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WEAK CONVERGENCE OF GENERALIZED EMPIRICAL PROCESSES RELATIVE TO d_q UNDER STRONG MIXING[1]

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Let $\{X_i : i \geqq 1\}$ be a strong-mixing sequence of uniform [0, 1] rv's and $\{C_i\}$ a sequence of constants, and define the generalized empirical process by $U_N(t) = (\sum_{i=1}^N C_i^2)^{-\frac{1}{2}} \sum_{i=1}^N C_i(I_{[X_i \leqq t]} - t)$, $0 \leqq t \leqq 1$. In this paper, the weak convergence, relative to the Skorohod metric, of (U_N/q) to a certain Gaussian process (U_0/q) is proved under certain conditions on the constants $\{C_i\}$, the strong-mixing coefficient and the function q defined on [0, 1]. The class of functions q considered in this paper include those of the type $q(t) = [t(1-t)]^\eta$, $\eta > 0$. The earlier results of Fears and Mehra [7] concerning empirical processes for ϕ-mixing sequences are also improved by weakening the conditions on the ϕ-mixing coefficient and the function q.

1. Introduction. Let $\{Y_N : N \geqq 1\}$ be a stationary sequence of real rv's with a continuous F as the common distribution function (df) and let F_N denote the sample df of Y_i, $1 \leqq i \leqq N$. The weak convergence relative to the Skorohod metric d, as $N \to \infty$, of the so-called one-sample empirical process $\{U_N(t) = N^{\frac{1}{2}}[F_N F^{-1}(t) - t] : 0 \leqq t \leqq 1\}$, an element of D (for definitions see [2] pages 109–111), has been widely studied in literature for sequences of independent as well as weakly dependent (mixing) rv's (see [2]). For a given function q on [0, 1], consider now the space (D_q, d_q) which is defined as follows: $x \in D_q$ iff $(x/q) \in D$ and $d_q(x, y) = d(x/q, y/q)$. The space (D_q, d_q) was first considered by Čibisov ([3] and [4]) who studied the weak convergence of $\{U_N(t) : 0 \leqq t \leqq 1\}$ relative to d_q, for a certain class of q functions, in the case of independent rv's. In [9] Govindarajulu et al. and in [17], [18], [21] Pyke and Shorack proved similar results for empirical processes and showed the usefulness of these results in proving, among others, the asymptotic normality of rank statistics and linear combinations of order statistics. In [7] Fears and Mehra and later in [15] Mehra and Rao (see also Mehra [14]) proved similar results for ϕ-mixing sequences, but for the restricted class of q functions whose behavior near 0 and 1 is of the type $[t(1-t)]^{\frac{1}{2}-\delta}$, for some $\delta > 0$ and which are bounded away from zero on $[\varepsilon, 1-\varepsilon]$ for every $\varepsilon > 0$.

It is important to note, however, that the condition of ϕ-mixing is in fact quite restrictive: For Gaussian processes, this condition is equivalent to that of

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m-dependence (see Theorem 17.3.2 of [12]). Such a restrictive property of m-dependence, on the other hand, does not follow in general for strong-mixing Gaussian sequences. The example of a first order stationary autoregressive Gaussian process $\{Y_N: N = 0, \pm 1, \cdots\}$ where $Y_N = \rho Y_{N-1} + e_N$ with $0 < \rho < 1$ and e_N's as i.i.d. $N(0, 1)$ rv's illustrates the point. This process is strong-mixing by Theorem 17.3.3 of [12], but it is not m-dependent since $\operatorname{Cov}(Y_k, Y_{k\pm N}) = \rho^{|N|}$. Further, as pointed out by Gastwirth and Rubin [8], ϕ-mixing imposes a severe restriction in general also on the important subclass of stationary autoregressive processes. The main object of the present paper, accordingly, is to extend the results of [7] to strong mixing sequences. In fact, in this paper we extend the results of [7] in three directions: Firstly, we consider the "generalized" empirical processes U_N defined by (2.3) (or see (2.9)); secondly, we prove the weak convergence of these processes relative to d_q for strong-mixing sequence $\{Y_N\}$; and thirdly, we improve the conditions of [7] on the class of functions q and the mixing coefficient ϕ. Our results, besides being of interest per se, are broad enough to lead to, under quite general conditions on the coefficients, the asymptotic normality of the normal-score versions of the linear rank statistics under strong-mixing. We shall, however, not consider any statistical applications in this paper.

In Section 2, we introduce the notation and prove some basic mixing inequalities. Section 3 deals with the weak convergence of U_N relative to d and contains some remarks on the conditions assumed. Section 4 contains the main results, namely, the weak convergence of U_N relative to d_q.

2. Mixing inequalities. For each fixed $N = 1, 2, \cdots$, let $\{Y_{Ni}: i \geqq 1\}$ be a sequence of real rv's defined on a probability space $(\Omega, \mathscr{A}, P)$. Let $\mathscr{M}_1^k (= \mathscr{M}_{N,1}^k)$ and $\mathscr{M}_{k+n}^\infty (= \mathscr{M}_{N,k+n}^\infty)$ denote the sub σ-algebras generated by $\{Y_{Ni}: 1 \leqq i \leqq k\}$ and $\{Y_{Ni}: i \geqq k + n\}$ respectively. Let α and ϕ be functions of nonnegative integers satisfying $\alpha(n) \downarrow 0$ and $\phi(n) \downarrow 0$. Then the sequences $\{Y_{Ni}: i \geqq 1\}$, $N = 1, 2, \cdots$, are said to be strong mixing (s.m.) if for all $A \in \mathscr{M}_1^k$ and $B \in \mathscr{M}_{k+n}^\infty$,

$$|P(A \cap B) - P(A)P(B)| \leqq \alpha(n),$$

where k and n are any positive integers; it is ϕ-mixing (ϕ.m. or uniform mixing) if the above inequality holds with $\phi(n)P(A)$ in place of $\alpha(n)$.

