The Annals of Probability
1976, Vol. 4, No. 1, 139–146

CONVERGENCE RATES FOR BRANCHING PROCESSES

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Almost sure estimates of the rate of convergence for the supercritical Galton–Watson process are obtained, e.g. $W - W_n = o(m^{-n/q})$ a.s. if and only if $E(Z_1{}^p \mid Z_0 = 1) < \infty$, where $1 < p < 2$, $1/p + 1/q = 1$. Extensions to the multitype and continuous time cases are outlined.

1. Introduction. Let $Z_0, Z_1, \cdots$ be a supercritical Galton–Watson process with offspring distribution F and mean $m = \int_0^\infty x\, dF(x)$, $1 < m < \infty$, and let $W_n = Z_n/m^n$, $W = \lim_n W_n$. It is well-known ([3], page 54), that for each n we have a decomposition

$$W - W_n = \frac{1}{m^n} \sum_{j=1}^{Z_n} U_{n,j}\,, \tag{1.1}$$

where the $U_{n,j}$'s are i.i.d. for fixed n, each distributed as $W - 1$ and thus with mean zero in the W nondegenerate case $\int_0^\infty x \log^+ x\, dF(x) < \infty$. Since Z_n is of the magnitude m^n, we might expect the convergence rate in the law of large numbers for random variables distributed as W to give an estimate of $W - W_n$. For example, if $\sigma^2 = \operatorname{Var} W < \infty$ and $F(0+) = 0$, then $Z_n \to \infty$ a.s. and one would expect $m^n(W - W_n)/\sigma(Z_n)^{\frac{1}{2}}$ to converge in distribution to the standard normal distribution and a law of the iterated logarithm,

$$\limsup m^n(W - W_n)/(2\sigma^2 Z_n \log\log Z_n)^{\frac{1}{2}} = 1 \quad \text{a.s.}$$

These results were established in [5], [6] and [7].

In the case of infinite variance, which is the subject of the present paper, the rate of convergence in the law of large numbers depends on the tail probabilities of W. Let e.g. $U^{(1)}, U^{(2)}, \cdots$ be i.i.d. each distributed as $W - 1$, let $\bar{U}^{(n)} = (U^{(1)} + \cdots + U^{(n)})/n$ and let $1 < p < 2$, $1/p + 1/q = 1$. Then it is known ([10], page 243) that $\bar{U}^{(n)} = o(n^{-1/q})$ a.s. if and only if $EW^p < \infty$, which in turn may be seen to be equivalent to $\int_0^\infty x^p\, dF(x) < \infty$. This together with (1.1) motivates the following result:

THEOREM 1. *Let* $1 < p < 2$, $1/p + 1/q = 1$. *Then*

$$W - W_n = o(m^{-n/q}) \quad \text{a.s.}, \qquad P(W > 0) > 0 \tag{1.2}$$

if and only if

$$\int_0^\infty x^p\, dF(x) < \infty\,. \tag{1.3}$$

The proof will be given in Section 2, where we also prove along the same lines

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Received March 3, 1975.

AMS 1970 *subject classifications.* Primary 60J80; Secondary 60J85.

Key words and phrases. Galton–Watson processes, supercritical, infinite variance, convergence rates.

THEOREM 2. *Let* $\alpha \geqq 0$. *Then*

$$\int_0^\infty x(\log^+ x)^{\alpha+1}\, dF(x) \tag{1.4}$$

implies that

$$W - W_n = o(n^{-\alpha}) \quad \text{a.s.} \tag{1.5}$$

and the a.s. *convergence of* $\sum_{n=1}^\infty (W - W_n)$ *for* $\alpha \geqq 1$. *Conversely, if*

$$\int_0^\infty x(\log^+ x)^{\alpha+1-\varepsilon}\, dF(x) = +\infty$$

for some $\varepsilon > 0$, *then* $\liminf n^\alpha(W - W_n) = -\infty$ a.s. *on* $\{W > 0\}$.

