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AN INVARIANCE PRINCIPLE FOR RANDOM WALK CONDITIONED BY A LATE RETURN TO ZERO

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Let $\{S_n : n \geqq 0\}$ denote the recurrent random walk formed by the partial sums of i.i.d. integer-valued random variables with zero mean and finite variance. Let $T = \min\{n \geqq 1 : S_n = 0\}$. Our main result is an invariance principle for the random walk conditioned by the event $[T = n]$. The limiting process is identified as a Brownian excursion on [0, 1].

1. Introduction. Let $X_1, X_2, \cdots$ be a sequence of i.i.d. random variables defined on a probability space $(\Omega, \mathscr{F}, P)$. In all that follows we assume the X_i are integer-valued with span 1 and $EX_i = 0$, $EX_i^2 = \sigma^2 < \infty$. Form the random walk $\{S_n : n \geqq 0\}$ by defining $S_0 = 0$ and $S_n = X_1 + \cdots + X_n$, $n \geqq 1$. Our main result describes the chance behavior of the random walk conditioned by a "late" return to the origin.

More precisely we define the hitting time T to be $\min\{n \geqq 1 : S_n = 0\}$ $(+\infty$ if no such n exists) and investigate the limiting behavior of the sequence $\{S_n\}$ conditioned by the event $[T = n]$. To provide additional motivation and background we discuss other related works.

Let $D \equiv D[0, 1]$ be the function space of all real-valued, right-continuous functions on $[0, 1]$ with left-hand limits and denote by $\mathscr{D}$ the sigma field of Borel subsets generated by the open sets of the Skorohod J_1-topology on D. Introduce another hitting time $T^+ = \min\{n \geqq 1 : S_n \leqq 0\}$ and consider the four sequences of probability measures $\{P_n\}$, $\{P_n^+\}$, $\{P_n^*\}$, $\{P_n^\circ\}$ defined on $(D, \mathscr{D})$ by assigning probability

$$\begin{aligned} &P_n : P[S_1/\sigma n^{\frac{1}{2}} = x_1, \cdots, S_n/\sigma n^{\frac{1}{2}} = x_n] \\ (1.1)\qquad &P_n^+ : P[S_1/\sigma n^{\frac{1}{2}} = x_1, \cdots, S_n/\sigma n^{\frac{1}{2}} = x_n \mid T^+ > n] \\ &P_n^* : P[S_1/\sigma n^{\frac{1}{2}} = x_1, \cdots, S_n/\sigma n^{\frac{1}{2}} = x_n \mid T > n] \\ &P_n^\circ : P[S_1/\sigma n^{\frac{1}{2}} = x_1, \cdots, S_n/\sigma n^{\frac{1}{2}} = x_n \mid T = n] \end{aligned}$$

to the random function $S_{[nt]}/\sigma n^{\frac{1}{2}}$, $0 \leqq t \leqq 1$. Denoting weak convergence of probability measures on $(D, \mathscr{D})$ by $\Rightarrow$, we discuss the weak limits of each of the above sequences under our assumptions.

The well-known theorem of Donsker gives $P_n \Rightarrow W$ where W is the probability measure corresponding to Brownian motion on [0, 1]. It follows from a theorem due to Iglehart (1974) that $P_n^+ \Rightarrow W^+$ under the additional assumption $E|X_i|^3 < \infty$, where W^+ corresponds to the process which he has entitled Brownian meander.

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The introduction to Brownian meander occurs apparently in Belkin (1972) where it is shown that $P_n^* \Rightarrow W^*$ and $W^+ = |W^*|$.

The main theorem of this paper was announced in Kaigh (1974) and is the result that $P_n^\circ \Rightarrow W^\circ$ where W° is the process entitled Brownian excursion (signed) by Itô and McKean (1965). We remark that Brownian excursion as well as each of the previously mentioned weak limits assigns probability one to continuous sample paths.

As for the other sequences the weak convergence of $\{P_n^\circ\}$ is attacked through the approach discussed in Chapters 3 and 4 of Billingsley (1968) which requires a demonstration of the appropriate convergence of the finite-dimensional distributions and a verification of tightness. It will be seen that the finite-dimensional distributions can be treated with relative ease, but that tightness requires considerable effort.