The following two basic inequalities (2.1) and (2.2) are needed throughout; for their proofs we refer the reader to Ibragimov [11] and Davydov [5]: Let ξ be $\mathscr{M}_1^k$-measurable and η be $\mathscr{M}_{k+n}^\infty$-measurable, then if $\{Y_{Ni}: i \geqq 1\}$ is ϕ-mixing

$$|\operatorname{Cov}(\xi, \eta)| \leqq 2\|\xi\|_a\|\eta\|_b[\phi(n)]^{1/a} \tag{2.1}$$

for all $\infty \geqq a, b \geqq 1$ with $(1/a) + (1/b) = 1$, and if $\{Y_{Ni}: i \geqq 1\}$ is strong-mixing

$$|\operatorname{Cov}(\xi, \eta)| \leqq 12\|\xi\|_a\|\eta\|_b[\alpha(n)]^{1/c} \tag{2.2}$$

for all $\infty \geqq a, b, c \geqq 1$ with $(1/a) + (1/b) + (1/c) = 1$. Clearly ϕ.m. implies s.m.

The following notation will be used throughout: $\{X_{Ni}: i \geqq 1\}$, $N = 1, 2, \cdots$, will denote a sequence (not necessarily stationary) of uniform [0, 1] rv's, $\{C_{Ni}: 1 \leqq i \leqq N\}$ a triangular array of constants, $\sum$ a denumerable sum, $\Rightarrow_\delta$ ($\Rightarrow$) weak convergence relative to a metric δ (the appropriate Euclidean metric), and K (with or without subscripts) will denote a generic constant throughout. We define a generalized empirical process (see also Koul [13]) by

$$U_N(t) = (\textstyle\sum_1^N C_{Ni}^2)^{-\frac{1}{2}} \sum_1^N C_{Ni}[I_{[X_{Ni} \leqq t]} - t]\,, \tag{2.3}$$

where $\{C_{Ni}: 1 \leqq i \leqq N\}$, $N = 1, 2, \cdots$, is a triangular array of arbitrary constants. For notational simplicity, we may suppress whenever convenient the subscript N from $\{Y_{Ni}: i \geqq 1\}$, $\{X_{Ni}: i \geqq 1\}$, $\{C_{Ni}\}$, etc. It may seem, then, that our results are proved for single infinite sequences only; in fact they cover also the case of triangular arrays $\{Y_{Ni}: 1 \leqq i \leqq N\}$, $N = 1, 2, \cdots$.

We shall now prove, using the inequalities (2.1) and (2.2), a few preliminary results leading up to the mixing inequalities of Lemma 2.6, and the central limit Theorem 3.1. The inequalities of Lemmas 2.4 and 2.6 are crucial to the main weak convergence results of Section 4 (Theorems 4.1 and 4.2). In fact in the strong-mixing case, the method of proof of Lemma 2.1 of [7] which uses the bound on the fourth moment of sums given by Lemma 1 ([2], page 195) does not yield a similar desired inequality. However, the sharper bound provided by Lemma 2.6(i) does yield this result (see inequality (4.1)). It is also worth pointing out that the inequality of Lemma 2.6(ii) has enabled us to extend Theorem 2.1 of [7] to a larger class $[Q^*(2)]$ of q functions (see Section 4 for definitions) and under a weaker condition on the mixing coefficient ϕ (cf. (1.1) of [7] with Condition I of (2.17) under which Theorem 4.2 is proved).

LEMMA 2.1. *If f is a nonnegative, nonincreasing function defined on* $\{0, 1, 2, \cdots\}$, *then* (i) $\sum_1^\infty f(m) < \infty$ *implies* $m^2 f(m) \to 0$ *as* $m \to \infty$ *and* $\sum_1^\infty m f(m) < \infty$, (ii) $\sum_1^\infty m^2[f(m)]^\delta < \infty$ *implies* $\sum_1^\infty [f(m)]^{\delta/2} < \infty$ *for any* $0 < \delta \leqq 1$.

PROOF. The proof is elementary, part (i) following from the nonincreasing property of f and part (ii) by using the Schwarz inequality. □

LEMMA 2.2. *Let* $\{Y_i: i \geqq 1\}$ *be strong mixing* (ϕ-*mixing*). *If* $|Y_i| \leqq 1$, *then* $\sum_1^\infty \alpha(j) < \infty$ ($\sum_1^\infty \phi(j) < \infty$) *implies*

$$\operatorname{Var}(\textstyle\sum_{M+1}^{M+m} C_i Y_i) \leqq K(\sum_{M+1}^{M+m} C_i^2)\,, \tag{2.4}$$

where M and m are arbitrary nonnegative integers and the constant K depends only on $\alpha(\phi)$.

PROOF. The proof follows using the expansion of $\operatorname{Var}(\sum C_i Y_i)$, the mixing inequality (2.2) and the inequality below.

$$\begin{aligned} \textstyle\sum_{i<j}^{M+m} |C_i C_j| \alpha(j-i) &= \textstyle\sum_{j=1}^{m-1} \alpha(j) \sum_{i=M+1}^{M+m-j} |C_i C_{i+j}| \\ &\leqq \textstyle\sum_{j=1}^{m-1} \alpha(j) (\sum_{i=M+1}^{M+m-j} C_i^2)^{\frac{1}{2}} (\sum_{i=M+1}^{M+m-j} C_{i+j}^2)^{\frac{1}{2}} \\ &\leqq (\textstyle\sum_{j=1}^{m-1} \alpha(j))(\sum_{i=M+1}^{M+m} C_i^2)\,. \end{aligned} \tag{2.5}$$