Let $\bar{U}^{(n)}$ be defined as above. Then it may be proved along the lines of [10], pages 242–243, that $\bar{U}^{(n)} = o((\log n)^{-\alpha})$ if and only if $EW(\log^+ W)^\alpha < \infty$. In [2] this latter condition is proved to be equivalent to $\int_0^\infty x(\log^+ x)^{\alpha+1}\, dF(x)$. Taking account of (1.1) and the fact that $\log Z_n \cong n$, one would thus expect (1.4) to be equivalent to (1.5). However, in Section 2 a counterexample is given, which shows that (1.5) may hold under slightly weaker conditions than (1.4).

The extension of Theorems 1 and 2 to the multitype and continuous-time Markovian case is stated and proofs outlined in Sections 3 and 4.

2. Proof of Theorems 1 and 2. We may write $Z_n = \sum_{j=1}^{Z_{n-1}} X_{n,j}$, where the $X_{n,j}$'s are i.i.d. and distributed according to the offspring distribution F. Thus

$$W_n - W_{n-1} = \frac{1}{m^n} \sum_{j=1}^{Z_{n-1}} (X_{n,j} - m)\,. \tag{2.1}$$

This identity will be used repeatedly in the sequel. Also, we let $\mathscr{F}_n = \sigma(X_{k,j},\ j = 1, 2, \cdots;\ k \leqq n)$ so that in particular Z_n is $\mathscr{F}_n$-measurable.

LEMMA 1. (1.3) *implies the* a.s. *convergence of*

$$\sum_{n=1}^\infty m^{n/q}(W_n - W_{n-1})\,.$$

PROOF. Let

$$\begin{aligned} Y_{n,j} &= X_{n,j} 1_{\{X_{n,j} \leqq m^{n/p}\}} - m\,, \\ S_n &= m^{n/q}(W_n - W_{n-1}) = m^{-n/p} \sum_{j=1}^{Z_{n-1}} (X_{n,j} - m)\,, \\ S_n' &= m^{-n/p} \sum_{j=1}^{Z_{n-1}} Y_{n,j}\,. \end{aligned}$$

Observing that $\sum_{k=1}^n (S_k' - E(S_k' \mid \mathscr{F}_{k-1}))$, $n = 1, 2, \cdots$, forms a martingale with respect to $\mathscr{F}_1, \mathscr{F}_2, \cdots$, it suffices by the Borel–Cantelli lemma and the convergence theorem for L^2-bounded martingales to show the convergence of each of the series

$$\sum_{n=1}^\infty P(S_n \neq S_n')\,, \qquad \sum_{n=1}^\infty \operatorname{Var}(S_n' - E(S_n' \mid \mathscr{F}_{n-1}))\,, \qquad \sum_{n=1}^\infty E(S_n' \mid \mathscr{F}_{n-1})\,,$$

the latter in the a.s. sense. First

$$\begin{aligned} \sum_{n=1}^\infty P(S_n \neq S_n') &= \sum_{n=1}^\infty E(P(S_n \neq S_n' \mid \mathscr{F}_{n-1})) \\ &\leqq \sum_{n=1}^\infty EZ_{n-1} \int_{m^{n/p}}^\infty dF(x) \\ &\leqq \int_0^\infty \left(\sum_{n=1}^\infty m^{n-1} 1_{\{x > m^{n/p}\}}\right) dF(x) \\ &= \int_0^\infty O(x^p)\, dF(x) < \infty\,, \end{aligned}$$

and

$$\begin{aligned}\sum_{n=1}^{\infty} \operatorname{Var}(S_n' - E(S_n' \mid \mathscr{F}_{n-1})) &= \sum_{n=1}^{\infty} E(\operatorname{Var}(S_n' \mid \mathscr{F}_{n-1})) \\ &= \sum_{n=1}^{\infty} EZ_{n-1} m^{-2n/p} \operatorname{Var} Y_{n,1} \\ &\leqq m^{-1} \int_0^\infty x^2 (\sum_{n=1}^{\infty} m^{n(1-2/p)} 1_{\{x \leqq m^{n/p}\}})\, dF(x) \\ &= m^{-1} \int_0^\infty x^2 O(x^{p(1-2/p)})\, dF(x) < \infty\,.\end{aligned}$$

