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ALMOST SURE CONVERGENCE OF THE QUADRATIC VARIATION OF MARTINGALES: A COUNTEREXAMPLE

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Let X_s be a continuous martingale and Q_ν be an increasing sequence of partitions of [0, 1]. Let

$$S^2(Q_\nu) = \sum_{t_i \in Q_\nu} (X_{t_i} - X_{t_{i-1}})^2 .$$

An example is given in which

$$\limsup_{\nu\to\infty} S^2(Q_\nu) = \infty .$$

1. Introduction. It is now a well-known fact that the quadratic variation of a right continuous martingale converges in measure. That is, if Q is a partition of [0, 1], then

$$S^2(Q) = \sum_{i=1}^n (X(s_i) - X(s_{i-1}))^2 \qquad s_i \in Q$$

converges in measure as max $(s_i - s_{i-1})$ goes to zero. For a review of this material see [2], [3] and [4]. In [2], Burkholder raises the following question. If Q_ν is an increasing sequence of partitions of [0, 1], what can be said about $\sup_\nu S^2(Q_\nu)$? In this paper an example of a right continuous martingale is given which has the property that

$$\limsup_\nu S^2(Q_\nu) = \infty \tag{1.1}$$

for a specified increasing sequence of partitions Q_ν. Thus, any results along the line of $S^2(Q_\nu)$ converging almost everywhere or $||\sup_\nu S^2(Q_\nu)||_p \leqq c_p||X^*||_p$ must make at least some demands on the martingale.

In fact, the example shows somewhat more than that. In [6], Taylor shows that the "correct" function for measuring the pathwise variation of Brownian motion is

$$\psi_1(s) = s^2/2 \log^* \log^* s$$

($\log^* s = \max\{1, |\log s|\}$) in the sense that $\sup_Q \sum \psi_1(W_{s_i} - W_{s_{i-1}})$ is a finite random variable but will not be finite if ψ_1 is replaced by any function ψ such that $\psi(s)/\psi_1(s) \to \infty$ as $s \to 0$. The martingale X and the increasing sequence of partitions Q_ν exhibited here have the property that

$$\limsup_\nu \sum_{Q_\nu} \psi_1(X_{s_i} - X_{s_{i-1}}) = 1 . \tag{1.2}$$

The paths of the martingale are continuous so as the mesh of the partitions Q_ν goes to zero, $|X_{s_i} - X_{s_{i-1}}| \to 0$. Since $\psi_1(s)/s^2 \to 0$ as $s \to 0$, (1.2) implies (1.1). Thus, it is the result (1.2) that will be proved.

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It should be noted that for any right continuous martingale with left-hand limits

$$\limsup_\nu \textstyle\sum_{Q_\nu} \psi_1(X_{s_i} - X_{s_{i-1}}) < \infty .$$

This follows from Taylor's result above and the fact that every such martingale is a time change of Brownian motion. See [5]. Thus in some sense, the martingale exhibited here is about as bad as a martingale can be.

2. Preliminaries. Let

$$\psi_1(s) = s^2/2 \log^* \log^* s \tag{2.1}$$

where $\log^* s = \max\{1, |\log s|\}$. Let Q_ν be an increasing sequence of partitions of $[0, 1]$, consisting of $s_{\nu,i}$, $0 \leqq i \leqq m_\nu$ where $0 = s_{\nu,0} < s_{\nu,1} < \cdots < s_{\nu,m} = 1$.

(2.2) THEOREM. *There exists a continuous martingale $(X_s, \mathscr{G}_s)$ and an increasing sequence of partitions, Q_ν, such that almost surely*

$$\limsup_\nu \textstyle\sum_{i=1}^{m_\nu} \psi_1(X_{s_{\nu,i}} - X_{s_{\nu,i-1}}) = 1 .$$

The martingale X_s constructed here is a time change of Brownian motion. The Brownian motion process will be denoted by $(W_t, \mathscr{F}_t)$. A continuous family of $\mathscr{F}_t$ stopping times T_s will be defined and X_s will be $W(T_s)$.

The two processes W_t and $X_s = W(T_s)$ will invite some confusion when discussing partitions of $[0, 1]$ so the following convention will be followed: partitions of the time parameter of W_t will be denoted by P and partitions of the time parameter of X_s will be denoted by Q. Thus if $Q = \{s_0, s_1, \cdots, s_n\}$ is a partition of the time parameter of X_s, then $P = \{T_{s_0}, T_{s_1}, \cdots, T_{s_n}\}$ is a partition of the time parameter of W_t. However, P is a random variable.

