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THE ERGODIC MAXIMAL FUNCTION WITH CANCELLATION[1]

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A variant of the ergodic maximal function is studied. This maximal function reduces to the usual one for $f \geqq 0$, but the cancellation between the positive and negative parts of f causes interesting behavior. In particular the maximal function can be in L^1 even when the function is not in $L \log^+ L$. A relation between this maximal function and the ergodic Hilbert transform is studied.

1. Introduction. Let (X, Σ, m) denote a complete nonatomic probability space, and T an invertible, ergodic measure preserving point transformation mapping X onto itself. In this setting, the ergodic maximal function is defined by

$$f^*(x) = \sup_n \frac{1}{n} \sum_{k=0}^{n-1} |f(T^k(x))| .$$

Ornstein (1971) proved that f^* is in $L^1(X)$ if and only if f is in the class $L \log^+ L$.

In the case where $\{X_i\}_{i=0}^{\infty}$ are independent identically distributed random variables, Burkholder (1962) proved $S^* = \sup_n |(1/n) \sum_{k=0}^{n-1} X_k(w)|$ is in L^1 if and only if X_0 is in $L \log^+ L$.

In view of these two results, it becomes natural to ask if a similar result holds for the operator

$$Mf(x) = \sup_{n>0} \left| \frac{1}{n} \sum_{k=0}^{n-1} f(T^k x) \right| .$$

It turns out that such a result is not true. Burgess Davis (1971) constructed a function on $[0, 1)$ with T the irrational shift, such that Mf is in L^1 but $f \notin L \log^+ L$. His example is generalized in Theorem 1, to show that for any ergodic transformation on a nonatomic probability space, there is an f such that $Mf \in L^1$, but $f \notin L \log^+ L$.

As a consequence of this result, and indeed part of the original motivation for studying the problem, it is possible to show that the class "ergodic H^1" introduced by Coifman and Weiss (1973) is strictly larger than $L \log^+ L$. The last section of this paper contains some additional results about the space H^1.

2. Results.

THEOREM 2.1. *For any invertible, ergodic measure preserving transformation of*

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a nonatomic probability space onto itself, there is a function f such that $Mf \in L^1$, but $f \notin L \log^+ L$.

The proof on Theorem 2.1 involves constructing a function f such that the action of T on f causes the positive and negative parts to cancel each other out. As a consequence Mf and f are nearly the same for many points x.

PROOF. Rokhlin's theorem says that given ε and N, we can find a set A_ε^N such that the sets $A_\varepsilon^N, TA_\varepsilon^N, \cdots, T^{N-1}A_\varepsilon^N$ are disjoint and

$$m\left(\bigcup_{k=0}^{N-1} T^k A_\varepsilon^N\right) > 1 - \varepsilon .$$

For given k, choose $\varepsilon = \varepsilon(k)$ and $N = N(k)$, such that

(1) N is even;

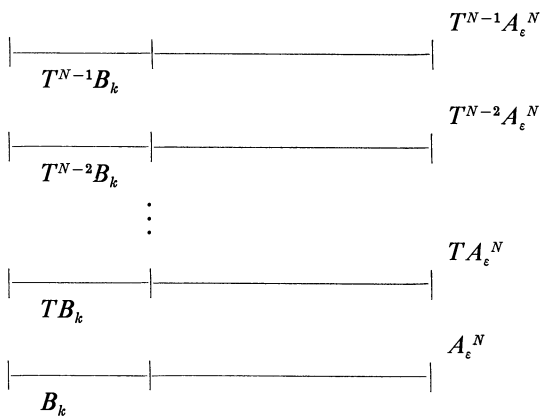
(2) $\varepsilon < \frac{1}{2}\left(\frac{1}{2^{4k}}\right)$;

(3) $\frac{1}{N} \log N < \frac{1}{2}\left(\frac{1}{2^{4k}}\right)$;

(4) $\frac{2}{N} < \frac{1}{2^{4(k+1)}}$.

For this N, ε make the Rokhlin construction. We note that the measure of A_ε^N is about $1/N$.

