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WEAK CONVERGENCE TO EXTREMAL PROCESSES

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$\{X_n, n \geqq 1\}$ are i.i.d. rv's with df F. Set $M_n = \max\{X_1, \cdots, X_n\}$. As a basic assumption, suppose normalizing constants $a_n > 0$, b_n, $n \geqq 1$ exist such that $\lim_{n\to\infty} P[M_n \leqq a_n x + b_n] = G(x)$, nondegenerate. Define the random function $Y_n(t) = (M_{[nt]} - b_n)/a_n$. By considering weak convergence of underlying two dimensional point processes, an alternate proof of the original Lamperti result that $Y_n \Rightarrow Y$ is given where Y is an extremal-G process. From the convergence of the point processes, other weak convergence results are shown. Let $x(t)$ be nondecreasing and $Nx(I)$ be the number of times x jumps in time interval I. Then $Y_n^{-1} \Rightarrow Y^{-1}$, $NY_n \Rightarrow NY$, $NY_n^{-1} \Rightarrow NY^{-1}$. From these convergences emerge a variety of limit results for record values, record value times and inter-record times.

1. Introduction and preliminaries. The well-known Donsker Theorem (cf. [2]) states that a sequence of suitably normalized random functions based on partial sums of independent, identically distributed (i.i.d.) random variables with finite variances converges weakly in the uniform topology to Brownian motion. An analogous result for maxima has long been known and was first proven by Lamperti [10] in 1964. Until recently comparatively little attention was paid to the structure of the converging processes or to that of the limiting extremal process with the result that the full potential of the basic weak convergence result was never, in our opinion, realized. In view of recent studies on the structure of maxima of i.i.d. random variables and extremal processes, another look at weak convergence questions seems justified.

Let $\{X_n, n \geqq 1\}$ be i.i.d. random variables with common distribution function (df) $F(\cdot)$ and set $M_n = \max\{X_1, \cdots, X_n\}$. Concepts necessary to elucidate the structure of $\{M_n\}$ are the following: Say X_j is a record value of the sequence $\{M_n\}$ (or $\{X_n\}$) if $X_j > M_{j-1}$. The indices at which record values occur are given by the record value times $\{L_n, n \geqq 0\}$ defined by

$$L_0 = 1 , \qquad L_n = \min\{j \mid j > L_{n-1}, X_j > X_{L_{n-1}}\}$$

and the record value sequence is $\{X_{L_n}, n \geqq 0\}$. The inter-record times are the random variables Δ_n defined by $\Delta_n = L_n - L_{n-1}$, $n \geqq 1$ and μ_n is the number of record values in the sequence $M_1, \cdots, M_n$.

For Donsker's Theorem, the appropriate limiting process is Brownian motion. When "sum" is replaced by "max" the right processes to consider are the extremal processes defined as follows: For the given df $F(x)$ define a consistent family of

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finite dimensional distributions by

$$F_{t_1,t_2,\cdots,t_k}(x_1, \cdots, x_k) = F^{t_1}(\min\{x_1, \cdots x_k\})F^{t_2-t_1}(\min\{x_2, \cdots, x_k\}) \cdots F^{t_k-t_{k-1}}(x_k)$$

for $0 < t_1 < \cdots < t_k$ and $x_1, \cdots, x_k$ real. There exists a process with these finite dimensional distributions called an extremal-F process. The process is denoted either as $\{Y(t), t > 0\}$, Y or if ambiguity must be precluded: Y_F. Y is continuous in probability and we fix a version with right continuous, nondecreasing sample paths. If the left and right end of F are defined respectively by $x_0 = \inf\{x \mid F(x) > 0\}$, $x_1 = \sup\{x \mid F(x) < 1\}$ then $\lim_{t\downarrow 0} Y(t) = x_0$, $\lim_{t\uparrow\infty} Y(t) = x_1$ a.s. The random counting measure $\nu(I)$ counts the number of jumps of Y in the time interval I $(I \subset R^+)$ and is a nonhomogeneous Poisson process with intensity t^{-1} in the case that F is continuous. See [4], [13] for details.