2. Preliminaries and a statement of the main result. A study of the conditioned random walk must focus initially on the behavior of the hitting time. From our assumptions T is finite a.s. and for $r_n = P[T > n]$ and $f_n = P[T = n]$

$$\lim_{n\to\infty} n^{\frac{1}{2}} r_n = (2/\pi)^{\frac{1}{2}}\sigma \; ; \tag{2.1}$$

$$\lim_{n\to\infty} n^{\frac{3}{2}} f_n = \sigma/(2\pi)^{\frac{1}{2}} \, . \tag{2.2}$$

The first expression is well-known and the second follows from a result of Kesten (1963).

Next we record a result of Belkin as

$$\lim_{n\to\infty} P[S_n/n^{\frac{1}{2}} \leqq x \,|\, T > n] = \int_{-\infty}^{x} (|y|/2\sigma^2) e^{-y^2/2\sigma^2}\, dy \, . \tag{2.3}$$

The proof of (2.3) appears in [1] and a statement is contained also in [2]. It is instructive to view (2.3) as a conditional central limit theorem.

It will be seen that the key to both tightness and convergence of the finite-dimensional distributions of $\{P_n^\circ\}$ is the following result contained in Kaigh (1975):

$$\lim_{n\to\infty} \sup_x |nP[T_{\{x\}} = n] - (|x|/\sigma n^{\frac{1}{2}})\phi(x/\sigma n^{\frac{1}{2}})| = 0 \, , \tag{2.4}$$

where $T_{\{x\}} = \min\{n \geqq 1 \,|\, S_n = x\}$ and $\phi(t)$ is the standard normal probability density function. The above expression actually is equivalent to a local version of (2.3). (See [7] for a more thorough discussion.)

Now we discuss briefly the Brownian excursion process W°. Itô and McKean (1965) provide two alternative derivations (we prove its existence as the weak limit of conditioned random walk) and show that W° is a Markov process on [0, 1] with nonstationary transition density p given by

$$p(0, 0; t, y) = [2\pi t^3(1-t)^3]^{-\frac{1}{2}} y^2 \exp[-y^2/2t(1-t)] \quad \text{for } 0 < t < 1 \text{ and } -\infty < y < \infty \; ;$$

$$p(t_1, y_1; t_2, y_2) = \frac{(1-t_1)^{\frac{3}{2}}}{(1-t_2)^{\frac{3}{2}}} \frac{y_2}{y_1} \frac{\phi(y_2/(1-t_2)^{\frac{1}{2}})}{\phi(y_1/(1-t_1)^{\frac{1}{2}})} \tag{2.5}$$

$$\times (t_2 - t_1)^{-\frac{1}{2}}\left[\phi\left(\frac{y_1 - y_2}{(t_2 - t_1)^{\frac{1}{2}}}\right) - \phi\left(\frac{y_1 + y_2}{(t_2 - t_1)^{\frac{1}{2}}}\right)\right]$$
$$\text{for } 0 < t_1 < t_2 < 1 \text{ and } y_1 y_2 > 0 .$$

The distributions of W° for $t = 0$ or 1 are degenerate at zero.

Finally we state the main result. The proof is deferred until the next section.

THEOREM 2.6. *The sequence of probability measures* $\{P_n{}^\circ\}$ *on* $(D, \mathscr{D})$ *defined by* (1.1) *converges weakly to a probability measure* W° *which assigns probability one to continuous sample paths. The weak limit* W° *corresponds to the Brownian excursion process on* $[0, 1]$.

3. Proof of Theorem 2.6. The proof will follow from Theorems 15.1 and 15.5 of Billingsley (1968). It is necessary to demonstrate convergence of the finite-dimensional distributions and tightness for $\{P_n{}^\circ\}$.

We consider first the convergence of the finite-dimensional distributions. It must be shown that for any finite collection $0 < t_1 < \cdots < t_k < 1$

$$\begin{aligned}\lim_{n\to\infty} P_n{}^\circ[\xi : \xi(t_1) \leqq x_1, \cdots, \xi(t_k) \leqq x_k] \\ \equiv \lim_{n\to\infty} P[S_{[nt_1]}/\sigma n^{\frac{1}{2}} \leqq x_1, \cdots, S_{[nt_k]}/\sigma n^{\frac{1}{2}} \leqq x_k \mid T = n] \\ = W^\circ[\xi : \xi(t_1) \leqq x_1, \cdots, \xi(t_k) \leqq x_k] \qquad \text{for all } x_1, \cdots, x_k \in \mathbb{R} .\end{aligned}$$