Denote by B_k some subset of A_ε^N such that $Nm(B_k) = 1/2^{4k}$.



Define

$$\begin{aligned} f_k(x) &= (-1)^i \qquad x \in T^i B_k \quad 0 \leqq i < N \\ &= 0 \qquad\quad\;\; x \notin \textstyle\bigcup_{i=0}^{N-1} T^i B_k . \end{aligned}$$

We do this for each k, and define

$$f(x) = \sum_{k=1}^{\infty} \frac{2^{4k}}{k^2} f_k(x) .$$

We first show that for this f, Mf is in L^1. We have that

$$\int_X Mf(x)\, dm(x) \leqq \sum_{k=1}^{\infty} \frac{2^{4k}}{k^2} \int_X Mf_k(x)\, dm(x) .$$

Consequently it is enough to show that

$$\int Mf_k(x)\,dm(x) \leqq \frac{c}{2^{4k}}\,.$$

Let $A = A_\varepsilon^N$ and recall that f_k is supported on the set $B = B_k \cup TB_k \cup \cdots \cup T^{N-1}B_k$. If $x \in B$ then $Mf_k(x) = 1$. If $x \in T^{N-1}(A - B_k)$ then $Mf(x) \leqq \frac{1}{2}$. If $x \in T^{N-2}(A - B_k)$ then $Mf_k(x) \leqq \frac{1}{3}$. In general, if $x \in T^{N-j}(A - B_k)$ then $Mf(x) \leqq 1/(j+1)$; $1 \leqq j \leqq N$. We will estimate $Mf(x)$ for x not in $A \cup TA \cup \cdots \cup T^{N-1}A$ by 1. We now add up the values times the size of the set they occur on. We get:

$$\begin{aligned}\int Mf_k(x)\,dm(x) &\leqq 1 \cdot \frac{1}{2^{4k}} + \frac{1}{2}\cdot\frac{1}{N} + \frac{1}{3}\cdot\frac{1}{N} + \cdots + \frac{1}{N+1}\cdot\frac{1}{N} + 1\cdot\varepsilon_k \\ &\leqq \frac{1}{2^{4k}} + \frac{c\log N}{N} + \frac{1}{2}\cdot\frac{1}{2^{4k}} \\ &\leqq \frac{c}{2^{4k}}\,.\end{aligned}$$

Hence $Mf \in L^1(X)$. We now show that $f \notin L\log^+ L$. First we note that

$$\begin{aligned}\left|\sum_{k=1}^{j-1}\frac{2^{4k}}{k^2}f_k\right| &\leqq \left|\sum_{k=1}^{j-1}\frac{2^{4k}}{k^2}\right| \\ &\leqq \frac{2^{4j}}{j^2}\,.\end{aligned}$$

Thus, for $x \in \{f_j \neq 0\}$, we have

$$\begin{aligned}\left|\sum_{k=1}^{j}\frac{2^{4k}}{k^2}f_k(x)\right| &\geqq \frac{2^{4j}}{j^2} - \frac{1}{2}\cdot\frac{2^{4j}}{j^2} \\ &\geqq \frac{1}{2}\cdot\frac{2^{4j}}{j^2}\,.\end{aligned}$$

We also have

$$\begin{aligned}\sum_{j=k+1}^{\infty} m\{f_k \neq 0\} &= \sum_{k=j+1}^{\infty}\frac{1}{2^{4k}} \\ &\leqq \frac{1}{2}\cdot\frac{1}{2^{4j}}\,,\end{aligned}$$

hence,

$$m\Big(\sum_{j=k+1}^{\infty} f_k \neq 0\Big) \leqq \frac{1}{2}\cdot\frac{1}{2^{4j}}\,.$$