An underlying assumption throughout this paper will be that there exist normalizing constants $a_n > 0$, b_n, $n \geqq 1$ such that for some nondegenerate $G(x)$:

$$(1) \qquad P[M_n \leqq a_n x + b_n] = F^n(a_n x + b_n) \to G(x)$$

as $n \to \infty$. When (1) holds we write $F \in D(G)$ and say F is in the domain of attraction of G. A basic result of extreme value theory ([3], [7]) states that G can belong to the types of one of three classes denoted by $\Lambda(x)$, $\Phi_\alpha(x)$, $\Psi_\alpha(x)$. For convenience we will here consider only the first two as the third never offers any new challenges. Recall $\Lambda(x) = \exp\{-e^{-x}\}$, $-\infty < x < \infty$ and $\Phi_\alpha(x) = \exp\{-x^{-\alpha}\}$ for $x \geqq 0$ and $= 0$ when $x < 0$. Here α is a positive parameter.

When (1) holds, we will be concerned with the convergence properties of the random function $Y_n(\cdot)$ defined by $M_0 = x_0(F)$ and

$$(2) \qquad Y_n(t) = (M_{[nt]} - b_n)/a_n .$$

In Section 2, we consider the weak convergence of the two dimensional point process with points $\{(k/n, a_n^{-1}(X_k - b_n)), k \geqq 1\}$ to a limiting Poisson random measure and thereby obtain a proof of the Lamperti result that for any $0 < a < b < \infty$: $Y_n \Rightarrow Y_G$; i.e., Y_n converges weakly to the extremal-G process in $D[a, b]$ with respect to the usual Skorohod J_1 topology. Generally, weak convergence notations and conventions are as in [2] except that "$\Rightarrow$" is used to denote weak convergence. Since the limiting extremal processes are stochastically continuous, weak convergence on $D[a, b]$ for any $a < b$ automatically extends to weak convergence on $D(0, \infty)$. (cf. [11].)

We also obtain as a result of the point process convergence that $Y_n^{-1} \Rightarrow Y^{-1}$. Throughout, inverses of nondecreasing functions are taken to be right continuous. In Section 3, we consider some implications of the basic convergences and obtain information about the asymptotic behavior of $\{X_{L_n}\}$, $\{L_n\}$, $\{\Delta_n\}$.

2. Basic convergences. What follows relies on an embedding technique based on the two-dimensional Poisson process which was devised by Pickands in [12].

We start by considering the point process T_n on $R_+ \times R$ with points $\{(k/n, a_n^{-1}(X_k - b_n)), k \geqq 1\}$ and show weak convergence to a limiting Poisson random measure. For background on convergence of point processes, the reader may consult Jagers [9] or Straf [19] but the situation here is quite simple and the following notion of convergence suffices: Say that point systems s_n in $R_+ \times R$ converge to a point system s if card $(s_n \cap A) \to$ card $(s \cap A)$ for all finite rectangles A for which $s \cap \partial A = \phi$. Extension of this concept to a.s. convergence of stochastic point processes is immediate.

The weak convergence of the T_n's may now be established by showing a.s. convergence of distributionally equivalent processes (cf. [18]).

THEOREM 1. *Let* $\{X_k, k \geqq 1\}$ *be* i.i.d. rv's *for which* (1) *holds and suppose* T_n *is the point process on* $R_+ \times R$ *with points* $\{(k/n, a_n^{-1}(X_k - b_n)), k \geqq 1\}$. *Then there exist point processes* U *and* U_n, $n \geqq 1$ *such that*

(i) $T_n =_d U_n$ *for* $n \geqq 1$

(ii) U *is Poisson random measure whose mean measure evaluated at* $[0, t] \times (x, \infty)$ *is* $-t \log G(x)$

(iii) *With probability* 1, $U_n \to U$ *in the sense described above.*

Some remarks on the proof: The construction of U_n and U are based on Pickand's embedding technique. Take a homogeneous Poisson random measure M on $R_+ \times R_+$ with points $\{(t_k, y_k), k \geqq 1\}$. Suppose the points of M in $R_+ \times [0, n]$ are $\{(\tau_k(n), \xi_k(n)), k \geqq 1\}$ ordered so that $\tau_1(n) < \tau_2(n) \cdots$. Note for each n, $\{\xi_k(n), k \geqq 1\}$ are i.i.d. rv's uniformly distributed on $[0, n]$. U_n is the process with points $\{(k/n, F_n^{-1}((1 - \xi_k(n)/n)^n)), k \geqq 1\}$ where $F_n(x) = F^n(a_n x + b_n)$ is the df of $a_n^{-1}(M_n - b_n)$ and U is the point process with points $\{(t_k, G^{-1}(e^{-y_k})), k \geqq 1\}$ so that U is M transformed. The transformation theory for Poisson random measures assures us that U has the mean measure described in (ii) above. Assertion (i) follows because

$$\{a_n^{-1}(X_k - b_n), k \geqq 1\} =_d \{F_n^{-1}((1 - \xi_k(n)/n)^n), k \geqq 1\}\,.$$