Let $\mathscr{P}$ be the set of partitions P such that:

(a) The only points in P are the form $k2^{-n}$ (n not fixed).
(b) If $t \in P$ and $k2^{-n} < t < (k + 1)2^{-n}$, then $k2^{-n} \in P$ and $(k + 1)2^{-n} \in P$ also.

All partitions of the time parameter of W_t will be in $\mathscr{P}$.

The goal is to select P such that it partitions $[0, 1]$ into intervals $[t_i, t_{i+1}]$ for which

$$\psi_1(W_{t_i} - W_{t_{i-1}}) > (1 - \varepsilon)(t_i - t_{i-1}) .$$

In fact, one wants an infinite sequence of such partitions P_ν, with $\varepsilon \to 0$ as $\nu \to \infty$. This will be done, almost, in Lemma (3.2). For this purpose, it is convenient to introduce some notation.

Let N_l be an increasing sequence of positive integers. Define $\mathscr{A}_l$ to be the intervals of the form $[k2^{-n}, (k + 1)2^{-n}]$ such that $n > N_l$, $k < 2^n$, and

$$\psi_1(W_{k2^{-n}} - W_{(k+1)2^{-n}}) > (1 - N_l^{-1})2^{-n} .$$

Let $\mathscr{A}_l^*$ be the intervals of the form $[k2^{-n}, (k + 1)2^{-n}]$ which are

(a) either in $\mathscr{A}_l$ or of length 2^{-N_l+1}, and
(b) not properly contained in any interval of the type described in (a).

Note that the intervals in $\mathscr{A}_l^*$ are disjoint and cover [0, 1]. The endpoints of the intervals in $\mathscr{A}_l^*$ therefore form a partition of [0, 1] which in turn defines $\mathscr{A}_l^*$. Thus it simplifies matters to consider $\mathscr{A}_l^*$ at once to be both a cover of [0, 1], and a partition of [0, 1].

For $N_l < n \leqq N_{l+1}$ and $k \leqq 2^n$, let

$$\begin{aligned} I_{k,n} &= 1 \quad \text{if} \quad [k2^{-n}, (k+1)2^{-n}] \Cap \mathscr{A}_l \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

Finally for each $t = j2^{-m}$ with j odd, $j \leqq 2^m$, $N_l < m \leqq N_{l+1}$, let $\mathscr{F}_{j,m}$ be the σ-field generated by the functions $I_{k,n}$, $n \leqq N_{l+1}$ and $(k+1)2^{-n} \leqq (j-1)2^{-m}$. Let $\mathscr{F}_l$ be the σ-field generated by all functions $I_{k,n}$, $n \leqq N_{l+1}$.

We will later use the fact that if k is odd, $k < 2^n$, then $\mathscr{F}_{k,n} \subset \mathscr{F}_{2k-1,n+1}$.

Observe that if F is an atom of $\mathscr{F}_l$, then $\mathscr{A}_l^*(\omega)$ is the same for all $\omega \in F$. Thus one can write without ambiguity $\mathscr{A}_l^*(F)$. In the same way, $\mathscr{A}_l^* \cap [0, (j-1)2^{-m}]$ depends only on the σ-field $\mathscr{F}_{j,m}$, so, if F is an atom of $\mathscr{F}_{j,m}$, one can write $\mathscr{A}_l^*(F) \cap [0, (j-1)2^{-m}]$.

3. Lemmas. The following simple variant of the law of the iterated logarithm is needed.

(3.1) LEMMA. *If $(W_t, \mathscr{F}_t)$ is a Brownian motion process, $t_0 \in (0, 1)$, and $\varepsilon > 0$, then almost surely there are infinitely many n such that*

$$\phi_1(W_{k2^{-n}} - W_{(k+1)2^{-n}}) > (1-\varepsilon)2^{-n}$$

where $k2^{-n} \leqq t_0 < (k+1)2^{-n}$.

