n p_i$. *Then*

$$0 \leqq e^{-\lambda} - \prod_{i=1}^n (1 - p_i) \leqq \tfrac{1}{2}e^{-\lambda} \sum_{i=1}^n p_i^2/(1 - p_i)\,.$$

We now state and prove the main theorem.

THEOREM 2.1. *Let $(\mathscr{X}, \mathscr{A})$ be a measurable Abelian group and let $\tilde{Q}$ and Q be given by* (1.1) *and* (1.2) *respectively. If there exists a constant K such that*

$$\mu_i \leqq K\mu_j \qquad \text{for} \quad i, j = 1, \cdots, n\,, \tag{2.1}$$

then for every nonnegative function h defined on $\mathscr{X}$ such that $\int h\, dQ < \infty$ and $m = 0, 1, \cdots, n$, we have

$$\begin{aligned} \int h\, d|\tilde{Q} - Q| \leqq{}& \tfrac{1}{2}K^2M\{\textstyle\sum_{i=1}^n p_i^2/(1 - p_i)\} \\ &\times \{(3 + 2\lambda^{-1}) \textstyle\int h\, dQ + \sum_{r=0}^{m-2} (e^{-\lambda}\lambda^r/r!) \int h\, d\mu^{r+2}\} \\ &+ \textstyle\sum_{r=m+1}^{\infty} (e^{-\lambda}\lambda^r/r!) \int h\, d\mu^r \end{aligned} \tag{2.2}$$

where r_0 is the largest integer not exceeding

$$K\{\lambda + \textstyle\sum_{i=1}^n p_i^2/(1 - p_i)\} + 1 \qquad \text{and} \qquad M = \{1 + K\lambda^{-1} \sum_{i=1}^n p_i^2/(1 - p_i)\}^{r_0}\,.$$

PROOF. We shall use arguments similar to those in Chen (1974) but in a much more general setting. To this end, we write

$$\tilde{Q} = \prod_{i=1}^n (1 - p_i) \prod_{i=1}^n [I + q_i\mu_i] = \sum_{r=0}^n \nu_r$$

where $q_i = p_i/(1-p_i)$, $\nu_0 = [\prod_{i=1}^n (1-p_i)]I$ and for $r \geqq 1$

$$\nu_r = \prod_{i=1}^n (1-p_i)\{\sum \cdots \sum_{i_1<\cdots<i_r} \prod_{k=1}^r q_{i_k}\mu_{i_k}\} .$$

We also define

$$\tilde{Q}^{(i)} = \prod_{j\neq i} [(1-p_j)I + p_j\mu_j]$$

and write

$$\tilde{Q}^{(i)} = \prod_{j\neq i}(1-p_j) \prod_{j\neq i}[I + q_j\mu_j] = \sum_{r=0}^{n-1} \nu_r^{(i)}$$

where $\nu_0^{(i)} = [\prod_{j\neq i}(1-p_j)]I$ and for $r \geqq 1$,

$$\nu_r^{(i)} = \prod_{j\neq i}(1-p_j)\{\sum \cdots \sum_{j_1<\cdots<j_r; j_1,\cdots,j_r\neq i} \prod_{k=1}^r q_{j_k}\mu_{j_k}\} .$$

We now derive generalizations of the identities and inequalities in Chen (1974). For $r \geqq 0$, we have

$$r\nu_r = \sum_{i=1}^n p_i\mu_i\nu_{r-1}^{(i)} \tag{2.3}$$

and

$$\nu_{r-1} = p_i\mu_i\nu_{r-2}^{(i)} + (1-p_i)\nu_{r-1}^{(i)} \tag{2.4}$$

where ν_r and $\nu_r^{(i)}$ are both taken to be the zero measure if r is negative. Combining (2.3) and (2.4), we obtain

$$r\nu_r = \lambda\mu\nu_{r-1} + \sum_{i=1}^n p_i^2\mu_i(\nu_{r-1}^{(i)} - \mu_i\nu_{r-2}^{(i)}) . \tag{2.5}$$