To check (iii) let $0 < a < b$, $0 < c < d$ and $I_n = (n(1 - F(a_n d + b_n)), n(1 - F(a_n c + b_n))]$ and note

$$\text{card}\,(U_n \cap (a, b] \times (c, d]) = \textstyle\sum_{a<k/n\leqq b} 1_{[\xi_k(n) \in I_n]}\,.$$

That card $(U_n \cap (a, b] \times (c, d]) \to$ card $(U \cap (a, b] \times (c, d])$ a.s. comes from the following reasoning: If (t, y) is a point of M then in order to construct U_n (for n bigger than the ordinate y) we relocate (t, y) to the new position $(n^{-1}\,\text{card}\,[0, t) \times [0, n) \cap M), y)$. The strong law of large numbers applied to homogeneous Poisson random measure shows that the new position converges to the original.

Next we make two observations. Let $g(\cdot)$ be a functional from point systems to $D(0, \infty)$ defined as follows: If the point system s has points $\{(t_k, y_k), k \geqq 1\}$ define $gs(t) = \sup\{y_k \mid t_k \leqq t\}$. Then it is easy to check that as a functional from point systems to $D[a, b]$ $0 < a < b$ (endowed of course with the Skorohod J_1

topology) g is continuous. Secondly, if we apply g to the Poisson random measure U described in Theorem 1 (ii), we obtain an extremal-G process: $gU =_d Y_G$. (In fact, this is a general method for generating any extremal process.) (Cf. [13].)

The conclusion from these two observations is that we can deduce Lamperti's theorem from Theorem 1 via the continuous mapping theorem:

THEOREM 2 (Lamperti). *If* (1) *holds and* Y_n *is defined by* (2) *then* $Y_n \Rightarrow Y_G$ *in* $D(0, \infty)$.

If Y is extremal-G where G has left and right ends x_0. x_1 respectively, we may consider along with Y the inverse process $\{Y^{-1}(x), x_0 < x < x_1\}$ defined by $Y^{-1}(x) = \inf\{z \mid Y(z) > x\}$. Y^{-1} is an additive process and has been studied in [5], [15], [17]. The method described between Theorems 1 and 2 can now be applied (except that g must be replaced by g' defined by $g's(x) = \inf\{t_k \mid y_k > x\}$) to yield:

THEOREM 3. *If* (1) *holds and* Y_n *is defined by* (2), *then* $Y_n^{-1} \Rightarrow Y_G^{-1}$ *in* $D(0, \infty)$ *if* $G = \Phi_\alpha$ *and in* $D(-\infty, \infty)$ *if* $G = \Lambda$.

Next for $x(\cdot) \in D(0, \infty)$ let $Nx(I)$ be the number of jumps of $x(\cdot)$ in the time interval I, $I \subset (0, \infty)$ so that $NY_n = NM_{[n\cdot]}$. Based on Theorem 1 and the continuous mapping theorem we can conclude

THEOREM 4. *If* (1) *holds then*

$$NM_{[n\cdot]} = NY_n \Rightarrow NY_G = \nu$$

and therefore in the notation of the introduction the counting functions

$$\mu_{[n\cdot]} - \mu_n \Rightarrow \nu(1, \cdot]$$

in $D[1, \infty)$.

Observe that Theorem 1 describes how to construct an almost surely convergent distributionally equivalent replacement for NY_n.