□

LEMMA 2.3. *Let* $\{Y_i : i \geqq 1\}$ *be* s.m. (ϕ.m.) *with* $\sum_{j=1}^{\infty} j^2\alpha(j) < \infty$ ($\sum_{j=1}^{\infty} j^2\phi(j) < \infty$) *and* $|Y_i| \leqq 1$, *then*

$$E[\sum_{M+1}^{M+m} C_i Y_i]^4 \leqq K[(\sum_{M+1}^{M+m} C_i^2)^2 + (\sum_{M+1}^{M+m} C_i^2)\psi^{*2}],$$

where $\psi^* = \max_{M+1 \leqq i \leqq M+m} |C_i|$, *M and m are as in Lemma* 2.2 *and K is a constant depending on* $\alpha(\phi)$ *only.*

PROOF. The proof of this lemma is similar to that of Lemma 2.6 below. Since $|Y_i| \leqq 1$ it is even simpler.

Let $[Q(r)\uparrow]$, for a fixed $r > 0$, denote the class of functions q defined on $[0, 1]$ satisfying

$$\text{(2.6)} \qquad \text{(i)} \quad q(0) \geqq 0, \quad \text{and} \quad q(t) > 0 \quad \text{for} \quad t > 0$$
$$\text{(ii)} \quad q(t) \quad \text{is nondecreasing} \quad (\uparrow) \quad \text{in} \quad t \quad \text{and} \quad tq^{-r}(t)\uparrow \quad \text{in} \quad t.$$

(Although not stated explicitly, (i) and (ii) imply the continuity of q; see Lemma 4.1(iii).)

Now let $\{X_{Ni} : i \geqq 1\}$, $N = 1, 2, \cdots$, be sequences of uniform $[0, 1]$ rv's and, for $0 < s, t < 1$, set

$$\text{(2.7)} \qquad g_t(X_i) = I_{[X_i \leqq t]} - t,$$
$$\eta_i = \frac{g_t(X_i)}{q(t)} - \frac{g_s(X_i)}{q(s)} \qquad \text{and} \qquad \xi_i = \xi_{Ni} = a_i \eta_i,$$

(at $t = 0, 1$ define $[g_t(X_i)/q(t)] = 0$) where we set

$$\text{(2.8)} \qquad a_i = a_{Ni} = C_{Ni}\left(\frac{1}{N} \sum_1^N C_{Ni}^2\right)^{-\frac{1}{2}} \qquad \text{and} \qquad \psi_N = \max_{1 \leqq i < N} |a_i|;$$

(note that $E(\eta_i) = 0$ and $\sum_{i=1}^N a_i^2 = N$). In this notation, the empirical process $\{U_N(t) : 0 \leqq t \leqq 1\}$ given by (2.3) is expressible as

$$\text{(2.9)} \qquad U_N(t) = N^{-\frac{1}{2}} \sum_{i=1}^N a_i[I_{[X_i \leqq t]} - t],$$

and

$$\left(\frac{U_N(t)}{q(t)} - \frac{U_N(s)}{q(s)}\right) = N^{-\frac{1}{2}} \sum_{i=1}^N \xi_i \qquad \text{for} \quad 0 \leqq s, \quad t \leqq 1.$$

LEMMA 2.4. *Let X be a uniform* $[0, 1]$ rv *and* $q \in [Q(r)\uparrow]$ *for a given* $r \geqq 1$. *Then for* $0 < s \leqq t < 1$

$$\text{(2.10)} \qquad E\left|\frac{g_t(X)}{q(t)} - \frac{g_s(X)}{q(s)}\right|^m \leqq \frac{(t-s)}{q^m(t)}(2^m + 2),$$

for all $1 \leqq m \leqq r$.

PROOF. By direct computation, we have for $s \leqq t$

$$\text{(2.11)} \qquad E\left|\frac{g_t(X)}{q(t)} - \frac{g_s(X)}{q(s)}\right|^m = x + y + z,$$

where

$$x = \left[\frac{(1-s)}{q(s)} - \frac{(1-t)}{q(t)}\right]^m s\,, \qquad y = \left[\frac{(1-t)}{q(t)} + \frac{s}{q(s)}\right]^m (t-s)\,,$$

$$z = \left[\frac{t}{q(t)} - \frac{s}{q(s)}\right]^m (1-t)\,.$$

Since $q \in [Q(r)\uparrow]$ implies $sq^{-1}(s)\uparrow$ in s,

$$y \leqq \left[\frac{(1-t)}{q(t)} + \frac{t}{q(t)}\right]^m (t-s) = \frac{(t-s)}{q^m(t)}\,, \tag{2.12}$$

and since $q(s)\uparrow$ in s,

$$z \leqq \left[\frac{t}{q(t)} - \frac{s}{q(s)}\right]^m \leqq \frac{(t-s)}{q^m(t)}\,. \tag{2.13}$$

Finally note that, since $q(s)/q(t) \geqq (s/t)$ and $sq^{-m}(s)\uparrow$ in s,

$$s\left[\frac{1}{q(t)} - \frac{1}{q(t)}\right]^m = \frac{s}{q^m(s)}\left[1 - \frac{q(s)}{q(t)}\right]^m \leqq \frac{t}{q^m(t)}\left(1 - \frac{s}{t}\right)^m,$$

so that using (2.13),

$$\begin{aligned} x &\leqq \left[\left(\frac{t}{q(t)} - \frac{s}{q(s)}\right) + s^{1/m}\left(\frac{1}{q(s)} - \frac{1}{q(t)}\right)\right]^m \\ &\leqq \frac{(t-s)}{q^m(t)}\left[1 + \left(1 - \frac{s}{t}\right)^{(m-1)/m}\right]^m \\ &\leqq 2^m\,\frac{(t-s)}{q^m(t)}\,; \end{aligned} \tag{2.14}$$

(2.10) follows using (2.12) to (2.14) in (2.11). □

Since $E(\xi_i) = 0$, the following lemma gives bounds on the covariance sum of ξ_i's.