Using (2.3) and (2.4) again, we obtain

$$\begin{aligned} r\nu_r &= \sum_{i=1}^n p_i\mu_i\nu_{r-1}/(1-p_i) - \sum_{i=1}^n p_i^2\mu_i^2\nu_{r-2}^{(i)}/(1-p_i) \\ &\leqq \lambda\mu\nu_{r-1} + \sum_{i=1}^n p_i^2\mu_i\nu_{r-1}/(1-p_i) \\ &\leqq \{1 + K\lambda^{-1}\sum_{i=1}^n p_i^2/(1-p_i)\}\lambda\mu\nu_{r-1} \end{aligned} \tag{2.6}$$

where it is noted that (2.1) implies $\mu_i \leqq K\mu$ for all i. By (2.1) again, the first inequality of (2.6) yields

$$r\nu_r \leqq K\{\lambda + \sum_{j=1}^n p_j^2/(1-p_j)\}\mu_i\nu_{r-1}$$

for $i = 1, \cdots, n$. This implies that for $i = 1, \cdots, n$,

$$\begin{aligned} (r-1)\nu_{r-1}^{(i)} &\leqq K\{\sum_{j\neq i} p_j + \sum_{j\neq i} p_j^2/(1-p_j)\}\mu_i\nu_{r-2}^{(i)} \\ &\leqq K\{\lambda + \sum_{j=1}^n p_j^2/(1-p_j)\}\mu_i\nu_{r-2}^{(i)} . \end{aligned}$$

Therefore, if $r \geqq r_0 + 1$, then for $i = 1, \cdots, n$, we have $\nu_{r-1}^{(i)} \leqq \mu_i\nu_{r-2}^{(i)}$. This together with (2.5) imply that, for $r \geqq r_0 + 1$, we have

$$r\nu_r \leqq \lambda\mu\nu_{r-1} . \tag{2.7}$$

Combining (2.6) and (2.7) and noting that $\nu_0 \leqq e^{-\lambda}I$, we have, for $r \geqq 0$,

$$\nu_r \leqq Me^{-\lambda}\lambda^r\mu^r/r! \tag{2.8}$$

which by Lemma 2.1 and noting that $K \geqq 1$

$$\leqq \{1 + K^2M(1+\lambda^{-1})\sum_{i=1}^n p_i^2/(1-p_i)\}e^{-\lambda}\lambda^r\mu^r/r! .$$

Adding the ν_r, we obtain

$$\tilde{Q} \leqq \{1 + K^2M(1 + \lambda^{-1}) \textstyle\sum_{i=1}^n p_i^2/(1 - p_i)\}Q . \tag{2.9}$$

Now (2.5) yields

$$r\nu_r \geqq \lambda\mu\nu_{r-1} - \textstyle\sum_{i=1}^n p_i^2\mu_i^2\nu_{r-2}^{(i)} . \tag{2.10}$$

By induction, (2.4), (2.8) and the fact that $\mu_i \leqq K\mu$ for all i, (2.10) in turn yields

$$\begin{aligned} \nu_r &\geqq \lambda^r\mu^r\nu_0/r! - \textstyle\sum_{i=1}^n p_i^2\mu_i^2\{\sum_{k=1}^{r-1} \lambda^{k-1}(r-k)!\, \mu^{k-1}\nu_{r-k-1}^{(i)}/r!\} \\ &\geqq \lambda^r\mu^r\nu_0/r! - \{\textstyle\sum_{i=1}^n p_i^2\mu_i^2/(1-p_i)\}\{\sum_{k=1}^{r-1} \lambda^{k-1}(r-k)!\, \mu^{k-1}\nu_{r-k-1}^{(i)}/r!\} \\ &\geqq \lambda^r\mu^r\nu_0/r! - \tfrac{1}{2}K^2M\{\textstyle\sum_{i=1}^n p_i^2/(1-p_i)\}e^{-\lambda}\lambda^{r-2}r(r-1)\mu^r/r! . \end{aligned} \tag{2.11}$$