REMARK. Instead of (1), we may prove Theorem 4 under the condition

$$\lim_{t \uparrow x_1(F)} (1 - F(t-))/(1 - F(t)) = 1\,. \tag{3}$$

This condition is satisfied certainly if F is continuous but also if F satisfies (1) ([3], [7]). To see that (3) is sufficient observe that $NM_{[n\cdot]} =_d NY_F([n\cdot]) =_d NS_F^{-1}(Y_\Lambda([n\cdot])) =_d NS_F^{-1}(Y_\Lambda([n\cdot]/n) + \log n)$ where $S_F(x) = -\log(-\log F(x))$ and also $\nu =_d NY_\Lambda =_d N(Y_\Lambda(\cdot) + \log n)$. For any $0 < a < b$ it is not too hard to show that

$$N(Y_\Lambda(\cdot) + \log n)(a, b] - N(S_F^{-1}(Y_\Lambda([n\cdot]/n) + \log n))(a, b] \to 0 \tag{4}$$

a.s. as $n \to \infty$. To see this, suppose $t_1, \cdots, t_k$ are the jump times of a path of $Y_\Lambda(t)$, $t \in (a, b]$. Essentially what happens is that a nonzero difference is created in (4) if two or more points from $\{Y_\Lambda(t_1) + \log n, \cdots, Y_\Lambda(t_k) + \log n\}$ lie in the same interval of constancy of the function S_F^{-1}. However (3) is equivalent to $S_F(x) - S_F(x-) \to 0$ which says that the lengths of intervals of constancy of $S_F^{-1}(x)$ for $x \geqq \log n$ shrink to zero as $n \to \infty$.

Theorem 4 has a companion result dealing with NY_n^{-1}. Interpret $NY_n^{-1}(z_1, z_2]$ as the number of states visited by Y_n in the subset of the range given by $(z_1, z_2]$. Define a point process $\xi(I)$ for intervals I relative to the original $\{X_n\}$ sequence by $\xi(I) = \#\{k \mid X_{L_k} \in I\}$ so that $NY_n^{-1}(I) = \xi(a_n I + b_n)$.

THEOREM 5. (1) *implies* $NY_n^{-1} \Rightarrow NY_G^{-1}$ *or equivalently*

$$\{I \to \xi(a_n I + b_n)\} \Rightarrow \{I \to NY^{-1}(I)\}.$$

Here NY^{-1} *is a nonhomogeneous Poisson process with mean measure* $S_G(I) = -\log(-\log G(I))$.

The last statement is Corollary 1 of [15]. The rest of the proof of Theorem 4 involves ideas already discussed in connection with the previous results.

Here is a convergence result related to Theorem 1 which could be combined with the technique of embedding $\{M_n\}$ in a suitable extremal process to yield a proof of Theorem 2 (cf. [21]). The technique of Theorem 1 could be used to prove this result but we offer an alternate method.

THEOREM 6. *If* F_n, $n \geqq 0$ *are* df's *and* $F_n(x) \to F_0(x)$ *at all* x *which are continuity points of* F_0 *then* $Y_{F_n} \Rightarrow Y_{F_0}$ *in* $D(0, \infty)$.

PROOF. Denoting the uniform distribution by U we have $Y_{F_n} =_d F_n^{-1} \circ Y_U$, $n \geqq 0$ so it suffices to show $F_n^{-1} \circ Y_U \to F_0^{-1} \circ Y_U$ a.s. and uniformly on compact subsets of $(0, \infty)$. The hypothesis implies $F_n^{-1}(x) \to F_0^{-1}(x)$ at all x in the continuity set of F_0^{-1}. From Proposition 6 of [13] we have since U is continuous: $P[Y_U$ hits discontinuity set of $F_0^{-1}] = 0$. Therefore for a.a. ω the range of $Y_U \subset$ continuity set of F_0^{-1} which entails by a theorem of Dini that $F_n^{-1} \to F_0^{-1}$ uniformly on finite subsets of the range of Y_U and finally we have $F_n^{-1} \circ Y_U \to F_0^{-1} \circ Y_U$ a.s. and uniformly on compact subsets of $(0, \infty)$. This suffices.

A final remark: As Pickands [12] has noted, the methods employed in this section extend without difficulty to the case where one studies weak convergence of kth largest variables in a sample of size n.

3. Applications to records, record times. Define the jump time functional as follows: For point process paths $x(\cdot)$ with a finite number of points in compact subintervals of $(0, \infty)$ $jx = (j^+x, j^-x) = (\cdots j_{-2}x, j_{-1}x, j_1x, j_2x, \cdots)$ gives the times of jumps before $t = 1$ and after $t = 1$; i.e., for $k = 1, 2, \cdots$

$$\begin{aligned} j_k x &= +\infty && \text{if } \{t > 1 \mid x(t) - x(1) = k\} = \phi \\ &= \inf\{t > 1 \mid x(t) - x(1) = k\}, && \text{otherwise} \end{aligned}$$