Consequently,

$$\begin{aligned}\int |f(x)|\log^+|f(x)|\,dm(x) &\geqq \sum_{j=1}^{\infty}\int_{\{f_j\neq 0,\,f_{j+k}=0 \text{ for all } k>0\}} |f(x)|\log^+|f(x)|\,dm(x) \\ &\geqq \sum_{j=1}^{\infty}\left[\left(\frac{1}{2}\cdot\frac{2^{4j}}{j^2}\right)\log\left(\frac{1}{2}\cdot\frac{2^{4j}}{j^2}\right)\right]\frac{1}{2}\cdot\frac{1}{2^{4j}} \\ &\geqq \sum_{j=1}^{\infty}\frac{1}{j^2}\log\frac{2^{4j}}{2j^2} \\ &= 0\left(\sum_{j=1}^{\infty}\left(\frac{1}{j}\right)\right) = \infty\,.\end{aligned}$$

Although Theorem 2.1 shows that the class $L \log^+ L$ is smaller than the class of functions with $MF \in L^1$, there is a very strong relationship between the two classes. We will say F is a rearrangement of f if F and f have the same distribution function. We then show, in a large number of cases, that given $f \notin L \log^+ L$, there is a rearrangement F of f such that $\int MF(x)\,dx = \infty$.

It is not hard to show that given $f \notin L \log^+ L$, and given N, there exists a rearrangement of f, denoted by F_N, such that $\int_X MF_N(x)\,dx \geqq N$. However, it is not clear how to take a limit and find F such that $\int_X MF(x)\,dx = \infty$. We can, however, by a different approach, and using powerful tools, solve the problem when T has positive entropy.

THEOREM 2.2. *If the entropy of T is greater than zero, then given $f \in L^1$, such that $f \notin L \log^+ L$, there exists a rearrangement F of f such that $MF \notin L^1$.*

The idea of the proof is to use Sinai's weak isomorphism theorem to find a rearrangement F of f, such that F, TF, T^2F, $\cdots$ are nearly independent. In this situation we apply Burkholder's result to complete the proof.

PROOF. Assume that $Mf \in L^1$, since if not we are already done. We can approximate f by a step function as follows. Let $A_k = \{2^{k-1} \leqq f(x) < 2^k\}$, $B_k = \{-2^k < f(x) \leqq 2^{k-1}\}$. Then

$$f(x) \cong s_1(x) = \sum_{k=1}^{\infty} 2^k(1_{A_k}(x) - 1_{B_k}(x))\,.$$

Since $f \in L^1$, $s_1 \in L^1$, and consequently

$$\sum_{k=1}^{\infty} 2^k[m(A_k) + m(B_k)] < \infty\,.$$

From this it follows that

$$0 < -\sum_{k=1}^{\infty} m(A_k) \log m(A_k) + m(B_k) \log m(B_k) < \infty\,.$$

If we denote the entropy of T by $H(T)$, then since $H(T) > 0$, there exists N_1 such that

$$\tfrac{1}{2}H(T) > [-\sum_{k=N_1}^{\infty} m(A_k) \log m(A_k) + m(B_k) \log m(B_k)]\,.$$

Since $h(x) = -x \log x$ is a continuous function, and $h(1) = 0$, we can find $\varepsilon < 1$ such that $\frac{1}{2}H(T) > h(\delta)$ for all $\delta \in [\varepsilon, 1]$. We now choose N_2 so large that

$$\varepsilon < \sum_{k=1}^{N_2} m(A_k) + m(B_k) < 1\,.$$

Let $N = \max(N_1, N_2)$, and let $C = \bigcup_{k=1}^{N-1} A_k \cup B_k$. We then have

$$H(T) \geqq -m(C) \log m(C) - (\sum_{k=N}^{\infty} m(A_k) \log m(A_k) + m(B_k) \log m(B_k)) > 0\,.$$

Now define

$$s(x) = \sum_{k=N}^{\infty} 2^k(1_{A_k}(x) - 1_{B_k}(x))\,.$$

It is clear that both s_1 and s are not in $L \log^+ L$.

Sinai's weak isomorphism theorem says that we can nowfin d sets $\bar{A}_k$, $\bar{B}_k$, $\bar{C}$, $k = N, N+1, \cdots$, such that

(1) $m(A_k) = m(\bar{A}_k)$, $m(B_k) = m(\bar{B}_k)$, and $m(C) = m(\bar{C})$.