Finally, the proof is completed by observing that

$$E(S_n' \mid \mathscr{F}_{n-1}) = -m^{-n/p} Z_{n-1} \int_{m^{n/p}}^\infty x\, dF(x) \leqq 0$$

and that

$$\begin{aligned}-\sum_{n=1}^{\infty} ES_n' &= \sum_{n=1}^{\infty} m^{-1} m^{n/q} \int_{m^{n/p}}^\infty x\, dF(x) \\ &= m^{-1} \int_0^\infty x O(x^{p-1})\, dF(x) < \infty\,.\end{aligned}$$

LEMMA 2. *Let $\alpha_1, \alpha_2, \cdots, \beta_1, \beta_2, \cdots$ be sequences of real numbers.*

(i) *If $\alpha_n \geqq 0$, $\alpha_n \uparrow \infty$ and the series $\sum_{n=1}^{\infty} \alpha_n \beta_n$ converges, then $\sum_{n=N+1}^{\infty} \beta_n = o(1/\alpha_N)$.*

(ii) *The convergence of $\sum_{n=1}^{\infty} n\beta_n$ implies that of $\sum_{n=1}^{\infty} \sum_{m=n}^{\infty} \beta_m$.*

(iii) *Let $\beta_n \geqq 0$, $\sum_{n=1}^{\infty} n^{\alpha-\varepsilon} \beta_n = +\infty$ for some ε in $0 < \varepsilon < \alpha$. Then*

$$\limsup N^\alpha \sum_{n=N+1}^{\infty} \beta_n = +\infty\,.$$

PROOF. Let $K_N = \sup_{p,q \geqq N} |\sum_{n=p}^{q} \alpha_n \beta_n|$, so that $K_N \to 0$, $N \to \infty$. From Abel's lemma ([4], page 54) $|\sum_{n=N+1}^{q} \beta_n| \leqq K_N/\alpha_N$ and (i) is clear by letting q tend to infinity. (ii) is clear from (i) and the identity $\sum_{n=1}^{N} \sum_{m=n}^{\infty} \beta_m = \sum_{n=1}^{N} n\beta_n + N \sum_{m=N+1}^{\infty} \beta_m$. For (iii), suppose $\sum_{n=N+1}^{\infty} \beta_n = O(N^{-\alpha})$. Then upon integration by parts

$$\begin{aligned}\sum_{n=1}^{\infty} n^{\alpha-\varepsilon} \beta_n &= \sum_{n=1}^{\infty} \beta_n + \int_1^\infty (\alpha - \varepsilon) t^{\alpha-\varepsilon-1} \sum_{n=[t]+1}^{\infty} \beta_n\, dt \\ &= \sum_{n=1}^{\infty} \beta_n + \int_1^\infty O(t^{-\varepsilon-1})\, dt < \infty\,.\end{aligned}$$

PROOF OF THEOREM 1. The "if"-part is clear by combining Lemmas 1 and 2 (i), with $\alpha_n = m^{n/q}$, $\beta_n = W_n - W_{n-1}$ so that $\sum_{n=N+1}^{\infty} \beta_n = W - W_N$. Suppose conversely that (1.2) holds. In particular, $W_n - W_{n-1} = o(m^{-n/q})$ a.s. and from $P(W > 0) > 0$ it follows, that $\inf Z_n/m^n > 0$ on $\{Z_n > 0$ for all $n\}$. Thus from (2.1)

$$\frac{1}{Z_{n-1}^{1/p}} \sum_{j=1}^{Z_{n-1}} (X_{n,j} - m) = \frac{m^{n/p}}{Z_{n-1}^{1/p}} m^{n/q} (W_n - W_{n-1}) \to 0 \quad \text{a.s.} \tag{2.2}$$

where we interpret the left-hand side of (2.2) as zero for $Z_n = 0$. Now take a sequence $U_1, U_2, \cdots$ of i.i.d. random variables distributed as $X_{1,1} - m$ and defined on some arbitrary probability space. Also, let $U_1^c, U_2^c, \cdots$ be i.i.d. such that U_1^c follows the symmetrized distribution of U_1 and let for $\varepsilon > 0$ $q_{n,\varepsilon} = P(n^{-1/p}|U_1 + \cdots + U_n| > \varepsilon)$, $q_{n,\varepsilon}^c = P(n^{-1/p}|U_1^c + \cdots + U_n^c| > \varepsilon)$.