LEMMA 2.5. *Let* $0 < \delta < 1$ *be arbitrary but fixed. Then for* $0 < s \leqq t \leqq 1$ *and* $1 \leqq i, j \leqq N$, *the sum* $\sum_{1\leqq i<j\leqq N} |E(\xi_i\xi_j)|$ *is bounded above by*

(i) $72N(t-s)^{1-\delta}q^{-2}(t)\sum_{j=1}^N [\alpha(j)]^\delta$ *if* $\{X_i\}$ *is* s.m. *and* $q \in [Q(2/(1-\delta))\uparrow]$,

(ii) $16N(t-s)q^{-2}(t)\sum_{j=1}^N [\phi(j)]^\delta$ *if* $\{X_i\}$ *is* ϕ.m. *and* $q \in [Q((1/\delta)\vee(1/(1-\delta)))\uparrow]$,

and

(iii) $8N(t-s)\sum_{j=1}^N \phi(j)$ *if* $\{X_i\}$ *is* ϕ.m. *and* $q \equiv 1$.

(Note that in (i) the class $[Q(\cdot)\uparrow]$ increases if $\delta\downarrow$, but the bound becomes less sharp. In (ii) the class $[Q(\cdot)\uparrow]$ is largest when $\delta = \frac{1}{2}$; if $\sum \phi^\delta(j) < \infty$ for some $\delta < \frac{1}{2}$, this does not help to enlarge the class of q functions beyond $[Q(2)\uparrow]$.)

PROOF. For part (i), applying (2.2) with $a = b = 2/(1-\delta)$ and using Lemma 2.4, we have

$$\begin{aligned} &\textstyle\sum_{i<j} |E\xi_i\xi_j| \\ &\leqq 12\textstyle\sum_{i<j} |a_i a_j| [E|\eta_i|^{2/(1-\delta)}]^{(1-\delta)/2}[E|\eta_j|^{2/(1-\delta)}]^{(1-\delta)/2}[\alpha(j-i)]^\delta \\ &\leqq \frac{72(t-s)^{1-\delta}}{q^2(t)}\textstyle\sum_{i<j} |a_i a_j| [\alpha(j-i)]^\delta\,, \end{aligned} \tag{2.15}$$

the last inequality following from the fact $[2^{2/(1-\delta)} + 2]^{1-\delta} \leqq 6$. The proof of part (i) follows now by using the inequality (2.5) and $\sum_{i=1}^{N} a_i^2 = N$. The proofs of part (ii) and (iii) are similar. □

Now set

$$(2.16) \qquad \begin{aligned} A_N(\phi) &= \textstyle\sum_{i<j<k\leqq N} \min\{\phi(k-j), \phi(j-i)\} \\ B_N(\phi) &= \textstyle\sum_{i<j<k<l\leqq N} \min\{\phi(l-k), \phi(k-j), \phi(j-i)\}, \end{aligned}$$

and

$$(2.17) \qquad \begin{aligned} &\text{Condition I:} \quad \textstyle\sum_{k=1}^{\infty} k^2\phi(k) < \infty \\ &\text{Condition II:} \quad \textstyle\sum_{k=1}^{\infty} k^2\alpha^\delta(k) < \infty \qquad \text{for a} \quad 0 < \delta < 1. \end{aligned}$$

LEMMA 2.6. *Let $0 < \delta < 1$ and $0 < s \leqq t \leqq 1$ be fixed.*

(i) *If $q \in [Q((2+\eta)/(1-\delta))\uparrow]$ for some $\eta \geqq 0$, $\{X_i\}$ is* s.m. *and Condition* II *holds, then*

$$E(\textstyle\sum_{i=1}^{N} \xi_i)^4 \leqq K_{\alpha,\delta}[N^2(t-s)^{2-\delta}q^{-4}(t) + N(t-s)^{1-\delta}q^{-2+\eta}(s)q^{-2-\eta}(t)\phi_N^2].$$

(ii) *If $q \in [Q(2)\uparrow]$, $\{X_i\}$ is ϕ.*m. *and Condition* I *holds, then*

$$E(\textstyle\sum_{i=1}^{N} \xi_i)^4 \leqq K_{\phi,\delta}[N^2(t-s)^2q^{-4}(t) + N(t-s)q^{-2}(s)q^{-2}(t)\phi_N^2].$$

(iii) *If $C_{Ni} \equiv 1$, $q(t) \equiv 1$, $\{X_i\}$ is ϕ.*m. *and $\sum \phi(j) < \infty$, then*

$$E(\textstyle\sum_{i=1}^{N} \xi_i)^4 \leqq K_\phi\{N^2(t-s)^2 + (t-s)[N + A_N(\phi) + B_N(\phi)]\}.$$

PROOF. The proof is based on Lemmas 2.4 and 2.5. For simplicity we prove part (i) for $\eta = 1$; the proof is similar for $\eta \neq 1$. First, note that

$$(2.18) \qquad E(\textstyle\sum_{i=1}^{N} \xi_i)^4 \leqq 4! \sum |E(\xi_i\xi_j\xi_k\xi_l)|,$$

where the sum on the right extends over all $1 \leqq i \leqq j \leqq k \leqq l \leqq N$, and further

$$(2.19) \qquad E(\xi_i) = 0, \qquad |\xi_i| \leqq 2a_i q^{-1}(s) \qquad \text{and} \qquad \textstyle\sum_{i=1}^{N} a_i^2 = N.$$