PROOF. Choose a sequence $n_m \to \infty$ such that $n_m/n_{m+1} \to 0$. Choose k_m such that $k_m 2^{-n_m} \leqq t_0 < (k_m+1)2^{-n_m}$, and, for convenience, denote $k_m 2^{-n_m}$ by t_m and $(k_m+1)2^{-n_m}$ by t_m'. Let

$$\begin{aligned} Z_m &= W_{t_m'} - W_{t'_{m+1}} + W_{t_{m+1}} - W_{t_m} \\ &= (W_{t_m'} - W_{t_m}) - (W_{t'_{m+1}} - W_{t_{m+1}}) . \end{aligned}$$

The random variables Z_m are independent and the usual Borel–Cantelli argument shows that for infinitely many m

$$|Z_m| > (1-\varepsilon/2)(2^{-n_m+1} \log^* \log^* 2^{n_m})^{\frac{1}{2}} .$$

See for instance [1], page 264. Moreover the law of the iterated logarithm says that for n_m sufficiently large

$$|W_{t'_{m+1}} - W_{t_{m+1}}| < 2(1+\varepsilon)(2 \cdot 2^{-n_{m+1}} \log^* \log^* 2^{n_{m+1}})^{\frac{1}{2}} .$$

Since $n_m/n_{m+1} \to 0$, it follows that

$$|W_{t_m'} - W_{t_m}| > |Z_m| - |W_{t'_{m+1}} - W_{t_{m+1}}| > (1-\varepsilon)(2^{-n_m+1} \log^* \log^* 2^{n_m})^{\frac{1}{2}}$$

for infinitely many m. Since

$$\phi_1((2t \log^* \log^* t)^{\frac{1}{2}}) \sim t \qquad \text{as} \quad t \to \infty ,$$

Lemma (3.1) follows.

Lemma (3.1) allows us to modify an argument given by Taylor [6].

(3.2) LEMMA. *The sequence* N_l *can be chosen so that*

$$P(\sum_{t_i \in \mathscr{A}_l^*} \psi_1(W_{t_i} - W_{t_{i-1}}) > 1 - N_l^{-1}) > 1 - N_l^{-1}.$$

PROOF. By induction, suppose that N_l has been selected. Lemma (3.1) and Fubini's theorem say that almost surely, $\mathscr{A}_l$ is a Vitali cover of [0, 1]. Thus if $\mathscr{C}$ denotes the intervals in $\mathscr{A}_l$ which are not contained in any other interval in $\mathscr{A}_l$, then almost surely

$$\sum_{[a,b] \in \mathscr{C}} \psi_1(W_a - W_b) > 1 - N_l^{-1}.$$

It is thus a simple matter to select N_{l+1} so large that

$$P(\sum_{t_i \in \mathscr{A}_l^*} \psi_1(W_{t_i} - W_{t_{i-1}}) > 1 - N_l^{-1}) > 1 - N_l^{-1}$$

which proves (3.2).

4. Definition of stopping times. In order to define the process $X_s = W(T_s)$ it is only necessary to define the stopping times T_s.

(4.1) LEMMA. *There is a countable set* $E \subset [0, 1]$ *and a family of stopping times* T_s, $s \in E$, *such that* T_s *is nondecreasing in s and*

(a) *The set* $E = \bigcup_{n=0}^{\infty} E_n$ *where* E_n *is an increasing sequence of finite sets. For each* $n \neq 0$, E_n *is the union of disjoint sets* E_n' *and* E_n'' *and* $E_{n-1} \subset E_n''$. *If the elements of* E_n *are arranged in increasing order, the elements of* E_n' *and* E_n'' *alternate. That is, an element of* E_n' *is preceded and followed by an element of* E_n''. *If* $s \in E_n'$, *the preceding element (in* E_n''*) is denoted by* $\rho(s)$ *and the following element (in* E_n''*) is denoted by* $\tau(s)$.

(b) *As s ranges over* E_n, T_s *takes on every value* $k2^{-n}$, $0 \leqq k \leqq 2^n$. *Thus if* s_1 *and* s_2 *are successive elements in* E_n, $T_{s_2} - T_{s_1}$ *is either* 2^{-n} *or zero.*

(c) *For each* $s \in E_n'$, *there is a unique odd* $k < 2^n$ *denoted by* k_s. *The set* H_s *defined by*

$$H_s = \{T_{\tau(s)} \neq T_{\rho(s)}\} = \{T_{\rho(s)} = (k_s - 1)2^{-n}, T_s = k_s 2^{-n}, T_{\tau(s)} = (k_s + 1)2^{-n}\}$$

is an atom of $\mathscr{F}_{k_s,n}$. *That is, if* $N_l < n \leqq N_{l+1}$, *the functions* $I_{j,m}$, $(j + 1)2^{-m} \leqq (k_s - 1)2^{-n}$ *and* $m \leqq N_{l+1}$ *are constant on* H_s.

