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AN EXAMPLE OF A WEAK MARTINGALE

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A sequence of integrable random variables $\{X_n, n \geqq 0\}$ is a weak martingale if $E(X_n \mid X_m) = X_m$ a.s. for all $0 \leqq m < n$. We present an example of a weak martingale which is not a martingale. It is bounded and has countably many paths.

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$$E(X_n \mid X_m) = X_m \quad \text{a.s.}, \qquad 0 \leqq m \leqq n .$$

P. Nelson (1970) gives examples of weak martingales that are not martingales (and also discusses the convergence of such processes). In this paper, we give an especially simple example that is bounded and has countably many paths.

Let $0 < \alpha < 1$ and $\Omega = \{f_0, f_1, \cdots\} \cup \{g_0, g_1, \cdots\}$ where for each $i \geqq 0$, f_i is the function on $0 \leqq t < \infty$ defined by

$$\begin{aligned} f_i(t) &= 0 , & t < i \\ &= \alpha^i - \alpha^{t-i} , & t \geqq i , \end{aligned}$$

and $g_i = -f_i$.

For the case $\alpha = \frac{1}{2}$, the nonconstant portions of the graphs of the functions in Ω are shown in Figure 1. (It is perhaps interesting to note that the increasing [decreasing] curves are all translates of one another.)

Let p be the probability mass function on Ω:

$$p(f_i) = p(g_i) = \tfrac{1}{2}(1 - \alpha)\alpha^i , \qquad i \geqq 0 .$$

Ω and p determine a probability space. On this space, let $\{X_n, n \geqq 0\}$ be the sequence of coordinate random variables:

$$X_n(f_i) = f_i(n) , \qquad X_n(g_i) = g_i(n) ; \qquad i \geqq 0, n \geqq 0 .$$

Clearly $|X_n| \leqq 1$ for all $n \geqq 0$. We show that $\{X_n, n \geqq 0\}$ is a weak martingale. It is obvious that the process is not a martingale.

The following two lemmas have straightforward proofs which are left to the reader.

Lemma 1. *If $r \neq 0$ and if $[X_m = r]$ is not empty, then there exists a unique i and a unique j such that $f_i(m) = r$ and $g_j(m) = r$; moreover, $i + j = m$.*

The content of this lemma is clearly visible in Figure 1 in the case $\alpha = \frac{1}{2}$. Its falsity when $r = 0$ is not apparent from Figure 1, as the constant zero portions of the graphs have been omitted.

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LEMMA 2. *If $i \leqq m$, $j \leqq m$, and $m < n$, then*

$$p(f_i)\{f_i(n) - f_i(m)\} + p(g_j)\{g_j(n) - g_j(m)\} = 0 .$$

PROPOSITION. *$\{X_n, n \geqq 0\}$ is a weak martingale.*

PROOF. Let r be real, $0 \leqq m < n$, and let $[X_m = r]$ be nonempty. We must show that

$$E(X_n \mid X_m = r) = r . \tag{1}$$

If $r = 0$, then (1) is a consequence of the symmetry of the process. If $r \neq 0$, then from Lemmas 1 and 2 it follows that

$$\int_{[X_m=r]} (X_n - X_m) = 0$$

which is equivalent to (1), and the proof is complete.

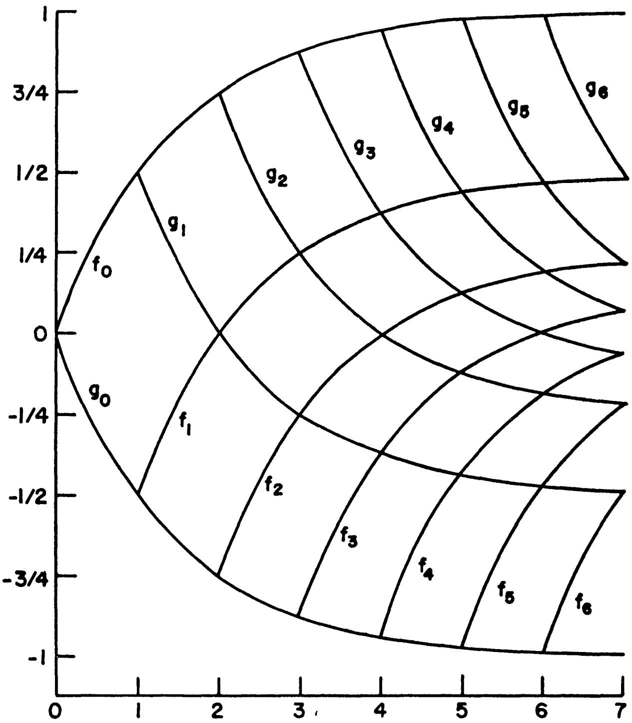

FIG. 1.

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