From (2.2) and an extension of the Borel–Cantelli lemma ([10], page 397) applied to the events

$$A_{n,\varepsilon} = \{Z_{n-1}^{-1/p} |\sum_{j=1}^{Z_{n-1}} (X_{n,j} - m)| > \varepsilon\},$$

it follows that

(2.3) $$\sum_{n=1}^{\infty} q_{Z_{n-1},\varepsilon} = \sum_{n=1}^{\infty} P(A_{n,\varepsilon} | \mathscr{F}_{n-1}) < \infty \quad \text{a.s.} \qquad \text{for any} \quad \varepsilon > 0 .$$

Thus also $\sum_{n=1}^{\infty} q_{Z_{n-1},\varepsilon}^c \leqq \sum_{n=1}^{\infty} 2q_{Z_{n-1},\varepsilon/2} < \infty$ a.s. Now take a sequence k_1, $k_2, \cdots$ of integers of the form $k_n = Z_{n-1}(\omega)$, where ω belongs to the set of positive probability, where $\lim_n Z_n(\omega)/m^n$ exists and is > 0 and where $\sum_{n=1}^{\infty} q_{Z_n(\omega),\varepsilon}^c < \infty$ for any rational $\varepsilon > 0$ (say). From the Borel-Cantelli lemma it follows that $V_{k_n} = k_n^{-1/p}(U_1^c + \cdots + U_{k_n}^c) \to 0$ a.s. Also, if we let $M_n = k_n^{-1/p} \sup_{1 \leq t \leq k_n} |U_1^c + \cdots + U_t^c|$, then by an inequality of P. Lévy ([10], page 247) $P(M_n > \varepsilon) \leqq 2q_{k_n,\varepsilon}$, and applying the Borel-Cantelli lemma once more we get $M_n \to 0$ a.s. Since $k_{n+1}/k_n \to m$, there is a $\gamma > 0$ such that $k_n \geqq \gamma k_{n+1}$ for all n, and for n large, $k_n < k_{n+1}$. For $k_n \leqq t \leqq k_{n+1}$

(2.4) $$\begin{aligned} |V_t| &\leqq t^{-1/p}|U_1^c + \cdots + U_{k_n}^c| + t^{-1/p}|U_{k_n+1}^c + \cdots + U_t^c| \\ &\leqq |V_{k_n}| + 2\gamma^{-1/p} M_{n+1}, \end{aligned}$$

so that $V_t \to 0$ a.s. for $t \to \infty$. But this is only possible if $E|U_1^c|^p < \infty$ ([10], page 213) and thus if $\int_0^\infty x^p \, dF(x) < \infty$.

Proof of Theorem 2. We let $S_n = n^\alpha(W_n - W_{n-1})$, $Y_{n,j} = X_{n,j} 1_{\{X_{n,j} \leq m^n/n^\alpha\}} - m$, $S_n' = (n^\alpha/m^n) \sum_{j=1}^{Z_{n-1}} Y_{n,j}$ in analogy with the proof of Lemma 1. By calculations similar to the ones employed there we get

(2.5) $$\begin{aligned} \sum_{n=1}^{\infty} P(S_n \neq S_n') &\leqq \int_0^\infty (\sum_{n=1}^{\infty} m^{n-1} 1_{\{x > m^n/n^\alpha\}}) \, dF(x) \\ &= \int_0^\infty O(x(\log^+ x)^\alpha) \, dF(x) ; \end{aligned}$$