(d) *For* $s \in E_n$, *let* $s' = \max\{v \in E_{n-1}, v < s\}$ *and* $s'' = \min\{v \in E_{n-1}; v > s\}$. *If* $s \in E_n'$, *either* $T_{\rho(s)} = T_{s'}$ *or* $T_{\tau(s)} = T_{s''}$, *and, on* H_s *both are true. For* $s \in E_n''\backslash E_{n-1}$, *either* $T_s = T_{s'}$ *or* $T_s = T_{s''}$.

Such a class of sets and stopping times T_s, $s \in E_n$, is easily defined by induction. One starts with $E_1 = \{0, \frac{1}{2}, 1\}$ and $T_0 = 0$, $T_{\frac{1}{2}} = \frac{1}{2}$ and $T_1 = 1$. If E_{n-1} and T_s have been defined for $s \in E_{n-1}$, one proceeds as follows. Let s' and s'' be any two successive elements in E_{n-1}. By (c) and (b), there is an odd $k < 2^n$ such that on the set $\{T_{s'} \neq T_{s''}\}$, $T_{s'} = (k - 1)2^{-n}$ and $T_{s''} = (k + 1)2^{-n}$. Now $\{T_{s'} \neq T_{s''}\}$ is in $\mathscr{F}_{(k+1)/2,n-1}$ if $(k + 1)/2$ is odd and $\{T_{s'} \neq T_{s''}\}$ is in $\mathscr{F}_{(k-1)/2,n-1}$ if $(k - 1)/2$ is odd. In either case we have $\{T_{s'} \neq T_{s''}\}$ is in $\mathscr{F}_{k,n}$. Write

$\{T_{s'} \neq T_{s''}\} = \bigcup_{i=1}^{m} \Lambda_i$ where the sets Λ_i are disjoint atoms in $\mathscr{F}_{k,n}$. Select $2m - 1$ points in the interval (s', s''), say $s_1 < s_2 < \cdots < s_{2m-1}$.

For $\omega \in \Lambda_l$, $T_{s_i} = (k-1)2^{-n}$ if $i < 2l - 1$, $T_{s_i} = (k+1)2^{-n}$ if $i > 2l - 1$ and $T_{si} = k2^{-n}$ if $i = 2l - 1$. Then for $\omega \in \Lambda_l$, T_s will increase only on the interval (s_{2l-2}, s_{2l}). The points $s_1, s_3, \cdots, s_{2m-1}$ are placed in E_n' and the points $s_2, s_4, \cdots, s_{2m-2}$ are placed in E_n'' as are all the points in E_{n-1}. The properties (a), (b), (c) and (d) are easily verified.

This defines the stopping times T_s for all $s \in E$. For any other $s \in [0, 1]$, let $T_s = \inf\{T_v : v \in E, v > s\}$. Since T_s increases in increments of 2^{-n} as s ranges over E_n, the stopping times thus defined are continuous. Thus the martingale X_t is defined.

5. Definition of partitions. The partitions Q_ν are subsets of E. In fact there is an increasing sequence ν_l such that $Q_{\nu_l} = E_{N_l}$. Starting with Q_{ν_l}, the elements of $\bigcup_{n=N_l+1}^{N_{l+1}} E_n$ will be added in an order that will be described shortly.

Toward the goal of defining Q_ν, order $\mathscr{P}$ as follows. If $P_1, P_2 \in \mathscr{P}$,

$$P_1 \prec P_2 \qquad \text{iff} \quad \min((P_1 \backslash P_2) \cup (P_2 \backslash P_1)) \in P_2 .$$

Let $s \in E_n'$, $N_l < n \leqq N_{l+1}$. Consider the set H_s. On H_s, $T_s = k_s 2^{-n}$ and the partitions $\mathscr{A}_{N_l}^* \cap [0, (k_s - 1)2^{-n}]$ are constant, as has been noted. Define

$$P_s = \bigcap \{P \in \mathscr{P}; P \cap [0, (k_s - 1)2^{-n}] = \mathscr{A}_l^*(H_s) \cap [0, (k_s - 1)2^{-n}] \text{ and } k_s 2^{-n} \in P\} .$$

For $s \in E_n'' \backslash E_{n-1}$ define (inductively)

$$P_s = P_{s'} \cup P_{s''}$$

where $s' = \max\{v \in E_{n-1}; v < s\}$ and $s'' = \min\{v \in E_{n-1}; v > s\}$.

