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AN ALMOST SURE INVARIANCE PRINCIPLE FOR THE EXTREMA OF CERTAIN SAMPLE FUNCTIONS[1]

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For a general class of statistics expressible as extrema of certain sample functions, an almost sure invariance principle, particularly useful in the context of the law of iterated logarithm and the probabilities of moderate deviations, is established, and its applications are stressed.

1. Introduction. Consider a general class of extrema of sample functions expressible as

(1.1) $$Z_n = \sup\{L(x, F_n(x)) : x \in A\} \qquad \text{for some} \quad A \subset E^p ,$$

where F_n is the empirical distribution function (df) of a random sample of size n from a continuous df F, defined on the $p(\geqq 1)$-dimensional Euclidean space E^p and L satisfies certain regularity conditions. An important member of this class (see [2, 8]) is the bundle strength of filaments where $p = 1$, the basic random variables (rv) are nonnegative, $L(x, y) = x(1-y)$, $Z_n = n^{-1}\max_{1\leqq i\leqq n}[(n-i+1)X_{n,i}]$ and the $X_{n,i}$ are the sample order statistics. For such statistics, asymptotic normality as well as the embedding of Wiener processes are established in [2, 5, 7, 8]. The object of the present investigation is to show that parallel to Theorem 1.4 of Strassen (1967) an almost sure (a.s.) invariance principle (particularly useful in the context of the law of iterated logarithm and probabilities of moderate deviations) holds for $\{Z_n\}$ under the same regularity conditions as (in [8]) pertaining to its asymptotic normality.

The main theorem along with the preliminary notions is presented in Section 2. Section 3 deals with certain basic lemmas which are incorporated in the proof of the main theorem in Section 4. The last section deals with some concluding remarks and applications.

2. The main theorem. Let $\{X_i, i \geqq 1\}$ be a sequence of independent and identically distributed (i.i.d.) rv's defined on a probability space $(\Omega, \mathscr{A}, P)$ where X_i has a continuous df $F(x)$, $x \in E^p$, for some $p \geqq 1$, and our desired $A \subset E^p$. We assume that for $x \in A$, $L(x, F(x))$ assumes a unique maximum θ at a unique point x_0, so that

(2.1) $$\theta = \sup\{L(x, F(x)) : x \in A\} = L(x_0, F(x_0)) \qquad \text{where} \quad x_0 \in A ,$$

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and for every $\varepsilon > 0$, there exists an $\eta > 0$, such that

(2.2) $$L(x, F(x)) < \theta - \varepsilon \quad \text{for every} \quad x: |x - x_0| > \eta .$$

Further, we assume that for some $\delta(> 0)$, sufficiently small, there exist four positive constants C_i, $k_i(\geqq 1)$, $i = 1, 2$, such that

(2.3) $$\theta - C_1|x - x_0|^{k_1} \leqq L(x, F(x)) \leqq \theta - C_2|x - x_0|^{k_2} \qquad k_1 \leqq k_2 ,$$

for all $x: |x - x_0| \leqq \delta$, where in (2.2)—(2.3), $|u|$ stands for the Euclidean norm. Also, we assume that F admits of a continuous density function $f(x)$ for all x: $|x - x_0| < \delta$, and

(2.4) $$0 < p_0 = F(x_0) < 1 , \qquad 0 < f(x_0) < \infty .$$

Let then $A^* = \{(x, y): x \in A \subset E^p, 0 \leqq y \leqq 1\}$, and for every $\delta \in [0, 1]$, let

(2.5) $$A_\delta^* = \{(x, y): x \in A, \max(0, F(x) - \delta) \leqq y \leqq \min(1, F(x) + \delta)\} .$$

We assume that $L(x, y)$ is defined for every $(x, y) \in A^*$, and for some $\delta > 0$, $(x, y) \in A_\delta^*$, $L(x, y)$ possesses a continuous partial (with respect to y) derivative $L_{01}(x, y)$ which satisfies the following conditions [due to Sen et al. (1973)]:

(2.6) $$|L_{01}(x, y)| \leqq g(x) , \quad \forall (x, y) \in A_\delta^* ,$$
$$\textstyle\int_A g^2(x)\, dF(x) \leqq \lambda^2 < \infty , \quad \forall A \subset E^p ;$$

(2.7) $$L_{01}(x_0, F(x_0)) = \xi_0 \qquad \text{where} \quad 0 < |\xi_0| < \infty .$$