The following facts are required to obtain the desired conclusion:

$$\lim_{n\to\infty} \int_{-\infty}^{x} r_n{}^{-1} P[S_{[nt]}/\sigma n^{\frac{1}{2}} \in dy;\, T > [nt]] = \int_{-\infty}^{x} \tfrac{1}{2} t^{-\frac{3}{2}} |y| e^{-y^2/2t}\, dy ; \tag{3.1}$$

$$\begin{aligned}\lim_{n\to\infty} \int_{-\infty}^{x} P^{\sigma n^{\frac{1}{2}} y_1}[S_{[nt]}/\sigma n^{\frac{1}{2}} \in dy_2;\, T > [nt]] \\ = \int_{-\infty}^{x} t^{-\frac{1}{2}} \left[\phi\left(\frac{y_1 - y_2}{t^{\frac{1}{2}}}\right) - \phi\left(\frac{y_1 + y_2}{t^{\frac{1}{2}}}\right)\right] I_{\{y_1 y_2 > 0\}}\, dy_2\end{aligned} \tag{3.2}$$

uniformly on compact sets (as a function of y_1);

$$\lim\, (r_n/f_n) P^{\sigma n^{\frac{1}{2}} y}[T = n - [nt]] = 2(1 - t)^{-\frac{3}{2}} |y| \sigma(y/(1 - t)^{\frac{1}{2}}) \tag{3.3}$$

uniformly in y. Both (3.1) and (3.2) were used by Belkin to prove $P_n{}^* \Rightarrow W^*$. A combination of (2.1) with (2.3) will yield (3.1) and a discussion of (3.2) appears on pages 53–55 and 61 of Belkin (1972). An application of (2.4) provides (3.3). It is of interest that (3.3) was not required in the proof that $P_n{}^* \Rightarrow W^*$.

Using (3.1), (3.2), and (3.3) and an induction similar to Belkin's (1972, pages 54 ff) we obtain convergence of the finite-dimensional distributions. Since the techniques required are very similar, we refer to either Belkin (1972) or Iglehart (1974) and omit the details.

Next we consider the tightness of the sequence $\{P_n{}^\circ\}$. The techniques again are similar to those required in showing tightness of $\{P_n{}^*\}$; however, the change of the conditioning events entails a significant increase in effort.

Our goal is to apply Theorem 15.5 of Billingsley (1968). For $x \in D$ and $T \subseteq {}_0[0, 1]$ set

$$W_x(T_0) = \sup\{|x(s) - x(t)| : s, t \in T_0\}$$

and define the modulus of continuity of x by

$$W_x(\rho) = \sup_{0\le t\le 1-\rho} W_x[t, t+p] \qquad \text{for} \quad 0<\rho<1 .$$

Then it suffices to show that for every $\varepsilon > 0$

$$\lim_{\rho\downarrow 0} \limsup_{n\to\infty} P_n{}^\circ[x: W_x(\rho) \geqq 2\varepsilon] = 0 . \tag{3.4}$$

To verify (3.4) we require two preliminary results.

LEMMA 3.5. *For fixed $\varepsilon > 0$, there corresponds to each $\delta \in (0, 1)$ a positive integer $n_0 = n_0(\delta, \varepsilon)$ such that*

$$P^{n^{\frac12}x}[T = n-k] \leqq 4(2\pi)^{\frac12}(\sigma/\varepsilon)e^{\varepsilon^2/2\sigma^2(1-\delta)}P^{n^{\frac12}\varepsilon}[T = n-[n\delta]]$$
$$\equiv c(\delta, \varepsilon)P^{n^{\frac12}\varepsilon}[T = n-[n\delta]]$$

for all x and all $k \leqq [n\delta]$ whenever $n > n_0$.