Thus for $m = 0, 1, \cdots, n$, we have

$$\begin{aligned} \tilde{Q} &\geqq \textstyle\sum_{r=0}^m \nu_r \\ &\geqq \textstyle\sum_{r=0}^m \lambda^r\mu^r\nu_0/r! - \tfrac{1}{2}K^2M\{\sum_{i=1}^n p_i^2/(1-p_i)\} \sum_{r=2}^m e^{-\lambda}\lambda^{r-2}\mu^r/(r-2)! \\ &= \textstyle\sum_{r=0}^m e^{-\lambda}\lambda^r\mu^r/r! - \sum_{r=0}^m [e^{-\lambda} - \prod_{i=1}^n (1-p_i)]\lambda^r\mu^r/r! \\ &\quad - \tfrac{1}{2}K^2M\{\textstyle\sum_{i=1}^n p_i^2/(1-p_i)\} \sum_{r=0}^{m-2} e^{-\lambda}\lambda^r\mu^{r+2}/r! \end{aligned} \tag{2.12}$$

which by Lemma 2.2

$$\begin{aligned} &\geqq \{1 - \tfrac{1}{2}[\textstyle\sum_{i=1}^n p_i^2/(1-p_i)]\} \sum_{r=0}^m e^{-\lambda}\lambda^r\mu^r/r! \\ &\quad - \tfrac{1}{2}K^2M\{\textstyle\sum_{i=1}^n p_i^2/(1-p_i)\} \sum_{r=0}^{m-2} e^{-\lambda}\lambda^{r-2}\mu^{r+2}/r! . \end{aligned}$$

Combining (2.9) and (2.12), and noting that $K \geqq 1$ and $M \geqq 1$, we prove (2.2) for bounded h. By the monotone convergence theorem, the theorem is proved.

It is noted that $M \to 1$ as $\sum_{i=1}^n p_{ni}^2 \to 0$ and K remains bounded, and therefore does not affect the order of the bound in (2.2). Because of the condition (2.1), we cannot deduce (1.3) from the theorem. However, (2.2) yields a bound on $||\tilde{Q}_n - Q_n||$ which is of the same order as that in (1.3) provided K does not depend on n and λ_n is bounded away from zero. We state this fact more precisely in the following corollary.

COROLLARY 2.1. *The inequality* (2.2) *yields*

$$||\tilde{Q} - Q|| \leqq 2K^2M(1 + \lambda^{-1}) \textstyle\sum_{i=1}^n p_i^2/(1 - p_i) . \tag{2.13}$$

PROOF. By choosing $m = n$, $h = 1$ and noting that $K \geqq 1$, it follows from (2.2) that

$$||\tilde{Q} - Q|| \leqq K^2M(2 + \lambda^{-1}) \textstyle\sum_{i=1}^n p_i^2/(1 - p_i) + \sum_{r=n+1}^\infty e^{-\lambda}\lambda^r/r! .$$

By Chebyshev's inequality, $\sum_{r=n+1}^\infty e^{-\lambda}\lambda^r/r! \leqq \lambda/(n+1) \leqq \lambda^{-1} \sum_{i=1}^n p_i^2$. This proves the corollary.

3. A limit theorem. It is noteworthy that the following limit theorem, which is a generalization of Chen (1974), is a consequence of the inequality (2.9).

THEOREM 3.1. *Let $\tilde{Q}_n$ and Q_n be given by* (1.1) *and* (1.2) *respectively. Suppose* *that there exists a constant K such that* $\mu_{ni} \leqq K\mu_{nj}$ *for all n and all* $i, j = 1, \cdots, n$ *and that*

$$\alpha_n = \sum_{i=1}^n p_{ni}^2 \to 0 \quad \text{as} \quad n \to \infty . \tag{3.1}$$