Now using (2.19), Lemmas 2.4, 2.5 and the mixing inequality (2.2) with $a = 3/(2(1-\delta))$, $b = 3/(1-\delta)$, it follows easily that the total contribution of the terms corresponding to $i = j = k = l$, $i = j \neq k = l$, $i = j = k \neq l$ to the sum in (2.18) is bounded above by

$$(2.20) \qquad K[N^2(t-s)^2q^{-4}(t) + N\phi_N^2(t-s)q^{-1}(s)q^{-3}(t)(1 + \textstyle\sum_{j=1}^{N} \alpha^\delta(j))],$$

where K is a generic constant. Also, the same technique coupled with the inequality

$$|E \textstyle\sum_{i<j<k}^{N} \xi_k^2\xi_i\xi_j| \leqq \sum_{i<j}^{N} |E(\xi_i\xi_j)| \sum_{k=1}^{N} E(\xi_k^2) + \sum_{i<j<k}^{N} \min\{|\mathrm{Cov}(\xi_i; \xi_j\xi_k^2)|, |\mathrm{Cov}(\xi_i\xi_j; \xi_k^2)|\},$$

and similar inequalities for other two sums $\sum_{k<i<j}$ and $\sum_{i<k<j}$ yields at once

$$(2.21) \qquad |\textstyle\sum_{i<j\neq k} E(\xi_i\xi_j\xi_k^2)| \leqq K_{\alpha,\delta}[N^2(t-s)^{2-\delta}q^{-4}(t) + N\phi_N^2(t-s)^{1-\delta}q^{-1}(s)q^{-3}(t)],$$

where we have also used $\sum_{i=1}^N a_i^2 = N$ and the fact that

$$\sum_{i<j<k}^N a_k^2 \min\{\alpha(j-i), \alpha(k-j)\} \leqq N \sum_{j'+k' \leqq N} \min\{\alpha(j'), \alpha(k')\} \leqq 2N \sum j\alpha(j) .$$

Finally, proceeding as for (2.21) and using (2.17 II) one gets the same bound for the remaining terms (corresponding to $i < j < k < l$) in the sum on the right of (2.18). This completes the proof of part (i). The proof for part (ii) is similar and uses (2.1), Lemma 2.4 and Lemma 2.5 (ii). If $q(t) \equiv 1$ and $C_i \equiv 1$, one can use instead Lemma 2.5 (iii) and (2.1) with $a_i \equiv 1$, so that the proof of part (iii) is straightforward. □

3. Weak convergence relative to *d*. The object of this section is to establish the weak convergence of U_N to a Gaussian process $\{U_0(t): 0 \leqq t \leqq 1\}$ tied down at 0 and 1. Our Theorems 2.3 and 3.3 below extend and improve the conditions of similar theorems proved earlier in the literature (see Remark 3.2). For this, we shall first prove a central limit theorem (Theorem 3.1) for strong mixing sequences.

THEOREM 3.1. *Suppose* $\{Y_{Nj}: j \geqq 1\}$, $N = 1, 2, \cdots$, *is* s.m. *with* $|Y_{Nj}| \leqq 1$ *and that* $\sum j^2\alpha(j) < \infty$. *Assume further that* $\liminf_{N\to\infty} \tau_N^2 (\sum_1^N C_{Ni}^2)^{-1} > 0$, *where* $\tau_N^2 = \operatorname{Var}(\sum_1^N C_{Ni} Y_{Ni})$ *and*

$$N^{-\frac{1}{2}}\psi_N = [\max_{1\leqq i\leqq N} |C_{Ni}|/(\sum_1^N C_{Ni}^2)^{\frac{1}{2}}] = o(1) , \qquad as \quad N \to \infty . \tag{3.1}$$

Then $\sum_i^N C_{Ni} Y_{Ni}/\tau_N \Rightarrow N(0, 1)$ *as* $N \to \infty$. (Note that the sequence $\{Y_{Nj}: j \geqq 1\}$ is not assumed to be stationary.)

PROOF. The proof is based on Theorem 3 of Philipp [16]. We only sketch the proof. First, by letting $b_N = N^{-\frac{1}{2}}\psi_N$, $K_N = b_N^{\frac{1}{4}}$ and noting that $\sum \alpha^{\frac{1}{2}}(j) < \infty$ (see Lemma 2.1 (ii)) it can be seen that the pair (K_N, b_N) is "admissible" (see Lemma 4 of [16]). Also, using the notation of [16], we get by Schwarz inequality and Lemma 2.3 (as in the proofs of Theorems 6 and 7 of [16]) that for every $\varepsilon > 0$

$$\begin{aligned} \sum_{j\leqq l} \int_{|y|>\varepsilon} y^2\, dF_{Nj} &\leqq \sum_{j\leqq l} [P(|y_j| \geqq \varepsilon)]^{\frac{1}{2}}[Ey_j^4]^{\frac{1}{2}} \\ &\leqq \sum_{j\leqq l} (Ey_j^2)^{\frac{1}{2}}[(h_j^*/\textstyle\sum_{i=1}^N C_{Ni}^2) + b_N^2] \\ &\leqq K_\alpha(\varepsilon) b_N^{\frac{1}{2}}(1 + b_N) , \end{aligned} \tag{3.2}$$

where $h_j^* = \sum_{i=\rho_j+1}^{\rho_j+h_j} C_{Ni}^2$ and F_{Nj} is the df of y_j (ρ_j, h_j^l, and y_j are as defined in (3.4) of [16]). The last inequality uses (3.7) of [16]. Now the result follows from (3.1) and (3.2). □

Now for $0 \leqq s, t \leqq 1$ define $\sigma(s, t)$, whenever the limit exists, by

$$\begin{aligned} \sigma(s, t) &= \lim_{N\to\infty} E[U_N(s)U_N(t)] \\ &= (s \wedge t)[1 - (s \vee t)] \\ &\quad + \lim_{N\to\infty} (\textstyle\sum_1^N C_i^2)^{-1} \sum_{i\neq j}^N C_i C_j E[g_t(X_i) g_s(X_j)] , \end{aligned} \tag{3.3}$$

where g and $U_N(t)$ are given by (2.7) and (2.9).