and similarly $j_{-k}x$ is the time of the kth jump to the left of 1 and is 0 if there is no such jump. j is continuous at those x all of whose interpoint distances are positive (cf. [8], page 57, [23]). When the conclusion of Theorem 4 holds, the continuous mapping theorem gives $jNY_n \Rightarrow jNY = j\nu$ where ν is nonhomogeneous Poisson with intensity t^{-1}. A simple transformation to a homogeneous Poisson process shows that $j^+\nu$ has the structure $\{\exp\{\sum_{i=1}^n Z_i\}, n \geqq 1\}$ where $\{Z_n, n \geqq 1\}$

are i.i.d. and $P[Z_i > x] = e^{-x}$, $x > 0$. The times when NY_n or Y_n jumps past $t = 1$ are L_k/n (in the terminology of Section 1). The conclusion is (cf. [21], pages 85–89).

COROLLARY 1. *If* (3) *holds, then* $\{L_k/n: L_k > n\} \Rightarrow \{\exp\{\sum_{i=1}^m Z_i\}, m \geqq 1\}$ *as* $n \to \infty$ *where* $\{Z_n, n \geqq 1\}$ *are* i.i.d. *exponential mean* 1 *random variables. Further*

$$\{\Delta_{k+1}/n: L_k > n\} \Rightarrow \{\exp\{\textstyle\sum_{k=1}^{m+1} Z_k\} - \exp\{\sum_{k=1}^m Z_k\}, m \geqq 1\}.$$

The last statement follows from the first by the continuous mapping theorem which also permits other variations:

$$\{L_k/L_{k+1}: L_k > n\} \Rightarrow \{U_m, m \geqq 1\} \equiv \{e^{-Z_m}, m \geqq 1\}$$
$$\{\log(L_k/n): L_k > n\} \Rightarrow \{\textstyle\sum_{j=1}^m Z_j, m \geqq 1\}.$$

Note $\{U_m\}$ are i.i.d. uniform (0, 1) random variables. Similarly $\{\Delta_k/\Delta_{k+1}: L_k > n\}$ has a weak limit.

These results may be compared with those in Section 4 of [16] and also with the fact that there is no normalization $\alpha_n > 0$, β_n, $n \geqq 1$ such that $\alpha_n^{-1}(L_n - \beta_n)$ has a weak limit. Cf. [1], [16], [20].

Returning again to the basic situation $jNY_n \Rightarrow j\nu$ we look at the time of the first jump after $t = 1$, the time of the jump immediately preceding $t = 1$ and the difference between the two jump times.

COROLLARY 2. *If* (3) *holds, then for* $x \geqq 0$:

$$\lim_{n\to\infty} P[(L_{\mu(n)+1} - n)/n \leqq x] = x(1+x)^{-1}$$
$$\lim_{n\to\infty} P[(n - L_{\mu(n)})/n \leqq x] = x \wedge 1$$
$$\lim_{n\to\infty} P[(L_{\mu(n)+1} - L_{\mu(n)})/n \leqq x] = x - \log(1+x) \qquad \text{if } x \leqq 1$$
$$= 1 - \log(x^{-1}(1+x)) \qquad \text{if } x > 1.$$

PROOF. By the continuous mapping theorem the limits are respectively the distributions of the forward and backward recurrence times of ν relative to $t = 1$ and the distribution of the length of the inter-point interval which covers 1. The two recurrence time distributions are easily computed and their convolution gives the third required limit distribution.

The techniques of Corollaries 1, 2 may be applied to Y_n^{-1} in conjunction with Theorem 5.

COROLLARY 3. *If* (1) *holds, then as* $T \to \infty$

(i) $F \in D(\Phi_\alpha)$ *entails* $\{X_{L_k}/T: X_{L_k} > T\} \Rightarrow \{\exp\{\alpha^{-1}\sum_{i=1}^m Z_i, m \geqq 1\}\}$.

(ii) $F \in D(\Lambda)$ *entails* $\{(X_{L_k} - b(1/e\bar{F}(T)))/a(1/\bar{F}(T)): X_{L_k} > T\} \Rightarrow \{1 + \sum_{i=1}^m Z_i, m \geqq 1\}$

where $\{Z_i\}$ *are* i.i.d. *exponential mean* 1 *random variables and* $\bar{F} = 1 - F$.

REMARK. (i) is obtained by Shorrock in [17].