PROOF. By (2.4) there exists N_1 such that

$$\sup_x |nP^{n^{\frac12}x}[T=n] - (|x|/\sigma)\phi(x/\sigma)| < 1$$

whenever $n > N_1$. Now choose n_1 so that $n - [n\delta] > N_1$ whenever $n > n_1$. Then for every $k \leqq [n\delta]$

$$\begin{aligned}\sup_x (n-k)P^{n^{\frac12}x}[T=n-k] &\leqq \sup_x |(n-k)P^{(n-k)^{\frac12}n^{\frac12}x/(n-k)^{\frac12}}[T=n-k] \\ &\quad - (n^{\frac12}|x|/\sigma(n-k)^{\frac12})\phi(n^{\frac12}x/\sigma(n-k)^{\frac12})| \\ &\quad + \sup_x (n^{\frac12}|x|/\sigma(n-k)^{\frac12})\phi(n^{\frac12}x/\sigma(n-k)^{\frac12}) \\ &\leqq 1 + \sup_y |y|\phi(y) \\ &= 1 + \phi(1) \leqq 2 \qquad \text{whenever} \quad n > n_1 .\end{aligned}$$

Again from (3.3)

$$\lim_{n\to\infty} (n-[n\delta])P^{n^{\frac12}\varepsilon}[T=n-[n\delta]] = (\varepsilon/\sigma)(1-\delta)^{-\frac12}\phi(\varepsilon/\sigma(1-\delta)^{\frac12})$$

so there exists n_2 such that

$$(n-[n\delta])P^{n^{\frac12}\varepsilon}[T=n-[n\delta]] \geqq (\varepsilon/2\sigma)(1-\delta)^{-\frac12}\phi(\varepsilon/\sigma(1-\delta)^{\frac12})$$

whenever $n > n_2$.

Now take $n_0 = \max\{n_1, n_2\}$. Then for every $k \leqq [n\delta]$

$$\frac{\sup_x P^{n^{\frac12}x}[T=n-k]}{P^{n^{\frac12}\varepsilon}[T=n-[n\delta]]}$$
$$\leqq \frac{n-[n\delta]}{n-k}\, 2[(\varepsilon/2\sigma)(1-\delta)^{-\frac12}\phi(\varepsilon/\sigma(1-\delta)^{\frac12})]^{-1} \leqq c(\delta, \varepsilon) .$$

In addition we require a result contained in the fourth formula on page 50 of [2].

LEMMA 3.6. *For fixed $\varepsilon > 0$, there corresponds to each $\delta \in (0, 1)$ a number $\lambda(\delta, \varepsilon) > 0$ and a positive integer $m_0 = m_0(\delta, \varepsilon)$ such that*

$$P^{n^{\frac12}x}[S_{[n\delta]-k}/n^{\frac12} \geqq x;\ T > [n\delta]-k] \geqq \lambda(\delta, \varepsilon)$$

for all $x \geqq \varepsilon$ whenever $[n\delta]-k \geqq m_0$. Also for each $\varepsilon > 0$ $\lim_{\delta\downarrow 0} \lambda(\delta, \varepsilon) > 0$.

We can proceed now to verify (3.4). Following the approach of Belkin (1972) define the random time

$$\tau = \tau(\rho, \varepsilon) = \inf\{t: \sup_{t \leqq s \leqq t+\rho} |x(s) - x(t)| \geqq 2\varepsilon\}$$

where the infimum of the empty set is $+\infty$.

Then for any $\delta \in (0, \frac{1}{2})$

$$P_n{}^\circ[W_x(\rho) \geqq 2\varepsilon] = P_n{}^\circ[\tau \leqq \delta] + P_n{}^\circ[\delta < \tau < 1 - \delta] + P_n{}^\circ[1 - \delta \leqq \tau \leqq 1],$$

and it suffices to show that each of the terms

$$\begin{aligned} &\limsup_{\rho \downarrow 0} \limsup_{n\to\infty} P_n{}^\circ[\tau \leqq \delta] \\ (3.7) \qquad &\limsup_{\rho \downarrow 0} \limsup_{n\to\infty} P_n{}^\circ[\delta < \tau < 1 - \delta] \\ &\limsup_{\rho \downarrow 0} \limsup_{n\to\infty} P_n{}^\circ[1 - \delta \leqq \tau \leqq 1] \end{aligned}$$

approaches zero as $\delta \downarrow 0$.