THEOREM 3.2. *Let* $\{X_{Ni}: i \geqq 1\}$, $N = 1, 2, \cdots$, *be* s.m. *and Condition* II *hold; then (whenever the limit in* (3.3) *exists)* $|\sigma(s, t)| < \infty$. *If also* $\sigma(s, t) > 0$ *for all* $0 < s, t < 1$ *and*

$$N^{-1}\psi_N^2 = O(N^{-\delta_0}), \qquad as\ N \to \infty, \tag{3.4}$$

for a $\delta_0 > (\delta/2)$, *then* $U_N \Rightarrow_d U_0$, *where* $U_0 = \{U_0(t): 0 \leqq t \leqq 1\}$ *is a Gaussian process tied down at* 0 *and* 1 *and defined by*

$$EU_0(t) = 0, \qquad E[U_0(s)U_0(t)] = \sigma(s, t), \qquad 0 \leqq s, \quad t \leqq 1. \tag{3.5}$$

Moreover $P[U_0 \in C] = 1$, *where C denotes the space of real continuous functions on* [0, 1].

THEOREM 3.3. *Let* $\{X_{Ni}: i \geqq 1\}$, $N = 1, 2, \cdots$, *be* ϕ.m. *Suppose Condition* I *holds and the limit in* (3.3) *exists. Then* $|\sigma(s, t)| < \infty$. *Further if* (3.1) *holds then* $U_N \Rightarrow_d U_0$ *as* $N \to \infty$ *and* $P[U_0 \in C] = 1$.

REMARKS 3.1. (i) Theorem 3.2 has been proved in the special case $C_{Ni} \equiv 1$ in Mehra and Rao [15], and proved earlier under stronger conditions in Deo [6] and Yokoyama [21]. This extends Theorem 22.1 of [1] and Theorem 3.1 of [20].

(ii) Theorem 3.3 is valid under the following weaker mixing conditions when $C_{Ni} \equiv 1 : N^{-2}[A_N(\phi) + B_N(\phi)] = o(1)$ as $N \to \infty$, where A_N and B_N are as defined in (2.16). This condition is satisfied if, for example, $\sum j(\log j)\phi(j) < \infty$. To see this, we note that the tightness of U_N follows from Lemma 4 of [19] and the finite dimensional convergence from Lemma 2.2 and Theorem 2.1 of [1]. Similar weakening of mixing conditions in Theorem 3.2 is also possible when $C_{Ni} \equiv 1$ but we omit it for brevity.

PROOFS. We shall first give proof for Theorem 3.2 and merely indicate modifications needed for Theorem 3.3: That $|\sigma(s, t)| < \infty$ under Condition II follows from Lemma 2.2. For proving $U_N \Rightarrow_d U_0$, we follow the line of proof of Theorem 22.1 of [2]: First we shall establish tightness for which, clearly, we may assume $C_i \geqq 0$ WLOG. By setting $q \equiv 1$, we have for $0 \leqq s < t \leqq 1$ by Lemma 2.6 (i)

$$E|U_N(t) - U_N(s)|^4 \leqq K_{\alpha,\delta}\left[(t-s)^{2-\delta} + (t-s)^{1-\delta}\frac{\psi_N^2}{N}\right], \tag{3.6}$$

where $K_{\alpha,\delta} < \infty$. Also, from (3.10) $\psi_N^2 N^{-1} \to 0$ as $N \to \infty$. Consequently, for given $\varepsilon > 0$ (and t, s), we can choose N sufficiently large such that

$$(\varepsilon\psi_N^2/N)^{1/(2\delta_0')} < (t - s) \qquad \text{where} \quad \frac{\delta}{2} < \delta_0' < (\tfrac{1}{2} \wedge \delta_0). \tag{3.7}$$

Noting that $2\delta_0' < 1$ and $\delta'' = 2\delta_0' - \delta > 0$, we have from (3.6) and (3.7)

$$E|U_N(t) - U_N(s)|^4 \leqq \frac{2K_{\alpha,\delta}}{\varepsilon}(t - s)^{1+\delta''}. \tag{3.8}$$

Further, let p be a number satisfying

$$(3.9)\qquad \left(\frac{\varepsilon\phi_N^2}{N}\right)^{1/(2\delta_0')} < p < \frac{\varepsilon N^{\frac{1}{2}}}{4(\sum_{i=1}^N a_{Ni})};$$

that such a choice of p is possible for large N can easily be verified using $\sum_{i=1}^N a_i \leqq N$ and that $\delta_0 > \delta_0'$. Now by applying Theorem 12.2 of [2] to the variables $U_N(s+ip) - U_N(s+(i-1)p)$, $i = 1, 2, \cdots, M$ and by using the inequality

$$(3.10)\qquad |U_N(t) - U_N(s)| \leqq |U_N(s+p) - U_N(s)| + 2pN^{-\frac{1}{2}} \textstyle\sum_1^N a_i$$

for $s \leqq t \leqq s + p$, we get from (3.8) and (3.9) (as in [2], page 199)

$$P[\sup_{s\leqq t\leqq s+mp} |U_N(t) - U_N(s)| \geqq \varepsilon] \leqq \frac{K_{\alpha,\delta}}{\varepsilon^5}(Mp)^{1+\delta''}.$$

This would prove the tightness of U_N as in Theorem 22.1. That the finite dimensional distributions of U_N converge, as $N \to \infty$, to the corresponding ones of U_0 under the hypothesis of the theorem follows from the multivariate extension of Theorem 3.1, which is an easy extension of Theorem 3.1 using standard arguments (for example, see D_3 of theorem on page 168 of [10]). The proof of Theorem 3.2 is complete in view of Theorems 15.5 and 15.1 of [2].