From [8], it follows that under (2.1)—(2.7), as $n \to \infty$

(2.8) $$\mathscr{L}(n^{\frac{1}{2}}[Z_n - \theta]) \to \mathscr{N}(0, \sigma^2) \qquad \text{where}$$
$$\sigma^2 = \xi_0^2 p_0(1 - p_0) \quad \text{so that} \quad 0 < \sigma < \infty .$$

Let $\phi = \{\phi(t): 0 \leqq t < \infty\}$ be a positive function with a continuous derivative $\phi'(t)$ such that (i) as $t \to \infty$ with $s/t \to 1$, $\phi'(s)/\phi'(t) \to 1$, (ii) for some $\frac{1}{2} < h < \frac{3}{5}$, $t^{-h}\phi(t)$ is $\downarrow$ but $\psi(t) = t^{-\frac{1}{2}}\phi(t)$ is $\uparrow$ in t, and (iii) the Kolmogorov–Petrovski–Erdös criterion holds, i.e., $\int_n^\infty t^{-1}\psi(t) \exp\{-\frac{1}{2}k^2\psi^2(t)\}\, dt < \infty$, $\forall k \geqq 1$ and $n \geqq 1$. Finally, we let $\nu_k(t) = \frac{1}{2}k^2\psi^2(t) - \log \psi(t) - \log\log t$, $k \geqq 1$, $t \geqq 1$, and assume that for every $\eta > 0$, there exists an $\varepsilon > 0$, such that

(2.9) $$|\nu_k(n)/\nu_1(n) - 1| < \eta \quad \text{for every} \quad 1 \leqq k \leqq 1 + \varepsilon , \quad \text{uniformly in} \quad n .$$

Let then $P_n(\psi) = P\{m^{\frac{1}{2}}|Z_m - \theta| > \sigma\psi(m)$ for some $m \geqq n\}$, and on replacing $|Z_m - \theta|$ by $(Z_m - \theta)$ and $(\theta - Z_m)$, we define the corresponding probabilities by $P_n^+(\psi)$ and $P_n^-(\psi)$, respectively. Then, our main theorem may be presented as follows.

THEOREM 1. *Under* (2.1)—(2.7) *and the conditions on* (ψ, ν) *stated above,*

(2.10) $$\lim_{n\to\infty} \{[\log P_n(\psi)]/\nu_1(n)\} = -1 ,$$

and, if, in addition, $\lim_{n\to\infty} (\log\log n)/\psi^2(n) = 0$, *then*

(2.11) $$\lim_{n\to\infty} \{[\log P_n(\psi)]/\psi^2(n)\} = -\tfrac{1}{2} .$$

The same results hold for $\{P_n^+(\psi)\}$ ***and*** $\{P_n^-(\psi)\}$.

3. Some useful lemmas. We consider the following.

LEMMA 3.1 [Sethuraman (1964)]. *For every Borel $B \subset E^p$ and $\varepsilon > 0$, there exists a $\rho = \rho(B, \varepsilon)$, such that $0 < \rho(B, \varepsilon) \leqq \rho(E^p, \varepsilon) < 1$, and*

$$\lim_{n\to\infty} [n^{-1} \log p \{\sup_{x\in B} |F_n(x) - F(x)| > \varepsilon\}] = \log \rho \, . \tag{3.1}$$

Let us now define $Z_n^* = n^{\frac{1}{2}}[L(x_0, F_n(x_0)) - \theta]/\sigma$, where x_0 and σ are defined by (2.1) and (2.8).

LEMMA 3.2. *Under the hypothesis of Theorem* 1,

$$\lim_{n\to\infty} [\{\nu_1(n)\}^{-1} \log P\{Z_m^* > \psi(m) \ \textit{for some} \ m > n\}] = -1 \, , \tag{3.2}$$

and in (3.2), *Z_m^* may also be replaced by $-Z_m^*$ or $|Z_m^*|$.*

PROOF. Note that $Z_n^* = \gamma_n\{n^{\frac{1}{2}}[F_n(x_0) - F(x_0)]/(p_0(1-p_0))^{\frac{1}{2}}\} = \gamma_n U_n$, say, where $\gamma_n = L_{01}(x_0, uF_n(x_0) + (1-u)F(x_0))/L_{01}(x_0, F(x_0))$, $0 < u < 1$, and U_n is the standardized form of a binomial random variable. Thus, for every $\varepsilon > 0$, we have