Initially we consider $P_n{}^\circ[\delta < \tau < 1 - \delta]$.

$$\begin{aligned} &P_n{}^\circ[\delta < \tau < 1 - \delta] \\ &\quad = \textstyle\sum_{x \neq 0} f_n{}^{-1} P[\delta < \tau < 1 - \delta;\, S_{n-[n\delta]}/n^{\frac{1}{2}} = x;\, T > n - [n\delta]] P^{n^{\frac{1}{2}}x}[T = [n\delta]]\,. \end{aligned}$$

From Lemma 3.5 we obtain for large n

$$\sup_x P^{n^{\frac{1}{2}}x}[T = [n\delta]] \leqq c(1 - \delta, \varepsilon) P^{n^{\frac{1}{2}}\varepsilon}[T = [n\delta]]\,.$$

It follows then for n sufficiently large

$$\begin{aligned} &P_n{}^\circ[\delta < \tau < 1 - \delta] \\ &\quad \leqq f_n{}^{-1} c(1 - \delta, \varepsilon) P^{n^{\frac{1}{2}}\varepsilon}[T = [n\delta]] \\ &\qquad \times \textstyle\sum_{x \neq 0} P[\delta < \tau < 1 - \delta;\, S_{n-[n\delta]}/n^{\frac{1}{2}} = x;\, T > n - [n\delta]] \\ &\quad \leqq f_n{}^{-1} c(1 - \delta, \varepsilon) P^{n^{\frac{1}{2}}\varepsilon}[T = [n\delta]] P[\delta < \tau < 1 - \delta;\, T > n - [n\delta]] \\ &\quad \leqq f_n{}^{-1} c(1 - \delta, \varepsilon) P^{n^{\frac{1}{2}}\varepsilon}[T = [n\delta]] P[\delta < \tau < 1 - \delta;\, T > [n\delta]] \\ &\quad \leqq f_n{}^{-1} c(1 - \delta, \varepsilon) P^{n^{\frac{1}{2}}\varepsilon}[T = [n\delta]] P[\sup_{\delta \leqq \tau \leqq 1-\delta} W_x[t, t + \rho] \geqq 2\varepsilon;\, T > [n\delta]] \\ &\quad \leqq f_n{}^{-1} c(1 - \delta, \varepsilon) P^{n^{\frac{1}{2}}\varepsilon}[T = [n\delta]] P[T > [n\delta]] P_n[W_x(\rho) \geqq 2\varepsilon]\,. \end{aligned}$$

From (2.1) and (3.3) $\limsup_{n\to\infty} f_n{}^{-1} r_{[n\delta]} P^{n^{\frac{1}{2}}\varepsilon}[T = [n\delta]] < \infty$. Since $P_n \Rightarrow W$ we obtain

$$(3.8) \qquad \limsup_{n\to\infty} P_n{}^\circ[\delta < \tau < 1 - \delta] = 0 \qquad \text{for every} \quad \delta \in (0, \tfrac{1}{2})\,.$$

The idea of the preceding argument roughly is that conditioning away from zero does not greatly affect the fluctuations in the middle portion of the process. From the Markov property of the random walk the condition $T = n$ affects the oscillations of the paths over $(\delta, 1 - \delta)$ only through the distribution of $S_{n-[n\delta]}$. The initial inequality obtained from Lemma 3.5 guarantees that the effect of the conditioning on the fluctuations is mild.

Now we consider $P_n{}^\circ[\tau \leqq \delta]$. Clearly we have

$$P_n{}^\circ[\tau \leqq \delta] \leqq P[\max_{t \leqq \delta} |S_{[nt]}/n^{\frac{1}{2}}| \geqq \varepsilon | T = n]$$

so it suffices to prove that

$$\limsup_{n\to\infty} f_n^{-1} P[\max_{t\leq\delta} |S_{[nt]}/n^{\frac{1}{2}}| \geqq \varepsilon;\ T = n] \to 0 \qquad \text{as } \delta \downarrow 0 .$$

To show this define $\tau_n = \min\{j : 1 \leqq j \leqq [n\delta];\ S_j/n^{\frac{1}{2}} \geqq \varepsilon\}$. Then for any fixed $m < [n\delta]$ we obtain from Lemma 3.5