Then for every nonnegative function h such that

$$\lim_{a\to\infty} \sup_n \int_{h>a} h\, dQ_n = 0 , \tag{3.2}$$

we have

$$\lim_{n\to\infty} \int h\, d|\tilde{Q}_n - Q_n| = 0 . \tag{3.3}$$

Proof. We first note that (3.1) implies $\bar{p}_n = \max_{1\leqq i\leqq n} p_{ni} \to 0$ as $n \to \infty$. From (2.9), $\tilde{Q}_n \leqq \{1 + K^2M(1-\bar{p}_n)^{-1}(\alpha_n + \bar{p}_n)\}Q_n$. Thus $\tilde{Q}_n$ is absolutely continuous w.r.t. Q_n such that the Radon–Nikodym derivative $\rho_n = d\tilde{Q}_n/dQ_n \leqq c_n \downarrow 1$. For every $\varepsilon > 0$, we have, by Chebyshev's inequality, $Q_n(|\rho_n - 1| > \varepsilon) \leqq \varepsilon^{-2} \int (\rho_n - 1)^2\, dQ_n = \varepsilon^{-2}(\int \rho_n^2\, dQ_n - 1) \leqq \varepsilon^{-2}(c_n^2 - 1)$. Therefore, for every $\varepsilon > 0$, we have $\limsup_n \int |\rho_n - 1|\, dQ_n \leqq \varepsilon + \limsup_n (c_n + 1)Q_n(|\rho_n - 1| > \varepsilon) \leqq \varepsilon + \varepsilon^{-2} \limsup_n (c_n + 1)(c_n^2 - 1) = \varepsilon$. This implies that $\lim_n \int |\rho_n - 1|\, dQ_n = 0$. Now let h be a nonnegative function satisfying (3.2). Then, for every $a > 0$, we have

$$\begin{aligned}
&\limsup_n \int h\, d|\tilde{Q}_n - Q_n| \\
&\qquad = \limsup_n \int h|\rho_n - 1|\, dQ_n \\
&\qquad \leqq \limsup_n (c_n + 1) \int_{h>a} h\, dQ_n + a \limsup_n \int |\rho_n - 1|\, dQ_n \\
&\qquad \leqq 2 \sup_n \int_{h>a} h\, dQ_n
\end{aligned}$$

which by (3.2) tends to zero as $a \to \infty$. The theorem is proved.

We note that the condition (3.2) implies $\sup_n \int h\, dQ_n = B < \infty$ and that

$$\begin{aligned}
\sum_{r=0}^{m-2} e^{-\lambda}(\lambda^r/r!) \int h\, d\mu^{r+2} &= \lambda^{-2} \sum_{r=2}^{m} e^{-\lambda}[r(r-1)\lambda^r/r!] \int h\, d\mu^r \\
&\leqq \lambda^{-2} m^2 B .
\end{aligned}$$

Thus the statement (3.3) can also be deduced from (2.2) by letting $m \sim \lambda_n^{\frac{1}{2}}\alpha_n^{-\frac{1}{4}}$ and applying (3.2) to the last term of (2.2), provided we impose the additional but weak condition $\bar{p}_n/\lambda_n \to 0$ as $n \to \infty$. This condition is satisfied by most interesting cases where λ_n is bounded away from zero.

4. Special cases. In this section, we considere the case where $\mathscr{X}$ is a subgroup of the additive group of real numbers and $\mathscr{A}$ the trace of Borel sets on $\mathscr{X}$, and deduce two different types of approximation theorem from Theorem 2.1. Let F and G be the distribution functions corresponding to $\tilde{Q}$ and Q respectively. The following corollary is an approximation theorem for the L_p norm, denoted by $\|\cdot\|_p$, of $F - G$ w.r.t. the Lebesgue measure where $1 \leqq p < \infty$. The normal counterpart of this problem has been considered by Erickson (1973).

Corollary 4.1. *Let* $\tilde{Q}$ *and* Q *be as in Theorem* 2.1. *Suppose* $\mathscr{X}$ *is a subgroup* *of the additive group of real numbers and* $\mathscr{A}$ *is the trace of Borel sets on* $\mathscr{X}$, *and* $\beta = \int |x|\, d\mu(x) < \infty$. *Then, for* $1 \leqq p < \infty$, *we have*

$$\|F - G\|_p \leqq C_p \textstyle\sum_{i=1}^n p_i^2/(1 - p_i) \tag{4.1}$$

where $C_p{}^p = C_1 C_\infty^{p-1}$, $C_\infty = 2$ *and* $C_1 = \beta K^2 M(3 + 2\lambda)$.