(2.6) $$\begin{aligned} \sum_{n=1}^{\infty} \operatorname{Var}(S_n' - E(S_n' | \mathscr{F}_{n-1})) &\leqq \int_0^\infty x^2(\sum_{n=1}^{\infty} n^{2\alpha}/m^n 1_{\{x \leq m^n/n^\alpha\}}) \, dF(x) \\ &= \int_0^\infty O(x(\log^+ x)^\alpha) \, dF(x) , \end{aligned}$$

(2.7) $$\begin{aligned} -\sum_{n=1}^{\infty} ES_n' &\leqq m^{-1} \int_0^\infty x(\sum_{n=1}^{\infty} n^\alpha 1_{\{x > m^n/n^\alpha\}}) \, dF(x) \\ &= m^{-1} \int_0^\infty x(\log^+ x)^{\alpha+1} \, dF(x) . \end{aligned}$$

Thus, arguing as before, (1.4) implies the a.s. convergence of $\sum_{n=1}^{\infty} n^\alpha(W_n - W_{n-1})$ and thus (1.5) by part (i) of Lemma 2. The a.s. convergence of $\sum_{n=0}^{\infty} (W - W_n)$ is clear for $\alpha > 1$ from (1.5), while for $\alpha = 1$ we need only to appeal to Lemma 2(ii), with $\beta_n = W_n - W_{n-1}$.

In order to show that $\int_0^\infty x(\log^+ x)^{\alpha+1-\varepsilon} \, dF(x) = +\infty$ for some $\varepsilon > 0$ implies that $\liminf n^\alpha(W - W_n) = -\infty$ a.s. on $\{W > 0\}$, it is no restriction to assume $P(W > 0) > 0$ so that at least $\int_0^\infty x \log^+ x \, dF(x) < \infty$. Also, by replacing α with a smaller α if necessary, we may assume $\int_0^\infty x(\log^+ x)^\alpha \, dF(x) < \infty$. Then from (2.6) and (2.5), $\sum_{n=1}^{\infty} (S_n' - E(S_n' | \mathscr{F}_{n-1}))$ and $\sum_{n=1}^{\infty} (S_n - E(S_n' | \mathscr{F}_{n-1}))$ converge a.s. Applying Lemma 2(i), with $\alpha_n = n^\alpha$, $\beta_n = W_n - W_{n-1} - n^{-\alpha}E(S_n' | \mathscr{F}_{n-1})$ we get

(2.8) $$W - W_N - \sum_{n=N+1}^{\infty} n^{-\alpha}E(S_n' | \mathscr{F}_{n-1}) = o(N^{-\alpha}) \quad \text{a.s.}$$

and we need only to show that $\liminf N^\alpha \sum_{n=N+1}^\infty n^{-\alpha} E(S_n' \mid \mathscr{F}_{n-1}) = -\infty$ a.s. on $\{W > 0\}$. But on $\{W > 0\}$ $\underline{W} = \inf Z_{n-1}/m^n > 0$ and the proof is concluded by Lemma 2(iii), with $\beta_n = -n^{-\alpha} E(S_n' \mid \mathscr{F}_{n-1})$, since

$$\begin{aligned} \sum_{n=1}^\infty n^{\alpha-\varepsilon}\beta_n &= \sum_{n=1}^\infty n^{\alpha-\varepsilon} Z_{n-1}/m^n \int_{m^n/n^\alpha}^\infty x\, dF(x) \\ &\geqq \underline{W} \int_0^\infty x\left(\sum_{n=1}^\infty n^{\alpha-\varepsilon} 1_{\{x > m^n/n^\alpha\}}\right) dF(x) \\ &= \underline{W} \int_0^\infty x O((\log^+ x)^{\alpha+1-\varepsilon})\, dF(x) = +\infty \qquad \text{on} \quad \{W > 0\}. \end{aligned}$$

From the last part of the proof, a counterexample on the implication (1.5) $\Rightarrow$ (1.4) is easily constructed. We need only to take an F such that $\int_0^\infty x(\log^+ x)^\alpha\, dF(x) < \infty$, $\int_0^\infty x(\log^+ x)^{\alpha+1}\, dF(x) = +\infty$, $\sum_{n=N+1}^\infty \int_{m^n/n^\alpha}^\infty x\, dF(x) = o(N^{-\alpha})$.