For proving Theorem 3.3, we can use part (ii) of Lemma 2.6 with $q = 1$ which, as above or by Lemma 4 of [19], yields the tightness of U_N. The convergence of finite dimensional distributions of U_N under Condition I follows from Theorem 3.1 as noted above. □

4. Main results: weak convergence relative to d_q. Let $[Q(r)\uparrow]$, for a fixed $r > 0$, be the class of q functions as defined by (2.6) and let $[Q(r)]$ denote the class of q functions such that $q(t) = q(1-t) = \bar{q}(t)$ for $0 \leqq t \leqq \frac{1}{2}$ for some function $\bar{q} \in [Q(r)\uparrow]$ and $\bar{q}(0+) = \bar{q}(0)$. Further denote by $[Q^*(r)\uparrow]$ and $[Q^*(r)]$ the corresponding classes of q functions which in addition to (2.6) also satisfy:

$$\textstyle\int_0^1 q^{-r}(t)\,dt < \infty \quad \text{(cf. Lemma 4.1).}$$

Note that $[Q^*(r)\uparrow] \subset [Q(r)\uparrow]$ and $[Q^*(r)] \subset [Q(r)]$. An important example of a function $q \in [Q^*(r)]$ is $q(t) = [t(1-t)]^\eta$, $0 \leqq t \leqq 1$, for an $0 < \eta < (1/r)$.

LEMMA 4.1. *Every $q \in [Q(r)\uparrow]$ satisfies the following properties*: (i) $tq^{-s}(t)\uparrow$ *in t for all $s < r$*; (ii) $\int_0^1 q^{-s}(t)\,dt < \infty$ *for all $s < r$ and $r > 2$*; (iii) *q is continuous on* $[0, 1]$.

PROOF. First note that $tq^{-r}(t)\uparrow$ and $q(r)\uparrow$ in t clearly imply $tq^{-s}(t)\uparrow$ for $s < t$. Also setting $\delta = 1 - (s/r) > 0$, we have

$$\textstyle\int_0^1 q^{-s}(t)\,dt = \int_0^1 \frac{1}{t^{1-\delta}}\left(\frac{t}{q^r(t)}\right)^{1-\delta} dt \leqq q^{-s}(1)\int_0^1 t^{\delta-1}\,dt < \infty.$$

This proves parts (i) and (ii). Part (iii) follows since for $h > 0$,

$$q(t+h) - q(t) = q(t+h)\left[1 - \frac{q(t)}{q(t+h)}\right] \leqq q(t+h)\left[1 - \left(\frac{t}{t+h}\right)^{1/r}\right] \to 0$$

as $h \downarrow 0$ and similarly for $h < 0$. □

We shall now prove the weak convergence of U_N relative to d_q, under appropriate conditions on the mixing property and the function q. Theorem 4.2 below generalizes Theorem 2.1 of [7] by way of weakening the condition on the mixing coefficient ϕ and by way of introducing constants C_{Ni}'s. Theorem 4.1, which proves the weak convergence of U_N relative to d_q under strong mixing, is considered by us to be the main new result of this paper.

Let (U_N/q) be defined equal to zero at 0 and 1. In the theorem below note that, since q is continuous, $(U_N/q) \in D$.

THEOREM 4.1. *Suppose* $\{X_{Ni}: i \geqq 1\}$ *satisfies the conditions of Theorem* 2.3 *and that* (3.10) *holds for a* $\delta_0 > [\delta \vee ((2-\eta)(1-\delta)/(2+\eta))]$ *where* $\eta \geqq 0$. *Then* $q \in [Q((2+\eta)/(1-\delta))]$ *implies* $U_N \Rightarrow_{d_q} U_0$, *as* $N \to \infty$, *where* U_0 *is as defined in Theorem* 3.2. *Furthermore* $P[(U_0/q) \in C] = 1$.

THEOREM 4.2. *Let* $\{X_{Ni}: i \geqq 1\}$ *satisfy the conditions of Theorem* 3.3 *and* $q \in [Q^*(2)]$. *Then* $U_N \Rightarrow_{d_q} U_0$, *as* $N \to \infty$ *and* $P[(U_0/q) \in C] = 1$.

REMARK 4.1. In view of Lemma 4.1(ii), the condition $\int_0^1 q^{-s}(t)\,dt < \infty$ for $s < r$ holds for each $q \in [Q(r)]$, $r > 2$, and this condition is adequate for proving Theorem 4.1. On the other hand, in proving Theorem 4.2 for $r = 2$, the additional condition $\int_0^1 q^{-2}(t)\,dt < \infty$ is needed, which does not hold in general for members of $[Q(2)]$. Hence we are able to prove Theorem 4.2 for $[Q^*(2)]$ instead of $[Q(2)]$.