(2) If $\bar{P}$ denotes the partition of X determined by $\{\bar{A}_k, \bar{B}_k, \bar{C}, k = N, N+1, \cdots\}$ the sequence partitions $\bar{P}$, $T\bar{P}$, $T^2\bar{P}$, $\cdots$ are all independent.

Now define

$$\bar{s}(x) = \sum_{k=N}^{\infty} 2^k(1_{\bar{A}_k}(x) - 1_{\bar{B}_k}(x)) .$$

Since $\bar{s}$ is measurable with respect to the partition $\bar{P}$, and the partitions $\bar{P}$, $T\bar{P}$, $T^2\bar{P}$, $\cdots$ are independent, the sequence $\bar{s}(\cdot)$, $\bar{s}(T(\cdot))$, $\bar{s}(T^2(\cdot))$ $\cdots$ can be thought of as independent identically distributed random variables. Invoking Burkholder's result, we see $M\bar{s} \in L^1$ if and only if $\bar{s} \in L \log^+ L$. As a consequence $M\bar{s} \notin L^1$. We now obtain F from $\bar{s}$. Define F on $\bar{A}_k$ so that the distribution of F on $\bar{A}_k$ is the same as the distribution of f on A_k. We do the same thing on $\bar{C}$ and $\{\bar{B}_k\}_{k=N}^{\infty}$. It is clear that this F satisfies the conclusion of the theorem.

3. Applications to H^1. The results in Section 2 can be used to give additional information about the space H^1 introduced by Coifman and Weiss (1973).

Let $\{U_t\}$, $-\infty < t < \infty$, be a one-parameter group of measure preserving transformations on X (i.e., an ergodic flow). The ergodic Hilbert transform $\tilde{f}_U(x)$; of a function $f \in L^p(X)$, $1 \leqq p \leqq \infty$ is defined as the a.e. limit of

$$H_\varepsilon f(x) = \frac{1}{\pi} \int_{\varepsilon<|t|<1/\varepsilon} f(U_t x) \frac{dt}{t} , \tag{3.1}$$

as $\varepsilon > 0$ tends to 0. The almost everywhere existence of this limit was first shown by Cotlar (1955). In the discrete case, Coifman and Weiss define the analogous operator, also called the Hilbert transform, by the formula

$$\tilde{f}_T(x) = \lim_{n\to\infty} \frac{1}{\pi} \sum_{k\neq 0; -n<k<n} f(T^k x) \frac{1}{k} . \tag{3.2}$$

In the classical situation H^1 can be characterized by the conditions $f \in L^1$ and $\tilde{f} \in L^1$, where $\tilde{f}$ is the classical Hilbert transform. With this in mind, Coifman and Weiss characterize ergodic H^1 as the class of $f \in L^1$ such that $\tilde{f}_T \in L^1$.

They also prove that $||f||_1 + ||\tilde{f}_T||_1 \leqq c||R^* f||_1$, where in the discrete setting,

$$R^* f(x) = 6 \sup_{n\neq 0} \left| \frac{1}{n} \sum_{k=1}^{n} \frac{k}{n} \left(1 - \frac{k}{n}\right) f(T^k x) \right| . \tag{3.3}$$

It is not hard to prove that (essentially) $||R^* f||_1 \leqq c||Mf||_1$. The argument is a special case of a general principle involving approximate identities, but uses a two-sided maximal function. The modifications are clear. As a consequence of this observation, and Theorem 2.1, we see that there is an f which is not in $L \log^+ L$ but is in H^1.

We now apply Theorem 2.2, along with some facts about independent identically distributed random variables, to show H^1 is not invariant under isomorphism. In particular we prove the following result.