For example, for $\alpha > 0$ we may take F with point probabilities of magnitude $(n^2(\log n)^{\alpha+2} \log\log n)^{-1}$. Then (2.8) holds and thus (1.5) since

$$-\sum_{n=N+1}^\infty n^{-\alpha} E(S_n' \mid \mathscr{F}_{n-1}) = \sum_{n=N+1}^\infty Z_{n-1}/m^n \int_{m^n/n^\alpha}^\infty x\, dF(x) = o(N^{-\alpha}) \quad \text{a.s.}$$

3. The multitype case. Let $Z_0, Z_1, \cdots$ be a k-type Galton–Watson process with offspring distributions $F_{i,j}$. That is, $Z_n = (Z_n^1 \cdots Z_n^k)$ and

$$Z_n^j = \sum_{i=1}^k \sum_{\nu=1}^{Z_{n-1}^i} X_{n,\nu}^{i,j} \qquad j = 1, \cdots, k \tag{3.1}$$

where the X's are independent and $X_{n,\nu}^{i,j}$ distributed according to $F_{i,j}$. Let M be the mean matrix with elements $m_{i,j} = \int_0^\infty x\, dF_{i,j}(x)$ so that $E(Z_n \mid Z_{n-1}) = Z_{n-1}M$. We consider the positive regular case as defined in [8] and let as usual ρ be the largest positive eigenvalue of M with associated right and left eigenvectors u and v, normalized by $vu' = 1$. For further details on the setup, see [8] or [3], Chapter 5.

In the supercritical case $\rho > 1$ there exists a one-dimensional random variable W such that $\lim_n Z_n/\rho^n = Wv$ a.s. To investigate the rate of this convergence, we shall content ourselves with the analogues of the direct parts of Theorems 1 and 2 and merely outline the proofs.

THEOREM 3. *Let* $1 < p < 2$, $1/p + 1/q = 1$. *Then*

$$\int_0^\infty x^p\, dF_{i,j}(x) < \infty\,, \qquad i,j = 1, \cdots, k\,, \tag{3.2}$$

implies that

$$W - Z_n u'/\rho^n = o(\rho^{-n/q}) \quad \text{a.s.} \tag{3.3}$$

Note that the sequence $Z_n u'/\rho^n$ forms a nonnegative martingale with limit $Wvu' = W$. In [9], some convergence rate results have been obtained in the case of finite variance by examining the limiting behaviour of $Z_n a'$ for vectors a with $va' = 0$. For example, for some ρ_1 with $0 < \rho_1 < \rho$, which may be arbitrarily close to ρ, $Z_n a'/\rho_1^n$ may have a nondegenerate limit. From this it follows that (3.3) could not be strengthened to $Wv - Z_n/\rho^n = o(\rho^{-n/q})$ a.s. However, a weaker estimate of $Wv - Z_n/\rho^n$ may be obtained under conditions corresponding to those of Theorem 2:

THEOREM 4. *Let* $\alpha \geqq 0$. *Then*

$$\int_0^\infty x(\log^+ x)^{\alpha+1}\, dF_{i,j}(x) < \infty\,, \qquad i,j = 1, \cdots, k \tag{3.4}$$

implies that

$$Wv - Z_n/\rho^n = o(n^{-\alpha}) \quad \text{a.s.} \tag{3.5}$$

and the a.s. *convergence of* $\sum_{n=0}^\infty (Wv - Z_n/\rho^n)$ *for* $\alpha \geqq 1$.