PROOFS. For simplicity we shall prove Theorem 4.1 when $\eta = 1$; the proof for any $\eta \geqq 0$ is exactly the same. Note that if $\eta = 1$, $q \in [Q(3/(1-\delta))]$. We shall first prove a result similar to Lemma 2.1 of [7]: Given $\varepsilon > 0$ and a $q \in [Q(3/(1-\delta))\uparrow]$, there is a $\theta = \theta(\varepsilon, \delta, \delta_0, q, \alpha) > 0$ and an $N_0 = N_0(\varepsilon, \delta, \delta_0, q, \alpha)$ such that

$$P\left[\sup_{0 \leqq t \leqq \theta}\left|\frac{U_N(t)}{q(t)}\right| \geqq \varepsilon\right] \leqq \varepsilon \qquad \text{for} \quad N \geqq N_0\,. \tag{4.1}$$

For this assuming $\delta_0 < 1$ WLOG, let δ' be such that

$$\left[0 \vee \frac{(1-4\delta)}{3}\right] < \delta' < (\delta_0 - \delta) < 1\,, \tag{4.2}$$

and note that, since $(4/(1+\delta')) < (3/(1-\delta))$ and $q \in [Q(3/(1-\delta))\uparrow]$, by Lemma 4.1(ii) for any $0 < s < t < 1$

$$\frac{(t-s)^{1+\delta'}}{q^4(t)} \leqq \left(\int_s^t [q(u)]^{-4/(1+\delta')}\,du\right)^{1+\delta'} < \infty\,, \qquad \text{and} \tag{4.3}$$

$$[q(t)/q(s)] \leqq (t/s)^{(1-\delta)/3}\,.$$

From Lemma 2.6(i), (4.2) and (4.3) we obtain

$$E\left|\frac{U_N(t)}{q(t)} - \frac{U_N(s)}{q(s)}\right|^4$$

$$(4.4)\qquad \leqq K_{\alpha,\delta}\left[1 + \frac{\phi_N^2}{N}(t-s)^{-(\delta+\delta')}\left(\frac{q(t)}{q(s)}\right)\right]\frac{(t-s)^{1+\delta'}}{q^4(t)}$$

$$\leqq K_{\alpha,\delta}\left[1 + \frac{\phi_N^2}{N}(t-s)^{-(\delta+\delta')}\left(\frac{t}{s}\right)^{(1-\delta)/3}\right](\textstyle\int_s^t q(u)^{-4/(1+\delta')}\,du)^{1+\delta'}\,.$$

Similarly,

$$(4.5)\qquad E\left|\frac{U_N(t)}{q(t)}\right|^4 \leqq K_{\alpha,\delta}\left[1 + \frac{\phi_N^2}{N}t^{-(\delta+\delta')}\right](\textstyle\int_0^t q(u)^{-4/(1+\delta')}\,du)^{1+\delta'}\,.$$

Now by letting, as in [7], $0 < s_1 < s_2 < \cdots < s_M = \theta < 1$, where $s = (l\theta/M)$, $1 \leqq l \leqq M$, we see from (3.10) and (4.3) that for $s_i \leqq t \leqq s_{i+1}$,

$$(4.6)\qquad \left|\frac{U_N(t)}{q(t)}\right| \leqq 4\max_{1\leqq i\leqq M}\left|\frac{U_N(s_i)}{q(s_i)}\right| + \frac{2N^{\frac{1}{2}}(\theta/M)}{q(\theta/M)}\,.$$

Further using (4.2) to (4.6) and the property $tq^{-4/(1+\delta')}\uparrow$ in t, and proceeding as in Lemma 2.1 of [7] it can be shown that

$$P\left[\sup_{0\leqq t\leqq(\theta/M)}\left|\frac{U_N(t)}{q(t)}\right| \leqq \frac{\varepsilon}{2}\right] \geqq 1 - (\varepsilon/2)\,,$$

and

$$P\left[\sup_{(\theta/M)<t\leqq\theta}\left|\frac{U_N(t)}{q(t)}\right| \geqq \frac{\varepsilon}{2}\right] \leqq \frac{\varepsilon}{2}\,, \qquad \text{for } N \text{ large,}$$

where $\theta > 0$ is independent of N and $M = O(N)$. In view of these inequalities the inequality (4.1) follows. Now since $q \in [Q(3/(1-\delta))]$ is continuous on [0, 1] by Lemma 4.1(iii), using the symmetry of the process $\{U_N/q\}$ about $t = \frac{1}{2}$ and (4.1) above, one can prove as in Theorem 2.1 of [7], the tightness of the sequence $\{U_N/q\}$, $N = 1, 2, \cdots$. Since the conditions of Theorem 4.1 imply those of Theorem 3.2, the convergence of finite dimensional distributions and $P[(U_0/q) \in C] = 1$ follow and the proof of Theorem 4.1 is complete. The proof of Theorem 4.2 is similar, where we use part (ii) in place of part (i) of Lemma 2.6 and Theorem 3.3 in place of Theorem 3.2. □

Consider now the process $\{U_N(t)\colon 0 \leqq t \leqq 1\}$ with $C_i = 1$ for $i \geqq 1$. For this usual empirical process, the weak convergencce result relative to d_q was proved in [7] for ϕ-mixing sequences and for functions of the type $q_0(t) = [t(1-t)]^{\frac{1}{2}-\mu}$ for every $0 < \mu < \frac{1}{2}$. The following corollary shows that this result holds for the same class of functions for strong mixing sequences as well, but under the mixing Condition II: $\sum_{k=1}^{\infty} k^2\alpha^\delta(k) < \infty$ with $\delta = 2\mu$.

COROLLARY 4.1. *Let $\{X_{Ni}\colon i \geqq 1\}$ satisfy the conditions of Theorem* 3.2, *and let*

$C_{Ni} = 1$ for all $i \geqq 1$. Then $U_N \Rightarrow_{d_q} U_0$, as $N \to \infty$, and $P[(U_0/q) \in C] = 1$ for all q of the type $q_0(t) = [t(1-t)]^{\frac{1}{2}-\mu}$, $0 \leqq t \leqq 1$, with $\mu \geqq \delta/2$.

PROOF. Since $C_{Ni} = 1$ for all $i \geqq 1$, (3.10) holds with $\delta_0 = 1$, so that the condition $\delta_0 > [\delta \vee ((2-\eta)(1-\delta)/(2+\eta))]$ is satisfied for $\eta = 0$. Further $q_0 \in [Q(2/(1-\delta))]$ and, therefore, the result follows from Theorem 4.1 with $\eta = 0$. □

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