$$\begin{aligned} &P\{Z_m^* \geqq (1+\varepsilon)\psi(m) \ \text{for some} \ m \geqq n\} \\ &\quad \leqq P\{Z_m^* \geqq (1+\varepsilon)\psi(m), \ \text{for some} \ m \geqq n, \gamma_m \leqq 1 + \tfrac{1}{2}\varepsilon, \forall\, m \geqq n\} \\ &\qquad + P\{\gamma_m > 1 + \tfrac{1}{2}\varepsilon, \ \text{for some} \ m \geqq n\} \\ &\quad \leqq P\{U_m \geqq (1+\varepsilon')\psi(m) \ \text{for some} \ m \geqq n\} \\ &\qquad + P\{\gamma_m > 1 + \tfrac{1}{2}\varepsilon \ \text{for some} \ m \geqq n\} \, , \end{aligned} \tag{3.3}$$

where $1 < (1+\varepsilon)/(1+\frac{1}{2}\varepsilon) = 1 + \varepsilon' < 1 + \varepsilon/2$. On the other hand,

$$\begin{aligned} &P\{Z_m^* > \psi(m) \ \text{for some} \ m \geqq n\} \\ &\qquad \geqq P\{Z_m^* > \psi(m) \ \text{for some} \ m \geqq n, \gamma_m \geqq 1 - \tfrac{1}{2}\varepsilon, \forall\, m \geqq n\} \\ &\qquad \leqq P\{U_m \geqq (1+\varepsilon'')\psi(m) \ \text{for some} \ m \geqq n\} \\ &\qquad\quad - P\{\gamma_m < 1 - \tfrac{1}{2}\varepsilon \ \text{for some} \ m \geqq n\} \, , \end{aligned} \tag{3.4}$$

where $1 < (1 - \frac{1}{2}\varepsilon)^{-1} = 1 + \varepsilon'' < 1 + \frac{1}{2}\varepsilon(1+\varepsilon)$ for every $0 < \varepsilon < 1$. Further, by Theorem 1.4 of Strassen (1967), we obtain that as $n \to \infty$,

$$\begin{aligned} &P\{U_m \geqq (1+\varepsilon')\psi(m) \ \text{for some} \ m \geqq n\} \\ &\qquad \sim (1+\varepsilon')(2\pi)^{-\frac{1}{2}} \textstyle\int_{n}^{\infty} \psi'(t) t^{-\frac{1}{2}} \exp\{-\frac{1}{2}(1+\varepsilon')\psi^2(t)\}\, dt \\ &\qquad = I_n(1+\varepsilon') \, , \quad \text{say,} \end{aligned} \tag{3.5}$$

and by Lemma 3.1 of Sen (1973b),

$$\lim_{n\to\infty} \{[\log I_n(1+\varepsilon')]/\nu_{1+\varepsilon'}(n)\} = -1 \, . \tag{3.6}$$

Finally, by virtue of the assumed continuity of L_{01} (in A_δ^*), for every $\eta > 0$ (sufficiently small), there exists a $\delta > 0$, such that $|L_{01}(x_0, y) - L_{01}(x_0, F(x_0))| < \eta$ whenever $|y - F(x_0)| < \delta$. Since $L_{01}(x_0, F(x_0)) = \xi_0 \neq 0$, η can always be so chosen that $\eta/|\xi_0| < \frac{1}{2}\varepsilon$. Hence, $\gamma_m < 1 + \frac{1}{2}\varepsilon$ whenever $|F_m(x_0) - F(x_0)| < \delta$.

Thus,

$$(3.7)\quad \begin{aligned} P\{\gamma_m > 1 + \tfrac{1}{2}\varepsilon \ \text{ for some } \ m \geqq n\} \\ &\leqq P\{|F_m(x_0) - F(x_0)| > \delta \ \text{ for some } \ m \geqq n\} \\ &\leqq C_\delta[\rho(\delta)]^n, \qquad 0 < \rho(\delta) < 1, 0 < C_\delta < \infty, \\ &= I_n(1 + \varepsilon')[o(1)], \end{aligned}$$

where the last step follows from the fact that $\nu_k(n)$ is bounded by $n^{\frac{1}{2}}$ for large n, while $[\rho(\delta)]^n = \exp\{n \log \rho(\delta)\}$ decreases exponentially. (3.1) then follows from (2.9) and (3.3) through (3.7). □

Note that in (2.3), $k_2 \geqq k_1 \geqq 1$. We write $d_n = \max(\log n, \psi^2(n))$, and let

$$(3.8)\quad B_n = B_n(x_0, \alpha) = \{x : |F(x) - F(x_0)| \leqq cn^{-\alpha} d_n\}, \qquad \alpha = (2k_2)^{-1}(< \tfrac{1}{2}) \ \text{and} \ c > 0,$$

$$(3.9)\quad G_n^* = \sup\{n^{\frac{1}{2}}|L(x, F_n(x)) - L(x, F(x)) - L(x_0, F_n(x_0)) + L(x_0, F(x_0))| : x \in B_n\}.$$