THEOREM 3.4. *Let T be an invertible measure preserving ergodic transformation* *on X. If T has positive entropy then there exists a transformation $\bar{T}$ isomorphic to T, and a function $f \in L^1(X)$ such that $\tilde{f}_T \in L^1(X)$ but $\tilde{f}_{\bar{T}} \notin L^1(X)$.*

As a consequence of this result in the discrete case, we can also state a result for measure preserving flows.

THEOREM 3.5. *There exist two isomorphic flows, U_t and $\bar{U}_t$, and a function $f \in L^1(X)$, such that $\tilde{f}_U \in L^1(X)$, but $\tilde{f}_{\bar{U}} \notin L^1(X)$.*

To see this we consider the space $X = [0, 1)xm$, where m is a probability space. Let T and $\bar{T}$ be isomorphic transformations. We then define U_t by

$$U_t(\alpha, x) = (\alpha + t - [\alpha + t], T^{[\alpha+t]}x) . \tag{3.6}$$

The fact that T is ergodic and measure preserving implies the same for U. The flow $\bar{U}_t$ is defined by (3.6) with T replaced by $\bar{T}$. The isomorphism between T and $\bar{T}$ can be extended to an isomorphism between U and $\bar{U}$. As pointed out by Coifman and Weiss, the operators defined by these flows, when operating on functions which depend only on their second variable, differ from their discrete versions by operators which are easily analyzed. These operators are easily shown to be bounded on all L^p, $1 \leqq p \leqq \infty$. Hence, the f from Theorem 3.4, extended in the obvious way also works for Theorem 3.5.

PROOF OF THEOREM 3.4. We begin by recalling some known facts about independent identically distributed random variables. Klass (1974) proves that if $\{Y_i\}_{i=1}^{\infty}$ are i.i.d. random variables, with mean 0, then

$$\lim_{n\to\infty} E\left(\left|\sum_{k=1}^{n} \frac{Y_k}{k}\right|\right) \tag{3.7}$$

and

$$E\left(\sup_{n\geq 1}\left|\frac{1}{n} \sum_{k=1}^{n} Y_k\right|\right) \tag{3.8}$$

are either both finite or both infinite. However 3.8 is finite if and only if $X_1 \in L \log^+ L$.

Clearly if $\{X_i\}_{i=-\infty}^{\infty}$ is a sequence of i.i.d. random variables, and $Y_k = X_k - X_{-k}$, then $\{Y_i\}_{i=1}^{\infty}$ is a sequence of i.i.d. mean 0 random variables. Further, $Y_k \in L \log^+ L$ if and only if $X_0 \in L \log^+ L$. Suppose $f(T^k x) = X_k(x)$, $-\infty < k < \infty$. Then

$$\begin{aligned} \tilde{f}_T(x) &= \lim_{n\to\infty} \frac{1}{\pi} \sum_{k\neq 0; -n<k<n} f(T^k x) \frac{1}{k} \\ &= \lim_{n\to\infty} \frac{1}{\pi} \sum_{k=1}^{n} (f(T^k x) - f(T^{-k}x)) \frac{1}{k} . \end{aligned} \tag{3.9}$$

We now combine these ideas. First we have $\tilde{f} \in L^1$ if and only if (3.9) is in L^1. However the L^1 norm of (3.9) is (3.7) in this case. Hence $\tilde{f} \in L^1$ if and only if (3.8) is finite, but (3.8) is finite if and only if $f \in L \log^+ L$.

We now return to the proof of Theorem 2.2. There we started with an f in

L^1 but not in $L \log^+ L$. We then constructed a step function $s(x)$, with the same integrability properties as f. The partition upon which this step function was defined had entropy less than or equal to that of T. By Ornstein's theorem, as modified by Smorodinsky (1972), we can find a new transformation $\bar{T}$, isomorphic to T, such that $\{s(\bar{T}^k(\cdot))\}_{k=-\infty}^{\infty}$ is a sequence of i.i.d. random variables. Consequently $\tilde{s}_{\bar{T}}$ is not in L^1, since s was not in $L \log^+ L$. Clearly this also implies that $\tilde{f}_{\bar{T}} \notin L^1$, and we are done.

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