PROOF. By $\|\cdot\|_p{}^p \leqq \|\cdot\|_1 \|\cdot\|_\infty^{p-1}$ and (1.3), it suffices to prove

$$\|F - G\|_1 \leqq \beta K^2 M(3 + 2\lambda) \textstyle\sum_{i=1}^n p_i^2/(1 - p_i)\,. \tag{4.2}$$

Since $F(z) - G(z) = \int_{(-\infty,z]} d(\tilde{Q} - Q) = - \int_{(z,\infty)} d(\tilde{Q} - Q)$, it follows that

$$\begin{aligned} \|F - G\|_1 &= \textstyle\int_{-\infty}^0 |\int_{(-\infty,z]} d(\tilde{Q} - Q| \, dz + \int_0^\infty |\int_{(z,\infty)} d(\tilde{Q} - Q)| \, dz \\ &\leqq \textstyle\int_{-\infty}^0 \int_{(-\infty,z]} d|\tilde{Q} - Q| \, dz + \int_0^\infty \int_{(z,\infty)} d|\tilde{Q} - Q| \, dz \end{aligned} \tag{4.3}$$

which by Fubini's theorem

$$= \textstyle\int_{-\infty}^\infty |x| \, d|\tilde{Q} - Q|(x)\,.$$

Therefore, by letting $h(x) = |x|$ and $m = n$, it follows from (2.2) and (4.3) that

$$\begin{aligned} \|F - G\|_1 \leqq \tfrac{1}{2} K^2 M \{\textstyle\sum_{i=1}^n p_i^2/(1 - p_i)\} & \\ \times \{(3 + 2\lambda^{-1}) \textstyle\int |x| \, dQ(x) + \sum_{r=0}^\infty (e^{-\lambda}\lambda^r/r!) \int |x| \, d\mu^{r+2}(x)\} & \\ + \textstyle\sum_{r=n+1}^\infty (e^{-\lambda}\lambda^r/r!) \int |x| \, d\mu^r(x) & \end{aligned}$$

where

$$\begin{aligned} \textstyle\int |x| \, d\mu^{r+k}(x) &= \textstyle\int \cdots \int |x_1 + \cdots + x_{r+k}| \, d\mu(x_1) \cdots d\mu(x_{r+k}) \\ &\leqq \beta(r + k) \end{aligned}$$

and

$$\begin{aligned} \textstyle\sum_{r=n+1}^\infty (e^{-\lambda}\lambda^r/r!) r &= \lambda \textstyle\sum_{r=n}^\infty e^{-\lambda}\lambda^r/r! \\ &\leqq \lambda^2/n \leqq \textstyle\sum_{i=1}^n p_i^2\,. \end{aligned}$$

Hence (4.2) follows and this proves the corollary.

If $\mathscr{X}$ is the additive group of integers and the μ_i's are concentrated at 1, then $\tilde{Q}$ is the distribution of $W = X_1 + \cdots + X_n$ where $X_1, \cdots, X_n$ are independent Bernoulli random variables with $P(X_i = 1) = 1 - P(X_i = 0) = p_i$ (the Poisson binomial distribution) and Q is the Poisson distribution with mean λ. Let Z be the Poisson random variable with mean λ. The following large deviation result is a consequence of Theorem 2.1.