PROOF OF THEOREM 3. Since

$$\begin{aligned} W - Z_n u'/\rho^n &= \sum_{m=n+1}^\infty Z_m u'/\rho^m - Z_{m-1}u'/\rho^{m-1} \\ &= \sum_{m=n+1}^\infty (Z_m - Z_{m-1}M)u'/\rho^m\,, \end{aligned}$$

(3.3) follows as in Section 1 once we have established the a.s. convergence of

$$\sum_{n=1}^\infty \rho^{n/q} \frac{(Z_n - Z_{n-1}M)}{\rho^n} = \sum_{n=1}^\infty \left(\frac{Z_n - Z_{n-1}M}{\rho^{n/p}}\right). \tag{3.6}$$

However, by (3.1) the jth component of the nth term of (3.6) equals

$$\sum_{i=1}^k \rho^{-n/p} \sum_{\nu=1}^{Z_{n-1}^i} (X_{n,\nu}^{i,j} - m_{i,j})$$

and the argument of the proof of Lemma 1 gives the convergence of each of the series

$$\sum_{n=1}^\infty \rho^{-n/p} \sum_{\nu=1}^{Z_{n-1}^i} (X_{n,\nu}^{i,j} - m_{i,j})$$

and thus of (3.6).

PROOF OF THEOREM 4. The a.s. convergence of

$$\sum_{n=1}^\infty n^\alpha (Z_n - Z_{n-1}M)$$

is obvious by combining the proofs of Theorems 2 and 3. Arguing as above, we get $W - Z_n u'/\rho^n = o(n^{-\alpha})$ and the a.s. convergence of

$$\sum_{n=0}^\infty (W - Z_n u'/\rho^n)$$

for $\alpha \geqq 1$. Since $\mathscr{R}^k$ is spanned by u and the vectors orthogonal to v, what we need to conclude the proof is thus (i) $Z_n a'/\rho^n = o(n^{-\alpha})$ a.s. for $va' = 0$, and, for the convergence of

$$\sum_{n=0}^\infty (Wv - Z_n/\rho^n)\,,$$

(ii) the a.s. convergence of

$$\sum_{n=0}^\infty Z_n a'/\rho^n$$

for $va' = 0$ and $\alpha \geqq 1$. Obviously both (i) and (ii) follow from

LEMMA 3. (3.4) *implies the* a.s. *convergence of* $\sum_{n=0}^\infty n^\alpha Z_n a'/\rho^n$ *for any* a *with* $va' = 0$.

The proof involves no essential new ideas. The details are, however, somewhat tedious and can be found in [1].

4. The continuous-time Markovian case. Let $\{Z_t;\, t \geqq 0\}$ be a Markov branching process with offspring distribution F and intensity β. Then in the supercritical case $\lambda = \beta \int_0^\infty (x - 1)\, dF(x) > 0$, $W = \lim_t W_t = \lim_t e^{-\lambda t} Z_t$ exists a.s.

THEOREM 5. *Let* $1 < p < 2$, $1/p + 1/q = 1$. *Then* $W - W_t = o(e^{-\lambda t/q})$ a.s., $P(W > 0) > 0$ *if and only if* $\int_0^\infty x^p\, dF(x) < \infty$.

PROOF. Since $\int_0^\infty x^p \, dF(x) < \infty$ if and only if $E(Z_1^p \mid Z_0 = 1)$, the "only if"-part is clear by applying Theorem 1 to $Z_0, Z_1, \cdots$. While the proof of the converse in discrete time was based upon martingale sequences of the form

$$\sum_{n=1}^N f(n)(W_n - W_{n-1}) , \qquad N = 1, 2, \cdots,$$

we shall here introduce a random measure dW_t with the property $\int_T^S dW_t = W_S - W_T$ for $T \leqq S$ and use integration with respect to dW_t to obtain suitable martingales. More precisely, we define dW_t in terms of the split times (cf. [3], Section III.9) $\tau_1, \tau_2, \cdots$ and the jumps $\xi_1, \xi_2, \cdots$ of the process. If $Z_0 = 1$, $S_n = Z_{\tau_n} = 1 + \xi_1 + \cdots + \xi_n$, then $Z_t = S_n$ for $\tau_n \leqq t < \tau_{n+1}$ and we let dW_t be the measure with atoms of weight $e^{-\lambda\tau_n}\xi_n$ at each τ_n and with density $-\lambda e^{-\lambda t}S_n$ with respect to Lebesgue measure dt on each of the intervals $[\tau_n, \tau_{n+1}]$. From the martingale property of W_t it is intuitively obvious and easily verified, that $(\int_0^T f(t)\, dW_t; \sigma(Z_t; t \leqq T))$ is a martingale for each f, which is measurable and bounded on finite intervals. Thus also the sequence $U_n = \int_0^{\tau_n} f(t)\, dW_t$ should be a martingale with respect to $\mathscr{F}_n = \sigma(\tau_1, \cdots, \tau_n; \xi_1, \cdots, \xi_n)$ and this follows from