$$\begin{aligned}
&P[\max_{t\leq\delta} S_{[nt]}/n^{\frac{1}{2}} \geqq \varepsilon;\ \tau_n \geqq [n\delta] - m;\ T = n] \\
&\quad \leqq c(\delta, \varepsilon) P^{n^{\frac{1}{2}}\varepsilon}[T = n - [n\delta]] \textstyle\sum_{k=n\delta-m}^{n\delta} \sum_{x\geq\varepsilon} P[\tau_n = k;\ S_k/n^{\frac{1}{2}} = x;\ T > k] \\
&\quad \leqq c(\delta, \varepsilon) P^{n^{\frac{1}{2}}\varepsilon}[T = n - [n\delta]] \textstyle\sum_{k=n\delta-m}^{n\delta} \sum_{x\geq\varepsilon} P[\tau_n = k;\ S_k/n^{\frac{1}{2}} = x;\ T > k] \\
&\qquad \times \frac{P[\min_{j\leq[n\delta]-k} S_j/n^{\frac{1}{2}} \geqq -\varepsilon/2]}{P[\min_{j\leq m} S_j/n^{\frac{1}{2}} \geqq -\varepsilon/2]} \\
&\quad = \{c(\delta, \varepsilon) P^{n^{\frac{1}{2}}\varepsilon}[T = n - [n\delta]]/P[\min_{j\leq m} S_j/n^{\frac{1}{2}} \geqq -\varepsilon/2]\} \\
&\qquad \times \textstyle\sum_{k=n\delta-m}^{n\delta} \sum_{x\geq\varepsilon} P[\tau_n = k;\ S_k/n^{\frac{1}{2}} = x;\ T > k] \\
&\qquad \times P^{n^{\frac{1}{2}}x}[\min_{j\leq[n\delta]-k} S_j/n^{\frac{1}{2}} \geqq x - \varepsilon/2] \\
&\quad \leqq \{c(\delta, \varepsilon) P^{n^{\frac{1}{2}}\varepsilon}[T = n - [n\delta]]/P[\min_{j\leq m} S_j/n^{\frac{1}{2}} \geqq -\varepsilon/2]\} \\
&\qquad \times \textstyle\sum_{k=n\delta-m}^{n\delta} \sum_{x\geq\varepsilon} P[\tau_n = k;\ S_k/n^{\frac{1}{2}} = x;\ T > k] \\
&\qquad \times P^{n^{\frac{1}{2}}x}[S_{[n\delta]-k}/n^{\frac{1}{2}} \geqq x - \varepsilon/2;\ T > [n\delta] - k] .
\end{aligned}$$

From the above it follows for large n

$$\begin{aligned}
&P[\max_{t\leq\delta} S_{[nt]}/n^{\frac{1}{2}} \geqq \varepsilon;\ \tau_n \geqq [n\delta] - m;\ T = n] \\
&\quad \leqq \{c(\delta, \varepsilon) P^{n^{\frac{1}{2}}\varepsilon}[T = n - [n\delta]]/P[\min_{j\leq m} S_j/n^{\frac{1}{2}} \geqq -\varepsilon/2\} \\
&\qquad \times P[S_{[n\delta]}/n^{\frac{1}{2}} \geqq \varepsilon/2;\ \tau_n \geqq [n\delta] - m;\ T > [n\delta]] .
\end{aligned} \tag{3.9}$$

Next we obtain a similar inequality for $\tau_n < [n\delta] - m$. For any fixed $m < [n\delta]$

$$\begin{aligned}
&P[\max_{t\leq\delta} S_{[nt]}/n^{\frac{1}{2}} \geqq \varepsilon;\ \tau_n < [n\delta] - m;\ T = n] \\
&\quad = \textstyle\sum_{k=1}^{n\delta-m-1} \sum_{x\geq\varepsilon} P[\tau_n = k;\ S_k/n^{\frac{1}{2}} = x;\ T > k] P^{n^{\frac{1}{2}}x}[T = n - k] .
\end{aligned}$$

From Lemmas 3.5 and 3.6 it follows that for sufficiently large n

$$\begin{aligned}
&P[\max_{t\leq\delta} S_{[nt]}/n^{\frac{1}{2}} \geqq \varepsilon;\ \tau_n < [n\delta] - m;\ T = n] \\
&\quad \leqq [c(\delta, \varepsilon)/\lambda(\delta, \varepsilon)] \textstyle\sum_{k=1}^{n\delta-m-1} \sum_{x\geq\varepsilon} P[\tau_n = k;\ S_k/n^{\frac{1}{2}} = x;\ T > k] \\
&\qquad \times P^{n^{\frac{1}{2}}x}[S_{[n\delta]-k}/n^{\frac{1}{2}} \geqq x;\ T > [n\delta] - k] P^{n^{\frac{1}{2}}\varepsilon}[T = n - [n\delta]] \\
&\quad \leqq [c(\delta, \varepsilon)/\lambda(\delta, \varepsilon)] P^{n^{\frac{1}{2}}\varepsilon}[T = n - [n\delta]] \\
&\qquad \times P[S_{[n\delta]}/n^{\frac{1}{2}} > \varepsilon;\ \tau_n < [n\delta] - m;\ T > [n\delta]] .
\end{aligned} \tag{3.10}$$