LEMMA 3.3. *Under the hypothesis of Theorem* 1, *for every* $\varepsilon > 0$, *as* $n \to \infty$,

$$(3.10)\quad P\{G_m^* > \varepsilon\psi(m) \ \textit{for some} \ m \geqq n\} = o(\exp\{-\psi^2(n)\}).$$

PROOF. On writing $\gamma_n(x) = L_{01}(x, uF_n(x) + (1 - u)F(x))$, $0 < u < 1$, $x \in B_n$, we have

$$(3.11)\quad \begin{aligned} G_n^* &= \sup\{n^{\frac{1}{2}}|\gamma_n(x)\{F_n(x) - F(x) - F_n(x_0) + F(x_0)\} \\ &\quad + [\gamma_n(x) - \gamma_n(x_0)]\{F_n(x_0) - F(x_0)\}| : x \in B_n\} \\ &\leqq \sup\{n^{\frac{1}{2}}|F_n(x) - F(x) - F_n(x_0) \\ &\quad + F(x_0)| : x \in B_n\} \sup\{|\gamma_n(x)| : x \in B_n\} \\ &\quad + n^{\frac{1}{2}}|L(x_0, F_n(x_0)) \\ &\quad - L(x_0, F(x_0))|[\sup\{|\gamma_n(x)/\gamma_n(x_0) - 1| : x \in B_n\}]. \end{aligned}$$

By virtue of the assumed continuity of $L_{01}(x, y)$ [for $(x, y) \in A_\delta^*$], (2.6)—(2.7), Lemma 3.1 and Lemma 3.2, on denoting by

$$(3.12)\quad G_n = \sup\{n^{\frac{1}{2}}|F_n(x) - F_n(x_0) - F(x) + F(x_0)| : x \in B_n\},$$

it suffices to show that for every $\varepsilon > 0$, as $n \to \infty$,

$$(3.13)\quad P\{G_m > \varepsilon\psi(m) \ \text{for some} \ m \geqq n\} = o(\exp\{-\psi^2(n)\}).$$

The proof of (3.13) follows along the line of the proof of Lemma 1 of Bahadur (1966) with modifications as in Theorem 3.2 of Sen (1973a), and hence, for intended brevity, the details are omitted.

Let us now define $Z_n^{(1)} = \sup\{L(x, F_n(x)) : x \in B_n\}$. Note that, $P\{m^{\frac{1}{2}}(Z_m^{(1)} - \theta) > \sigma\psi(m)$ for some $m \geqq n\} \leqq P\{\sup[L(x, F_n(x)) - L(x_0, F(x_0)) : x \in B_m] > m^{-\frac{1}{2}}\sigma\psi(m)$ for some $m \geqq n\} = P\{Z_m^* + G_m^{*'} > \sigma\psi(m)$ for some $m \geqq n\}$, where G_m^* is defined by (3.9). Hence, by Lemma 3.2 and Lemma 3.3, we arrive at the following.

LEMMA 3.4. *Under the hypothesis of Theorem* 1,

$$\lim_{n\to\infty}[\{\nu_1(n)\}^{-1}\log P\{m^{\frac{1}{2}}|Z_m^{(1)}-\theta|>\sigma\psi(m) \text{ for some } m\geqq n\}]=-1\,, \tag{3.14}$$

and in (3.14), $|Z_m^{(1)}-\theta|$ *may be replaced by* $(Z_m^{(1)}-\theta)$ *or by* $(\theta-Z_m^{(1)})$.

Finally, let $B_n(\delta)=\{x: |x-x_0|\leqq\delta$ *but* $x\notin B_n(x_0,\alpha)\}$ *and* $H_n^*=\sup\{n^{\frac{1}{2}}|L(x, F_n(x))-L(x,F(x)): x\in B_n(\delta)\}$, *where* δ *satisfy* (2.3). *Also, let* $\{a_n\}$ *be any sequence of positive numbers. Then, by virtue of* (2.5)—(2.7) *and Theorem* 1—*m of Kiefer* (1961) *we obtain that for every* $n(\geqq 1)$,

$$P\{H_n^*\geqq a_n\}\leqq c_1\exp\{-c_2a_n^2\}\,, \tag{3.15}$$

where c_1 *and* c_2 *are both finite positive numbers, and for large* n, c_2 *may be replaced by* $(2-\varepsilon)/\xi_0^2$, *for some* $\varepsilon>0$.