PROOF. From Theorem 5 we have $jNY_n^{-1} \Rightarrow jNY_G^{-1}$.

Note that $j^+NY_n^{-1} = \{a_n^{-1}(X_{L_k} - b_n): X_{L_k} > a_n + b_n\}$ so that

$$\{a_n^{-1}(X_{L_k} - b_n): X_{L_k} > a_n + b_n\} \Rightarrow j^+NY_G^{-1} \tag{5}$$

where NY_G^{-1} is a Poisson process with mean measure $S_G(I) = -\log(-\log G(I))$ ([15]). When $F \in D(\Phi_\alpha)$ we may choose $b_n = 0$, $a_n = F^{-1}(1 - n^{-1})$ ([3], [7]). Putting this in (5) and changing variables gives (i). In the case $F \in D(\Lambda)$ we set $b_n = F^{-1}(1 - n^{-1})$, $a_n = F^{-1}(1 - (ne)^{-1}) - F^{-1}(1 - n^{-1})$ ([3], [7]) so that $a_n + b_n = F^{-1}(1 - (ne)^{-1})$. Now combine (5), a change of variable and the fact that $a(\cdot)$ is slowly varying ([3]) to obtain (11).

Some further results in this vein obtainable from the continuous mapping theorem: If $F \in D(\Phi_\alpha)$ then

$$\{X_{L_k}/X_{L_{k+1}}: X_{L_k} > T\} \Rightarrow \{U_m^{1/\alpha}, m \geqq 1\}$$

where $\{U_m, m \geqq 1\}$ are i.i.d. uniform (0, 1). Also

$$\{\log\{X_{L_k}/T\}: X_{L_k} > T\} \Rightarrow \{\alpha^{-1} \textstyle\sum_{i=1}^m Z_i, m \geqq 1\}\,.$$

Typical when $F \in (\Lambda)$ is the result from (ii):

$$\{(X_{L_{k+1}} - X_{L_k})/a(1/\bar{F}(T)): X_{L_k} > T\} \Rightarrow \{Z_m, m \geqq 1\}\,.$$

We may particularize Corollary 3 by looking at the first jump past 1 or more generally past any s. For convenience define $\tau(v) = \inf\{n \mid M_n > v\}$.

COROLLARY 4. (i) *If $F \in D(\Lambda)$ then for any s, $x > 0$*

$$\lim_{n\to\infty} P[a_n^{-1}(M_{\tau(a_n s + b_n)} - (a_n s + b_n)) \leqq x] = 1 - e^{-x}\,.$$

(ii) *If $F \in D(\Phi_\alpha)$ then for $x > 0$*

$$\lim_{v\to\infty} P[(M_{\tau(v)} - v)/v \leqq x] = 1 - (1 + x)^{-\alpha}\,.$$

Variants of Corollaries 3, 4 are obtained by looking at jump heights past time 1. The results: If $F \in D(\Lambda)$ then

$$\{a_n^{-1}(X_{L_k} - b_n): L_k > n\} \to \{Y(1) + \textstyle\sum_{i=1}^m Z_i, m \geqq 1\}$$

and

$$\{a_n^{-1}(X_{L_{k+1}} - X_{L_k}): L_k > n\} \to \{Z_m, m \geqq 1\}$$

as $n \to \infty$ where $\{Z_m, m \geqq 1\}$ are i.i.d. exponential random variables independent of $Y(1)$ where $P[Y(1) \leqq x] = \Lambda(x)$. (The limit is evaluated using Theorem 3, [13].) Observe that these results entail

$$a_n^{-1}(X_{L_{\mu(n+1)}} - b_n) \Rightarrow Y(1) + Z_1$$
$$a_n^{-1}(X_{L_{\mu(n+1)}} - X_{L_{\mu(n)}}) \Rightarrow Z_1$$

so that the size of the first jump after index n in the Markov chain $\{M_j, j \geqq 1\}$ is asymptotically exponentially distributed.