COROLLARY 4.2. *For every nonnegative integer* $z \leqq n - 1$, *we have*

$$\left|\frac{P(W > z)}{P(Z > z)} - 1\right| \leqq \tfrac{1}{2} S\{4(1 + \lambda^{-1}) + \lambda^{-2}(z + 1)(z + \lambda)\} \textstyle\sum_{i=1}^n p_i^2/(1 - p_i) \tag{4.4}$$

where t *is the largest integer not exceeding* $\lambda + 1 + \sum_{i=1}^n p_i^2/(1 - p_i)$ *and*

$$S = \{1 + \lambda^{-1} \textstyle\sum_{i=1}^n p_i^2/(1 - p_i)\}^t\,. \tag{4.5}$$

PROOF. By letting $m = n$ and $h(w) = 1$ if $w > z$ and $= 0$ if $w \leqq z$, it follows from (2.2) that

$$|P(W > z) - P(Z > z)| \leqq \tfrac{1}{2}S\{\textstyle\sum_{i=1}^n p_i^2/(1-p_i)\}\{(3 + 2\lambda^{-1})P(Z > z) + P(Z > z-2)\} + P(Z > n)$$

where it is noted that $K = 1$. Next we see that $P(Z > z - 2) = P(Z > z) + \lambda^{-2}(z+1)(z+\lambda)P(Z = z+1) \leqq \{1 + \lambda^{-2}(z+1)(z+\lambda)\}P(Z > z)$ and that $P(Z > n) \leqq EZ\chi(Z > n)/n = \lambda P(Z > n-1)/n \leqq (\lambda^{-1}\sum_{i=1}^n p_i^2)P(Z > z)$. The corollary follows.

5. Application of a method of Poisson approximation. In this section, we shall indicate that, in the case where $\mathscr{X}$ is the additive group of integers and the μ_i's are concentrated at 1, Theorem 2.1 can be proved, with possible improvement in the absolute constants in the bound, by using the equation (2.6) of Chen (1975), in which a method of Poisson approximation is established. Perhaps it should be mentioned that there is a subtle difference between the method in this paper and that in Chen (1975). The former uses recursive identities whereas the latter is based on a perturbation argument.

Let W and Z be as in Section 4 and let $W^{(i)} = \sum_{k\neq i} X_k$. Using independence and $m = 0$ in the equation (2.6) of Chen (1975), we obtain

$$Eh(W) = Eh(Z) - \textstyle\sum_{i=1}^n p_i^2 E\Delta S_\lambda h(W^{(i)} + 1) = Eh(Z) - \sum_{i=1}^n p_i^2 EA_i(Z)\Delta S_\lambda h(Z+1) \tag{5.1}$$

where h is any bounded function defined on the nonnegative integers, $\Delta f(w) = f(w+1) - f(w)$,

$$S_\lambda h(w) = (w-1)!\,\lambda^{-w}\textstyle\sum_{k=w}^\infty [h(k) - Eh(Z)]\lambda^k/k! \tag{5.2}$$

and $A_i(k) = P(W^{(i)} = k)/P(Z = k)$. By letting $h(w) = 1$ if $w = r$ and $= 0$ if $w \neq r$, (5.1) yields

$$\begin{aligned} P(W = r) = P(Z = r) + \lambda^{-2}\textstyle\sum_{i=1}^n p_i^2\{&\lambda \textstyle\sum_{k=0}^{r-1} A_i(k)P(Z = r) \\ &- \lambda\textstyle\sum_{k=0}^{n-1} A_i(k)P(Z \geqq k+1)P(Z = r) \\ &- \textstyle\sum_{k=0}^{r-2} A_i(k)(k+1)P(Z = r) \\ &+ \textstyle\sum_{k=0}^{n-1} A_i(k)(k+1)P(Z \geqq k+2)P(Z = r)\} \end{aligned} \tag{5.3}$$

where it is noted that $A_i(k) = 0$ if $k \geqq n$.

To prove Theorem 2.1 in this special case, first use (2.8) and $(1 - p_i)P(W^{(i)} = r)/P(W = r) = P(W^{(i)} = r \mid W = r) \leqq 1$ to show that $A_i(k) \leqq S/(1 - p_i)$ where S is given by (4.5). Then use (2.7) and (5.3) to obtain upper and lower bounds for $P(W = r)$ for $r = 0, 1, \cdots, n$. The rest of the proof is clear.

As a closing note, we would like to mention that it seems very likely that an L_p approximation theorem can be proved for ϕ-mixing Bernoulli random variables using (2.6) and other results of Chen (1975) without much difficulty. However, we shall not discuss it here.

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