A combination of (3.9) and (3.10) will give

$$\begin{aligned}
&P[\max_{t\leq\delta} S_{[nt]}/n^{\frac{1}{2}} \geqq \varepsilon;\ T = n] \\
&\quad \leqq c(\delta, \varepsilon) P^{n^{\frac{1}{2}}\varepsilon}[T = n - [n\delta]] P[S_{[n\delta]}/n^{\frac{1}{2}} \geqq \varepsilon/2;\ T > [n\delta]] \\
&\qquad \times \{1/\lambda(\delta, \varepsilon) + 1/P[\min_{j\leq m} S_j/n^{\frac{1}{2}} \geqq -\varepsilon/2]\} .
\end{aligned} \tag{3.11}$$

By (3.3)

$$\lim_{n\to\infty} n P^{n^{\frac{1}{2}}\varepsilon}[T = n - [n\delta]] = (1 - \delta)^{-\frac{3}{2}} (\varepsilon/\sigma) \phi(\varepsilon/\sigma(1 - \delta)^{\frac{1}{2}}) .$$

From (2.3)

$$\lim_{n\to\infty} n^{\frac{1}{2}} P[S_{[n\delta]}/n^{\frac{1}{2}} \geqq \varepsilon/2;\, T > [n\delta]] = \sigma(2/\pi)^{\frac{1}{2}}\delta^{-\frac{1}{2}} \int_{\varepsilon/2\delta^{\frac{1}{2}}}^{\infty} (1/2\sigma^2)|y|e^{-y^2/2\sigma^2}\,dy$$
$$= \sigma(2\pi\delta)^{-\frac{1}{2}} e^{-\varepsilon^2/8\sigma^2\delta}\,.$$

Since m is fixed $\lim_{n\to\infty} P[\min_{j\leqq m} S_j/n^{\frac{1}{2}} \geqq -\varepsilon/2] = 1$. Using these facts and (2.2) provides

$$\limsup_{n\to\infty} P[\max_{t\leqq\delta} S_{[nt]}/n^{\frac{1}{2}} \geqq \varepsilon \mid T = n]$$
$$\leqq \sigma^{-1}c(\delta, \varepsilon)\{1/\lambda(\delta, \varepsilon) + 1\}\varepsilon\delta^{-\frac{1}{2}}(1-\delta)^{-\frac{3}{2}}\phi(\varepsilon/\sigma(1-\delta)^{\frac{1}{2}})e^{-\varepsilon^2/8\sigma^2\delta}\,.$$

For fixed ε, $\lim_{\delta\downarrow 0} c(\delta, \varepsilon) < \infty$ and $\lim_{\delta\downarrow 0} \lambda(\delta, \varepsilon) > 0$ so

$$\lim_{\delta\downarrow 0} \limsup_{n\to\infty} P[\max_{t\leqq\delta} S_{[nt]}/n^{\frac{1}{2}} \geqq \varepsilon \mid T = n] = 0\,.$$

A symmetry argument will show

$$\lim_{\delta\downarrow 0} \limsup_{n\to\infty} P[\max_{t\leqq\delta} |S_{[nt]}/n^{\frac{1}{2}}| \geqq \varepsilon \mid T = n] = 0$$

which completes the proof that

$$\limsup_{\rho\downarrow 0} \limsup_{n\to\infty} P_n{}^\circ[\tau \leqq \delta] \to 0 \qquad \text{as } \delta \downarrow 0\,.$$

We consider finally $P_n{}^\circ[1-\delta \leqq \tau \leqq 1]$. The chance behavior of the final portion of the conditioned random walk is the same as the initial portion so consideration of the reversed random walk and the result just proved will show

$$\lim_{\delta\downarrow 0} \limsup_{n\to\infty} P[\max_{1-\delta\leqq t\leqq 1} |S_{[nt]}/n^{\frac{1}{2}}| \geqq \varepsilon \mid T = n]$$

and the verification of (3.7) is complete.

From Theorem 15.1 in [3] we obtain $P_n{}^\circ \Rightarrow W^\circ$.

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