Next consider the function h defined for $x \in D(0, \infty)$ or $D(-\infty, \infty)$ by

$$(hx)(t) = \sup_{0<s\leqq t}\{x(s) - x(s-)\}\,.$$

h is continuous so by Theorem 2: $hY_n^{-1} \to hY^{-1}$. Since Y^{-1} had independent increments, hY^{-1} is a variant of the class of extremal processes considered here: hY^{-1} is a nonhomogeneous Markov process of the type studied in [22]. (Cf. [13], Section 2.) Due to the additive structure of Y^{-1}, the Lévy measure can be computed. This was done in [15], Theorem 2. Consequently

$$P[(hY^{-1})(t) \leqq x] = P[Y^{-1}(u) \text{ has no jump of size } > x, 0 < u \leqq t]$$
$$= \exp\{-[\textstyle\int_x^\infty e^{-Q(t)s} s^{-1}\, ds - \int_x^\infty e^{-Q(0)s} s^{-1}\, ds]\}$$

where $Q(s) = -\log G(s)$.

Next we identify $(hY_n^{-1})(t)$ as the maximum holding time of $Y_n(u)$, $0 < u \leqq Y_n^{-1}(t) = \sup\{\Delta_j/n \mid L_j \leqq \tau(a_n t + b_n)\}$. Recall $\tau(a) = \inf\{n \mid M_n > a\}$ and $Y_n^{-1}(t) = \tau(a_n t + b_n)/n$. In the case that $G = \Phi_\alpha$, we set $b_n = 0$, $a_n = F^{-1}(1 - n^{-1})$ and $t = 1$ and set $s = F^{-1}(1 - n^{-1})$. The result:

COROLLARY 5. *If* (1) *holds then*

(i) *If* $G = \Phi_\alpha$:

$$\sup\{\Delta_j \bar{F}(s) \mid L_j \leqq \tau(s)\} \to \exp\{-\textstyle\int_x^\infty e^{-z} z^{-1}\, dz\}$$

as $s \to \infty$ *for* $x \geqq 0$.

(ii) *If* $G = \Lambda$ *then*

$$\sup\{\Delta_j/n \mid \tau(b_n) < L_j \leqq \tau(a_n + b_n)\} \to \exp\{-[\textstyle\int_x^\infty e^{-e^{-1}z} z^{-1}\, dz - \int_x^\infty e^{-z} z^{-1}\, dz]\}$$

as $n \to \infty$ *for* $x \geqq 0$.

Attempts to derive explicitly the df of the largest jump of Y in the time interval $(1, t]$, $t > 1$, have not been successful. Consider instead the following continuous functional: For $x(\cdot)$ a nondecreasing function define

$$(h'x)(t) = \max\{x(x^{-1}(0)), \sup\{x(s) - x(s-) \mid 0 < x(s-) \leqq x(s) \leqq t\}\}.$$

By the continuous mapping theorem: $(h'Y_n)(t) \to (h'Y)(t)$. We first compute the df of $(h'Y)(t)$. Suppose Y is extremal-Λ so that the point process induced by the range of Y is homogeneous Poisson, rate 1. If then $\{Z_j, j \geqq 1\}$ are i.i.d. exponential mean 1 random variables and $S_n = \sum_{j=1}^n Z_j$ then

$$(h'Y)(t) =_d \max\{Z_j \mid S_j \leqq t\} \equiv J(t)$$

where max of an empty set is zero. We show for $y > 0$:

$$P[J(t) \leqq y] = 1 + e^{-t} \sum_{j=1}^\infty \frac{(-1)^j (t - jy)_+^j}{j!} e^{(t-jy)_+}$$

where $s_+ = s$ if $s \geqq 0$, $= 0$ if $s < 0$. The derivation uses some facts presented in [6]. Conditioning on the last S_n before t we have:

$$P[J(t) \leqq y] = \sum_{n=1}^\infty \int_0^t P[J(t) \leqq y \mid S_n = u] \frac{e^{-(t-u)} e^{-u} u^{n-1}}{(n-1)!}\, du + P[S_1 > t].$$

Note $P[J(t) \leqq y \mid S_n = u]$ is the probability that of the n subintervals induced by

$n - 1$ points chosen at random from $[0, u]$ none is longer than y. This probability is computed on pages 28–29 of [6]. The rest is routine manipulation involving reversal of the order of summations.

This derivation enables one to conclude:

COROLLARY 6. *If* (1) *holds with* $G = \Lambda$ *then*

$$\max\{a_n^{-1}(M_{\tau(0)} - b_n), \sup\{a_n^{-1}(X_{L_{k+1}} - X_{L_k}) \mid 0 < X_{L_k} \leqq X_{L_{k+1}} \leqq t\}\} \to J(t)$$

where the df *of* $J(t)$ *is given by* (25).

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