Order $s \in \bigcup_{n=N_l+1}^{N_{l+1}} E_n$ by

$$s_1 \prec s_2 \qquad \text{if } P_{s_1} \prec P_{s_2} \text{ or if } P_{s_1} = P_{s_2} \text{ and } s_1 < s_2 .$$

This is the desired order in which the partitions will be augmented.

(5.1) LEMMA. *Let F be any atom of the σ-field $\mathscr{F}_l$. Let*

$$Q_F = \{s \in E; P_s \preceq \mathscr{A}_l^*(F)\} .$$

Then

$$\{T_s(\omega); s \in Q_F, \omega \in F\} = \mathscr{A}_l^*(F) .$$

(Here, $P_1 \preceq P_2$ means either $P_1 \prec P_2$ or $P_1 = P_2$.)

PROOF. To show that $\mathscr{A}_l^*(F) \subset \{T_s(\omega); s \in Q_F, \omega \in F\}$, let $k2^{-n} \in \mathscr{A}_l^*(F)$. We can assume that k is odd. Then for some $s \in E_n$ with $n \leqq N_{l+1}$, $k = k_s$ and $H_s \supset F$. It follows easily that $P_s \subset \mathscr{A}_l^*(F)$ so $P_s \preceq \mathscr{A}_l^*(F)$. That is, $s \in Q_F$ and on $F \subset H_s$, $T_s = k_s 2^{-n} = k2^{-n}$.

To show that $\{T_s(\omega); s \in Q_F, \omega \in F\} \subset \mathscr{A}_l^*(F)$, suppose the contrary. Choose the smallest n such that for some $k < 2^n$, $k2^{-n} \notin \mathscr{A}_l^*(F)$ but $T_s(\omega) = k2^{-n}$ for some $s \in Q_F$ and $\omega \in F$.

Then either $s \in E_n'$ or $s \in E_n''\backslash E_{n-1}$. But if $s \in E_n''\backslash E_{n-1}$, by (d) of (4.1) either $T_s = T_{s'}$ or $T_s = T_{s''}$ where $s' = \max\{v \in E_{n-i}; v < s\}$ and $s'' = \min\{v \in E_{n-1}; v > s\}$ which contradicts the choice of n, since $T_{s'}$ and $T_{s''}$ take on values of the form $j2^{-(n-1)}$. Thus $s \in E_n'$. Moreover, $F \subset H_s$. Otherwise, $F \cap H_s = \emptyset$, since F is an atom, and $T_{\tau(s)} = T_{\rho(s)}$ on F. But again by (d) of (4.1) either $T_s = T_{s'}$ or $T_s = T_{s''}$ contradicting the choice of n. Thus $F \subset H_s$.

Now it follows that $k = k_s$ and

$$\mathscr{A}_l^*(F) \cap [0, (k-1)2^{-n}] = \mathscr{A}_l^*(H_s) \cap [0, (k-1)2^{-n}]$$

so

$$P_s \cap [0, (k-1)2^{-n}] = \mathscr{A}_l^*(F) \cap [0, (k-1)2^{-n}] .$$

But $k2^{-n} = k_s 2^{-n} \in P_s$ and $P_s \preceq \mathscr{A}_l^*(F)$. Thus there is a $t \in \mathscr{A}_l^*(F)$ such that $(k-1)2^{-n} < t < k2^{-n}$. But since $\mathscr{A}_l^*(F) \in \mathscr{P}$, this means that $k2^{-n} \in \mathscr{A}_l^*(F)$ which contradicts the original assumption on n. Thus $\{T_s; s \in Q_F, \omega \in F\} \subset \mathscr{A}_l^*(F)$.

The proof of Theorem 2.2 is now immediate. For any atom F in $\mathscr{F}_l$, let $s_0 = \sup_< Q_F$ and observe that

$$\{s; s \preceq s_0\} = Q_F .$$

Thus $Q_F = Q_\nu$ for some ν and Lemma 5.1 says that

$$\sum_{i=1}^{m_\nu} \phi_1(X(s_{\nu,i}) - X(s_{\nu,i-1})) = \sum_{t_i \in \mathscr{A}_l^*} \phi_1(W(t_i) - W(t_{i-1})) .$$

Lemma 3.2 completes the proof.

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