$$\begin{aligned} E(&\textstyle\int_{\tau_n}^{\tau_{n+1}} f(t)\, dW_t \mid \mathscr{F}_n) \\ &= E(f(\tau_{n+1})e^{-\lambda\tau_{n+1}}\xi_{n+1} - \lambda S_n \textstyle\int_{\tau_n}^{\tau_{n+1}} f(t)e^{-\lambda t}\, dt \mid \mathscr{F}_n) \\ &= \textstyle\int_0^\infty f(\tau_n + t)e^{-\lambda(\tau_n+t)}\beta S_n e^{-\beta S_n}\, dt \cdot \lambda/\beta \\ &\quad - \lambda S_n \textstyle\int_0^\infty f(\tau_n + t)e^{-\lambda(\tau_n+t)}P(\tau_{n+1} - \tau_n > t \mid \mathscr{F}_n)\, dt = 0 \end{aligned}$$

since $\tau_{n+1} - \tau_n$ is exponential with rate βS_n.

If $f(t) = e^{\lambda t/q}$ and $E|\xi_1|^p < \infty$, the a.s. existence of $\lim_n U_n$ may be shown by truncating ξ_n at $n^{1/p}$; this step involves some standard calculations, which are similar to those of the proof of Lemma 1 and will be omitted. For $\tau_n \leqq T < \tau_{n+1}$, $|\int_0^T f(t)\, dW_t - U_n| = \lambda S_n \int_{\tau_n}^T f(t)e^{-\lambda t}\, dt \leqq \lambda S_n(\tau_{n+1} - \tau_n)e^{-\lambda\tau_n/p}$.

This last expression is easily seen to tend to zero as $n \to \infty$, using the Borel–Cantelli lemma and the facts that $e^{-\lambda\tau_n/p} = O(n^{-1/p})$ ([3], page 120) and that $S_1(\tau_2 - \tau_1), S_2(\tau_3 - \tau_2), \cdots$ are i.i.d. with exponential distribution. Thus the a.s. existence of $\lim_T \int_0^T f(t)\, dW_t$ follows by letting n tend to infinity. We can now conclude the proof by an argument similar to the one leading to part (i) of Lemma 2. Let $K_T = \sup_{s \geqq t \geqq T} |\int_t^s f(\tau)\, dW_\tau|$, so that $K_T \to 0$ a.s. for $T \to \infty$, and let $g(t) = d/dt(1/f(t))$. Since $g(t) < 0$, we get for $T \leqq S$

$$\begin{aligned} |W_S - W_T| = |\textstyle\int_T^S dW_t| &= \left|\textstyle\int_T^S \left(\dfrac{1}{f(S)} - \textstyle\int_t^S g(\tau)\, d\tau\right) f(t)\, dW_t\right| \\ &\leqq \frac{1}{f(S)} K_T + |\textstyle\int_T^S g(\tau)\, d\tau \int_T^\tau f(t)\, dW_t| \\ &\leqq K_T\left(\frac{1}{f(S)} - \textstyle\int_T^S g(\tau)\, d\tau\right) = K_T \frac{1}{f(T)} . \end{aligned}$$

Thus $|W - W_T| \leqq K_T/f(T) = o(1/f(T)) = o(e^{-\lambda T/q})$.

Acknowledgments. The author is indebted to Krishna B. Athreya, Søren Johansen and Niels Keiding for some valuable comments, from which the presentation of the paper has benefited very much. Also, one of the ideas in the proof of Theorem 5 grew out of discussions with Martin Jacobsen.

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