4. Proof of the main theorem. We prove (2.10)—(2.11) only for $\{P_n^+(\psi)\}$; the proof for $\{P_n^-(\psi)\}$ follows on parallel lines, while by noting that $P_n^+(\psi)\leqq P_n(\psi)\leqq P_n^+(\psi)+P_n^-(\psi)$, for all n, the proof for $\{P_n(\psi)\}$ follows immediately.

Note that, by definition, $P_n^+(\psi)=P\{m^{\frac{1}{2}}(Z_m-\theta)>\sigma\psi(m)$ for some $m\geqq n\}\geqq P\{Z_m^*>\psi(m)$ for some $m\geqq n\}$, as $Z_m\geqq L(x_0, F_n(x_0))=\theta+\sigma n^{-\frac{1}{2}}Z_m^*$. Hence, by (4.1) and Lemma 3.2, we obtain that

$$\liminf_n\{[\log P_n^+(\psi)]/\nu_1(n)\}\geqq -1\,. \tag{4.1}$$

Note that by virtue of (2.6), for every $(x,y)\in A_\delta^*$,

$$|L(x,y)-L(x,F(x))|\leqq g(x)|y-F(x)|\,, \tag{4.2}$$

so that whenever $\sup_x|F_n(x)-F(x)|<\delta$,

$$|L(x,F_n(x))-L(x,F(x))|\leqq g(x)|F_n(x)-F(x)|\,. \tag{4.3}$$

As such, by a straightforward generalization of Theorem 2.1 of Sen (1973a) [under our (2.6)], it follows that for every $\varepsilon>0$, there exist positive constants $C(<\infty)$, $\rho^*(\varepsilon): 0<\rho^*(\varepsilon)<1$ and an integer $n_0(\varepsilon)$, such that for $n\geqq n_0(\varepsilon)$,

$$P\{\sup_{x\in\tilde{A}_\delta}|L(x,F_n(x))-L(x,F(x))|>\tfrac{1}{2}\varepsilon\}\leqq C[\rho^*(\varepsilon)]^n\,, \tag{4.4}$$

where $\tilde{A}_\delta=\{x: x\in A$ but $|x-x_0|>\delta\}$. As a result,

$$\begin{aligned} &P\{\sup_{x\in\tilde{A}_\delta}L(x,F_m(x))>\theta-\tfrac{1}{2}\varepsilon \text{ for some } m\geqq n\}\\ &\quad\leqq \textstyle\sum_{m=n}^\infty P\{\sup_{x\in\tilde{A}_\delta}L(x,F_m(x))>\theta-\tfrac{1}{2}\varepsilon\}\\ &\quad= \textstyle\sum_{m=n}^\infty P\{\sup_{x\in\tilde{A}_\delta}[L(x,F(x))\\ &\qquad\qquad+\{L(x,F_m(x))-L(x,F(x))\}]>\theta-\tfrac{1}{2}\varepsilon\}\\ &\quad\leqq \textstyle\sum_{m=n}^\infty P\{\sup_{x\in\tilde{A}_\delta}\{L(x,F_m(x))-L(x,F(x))\}>\tfrac{1}{2}\varepsilon\}\quad\text{(by (2.2))}\\ &\quad\leqq C^*[\rho^*(\varepsilon)]^n\,,\qquad\text{where}\quad C^*=C\{1-\rho^*(\varepsilon)\}^{-1}\quad(<\infty)\,. \end{aligned} \tag{4.5}$$

Let us then define $Z_n^{(1)}$ as in Section 3, and let

$$\begin{aligned} Z_n^{(2)}&=\sup\{L(x,F_n(x)): x\in B_n(\delta)\}\,,\\ Z_n^{(3)}&=\sup\{L(x,F(x)): x\in\tilde{A}_\delta\}\,, \end{aligned} \tag{4.6}$$

so that $Z_n = \max(Z_n^{(1)}, Z_n^{(2)}, Z_n^{(3)}; B_n(\delta))$ being defined after (3.14). Then, by definition and by (3.8),

$$Z_n^{(2)} \leqq \theta - c_2 n^{-\frac{1}{2}} d_n^{k_2} + \sup\{|L(x, F_n(x)) - L(x, F(x))| : x \in B_n(\delta)\}\,, \tag{4.7}$$

so that by (4.7) and (3.15), we obtain that

$$\begin{aligned} &P\{Z_m^{(2)} \geqq \theta - \tfrac{1}{2}c_2 m^{-\frac{1}{2}} d_m^{k_2} \text{ for some } m \geqq n\} \\ &\quad \leqq P\{\sup[m^{\frac{1}{2}}|L(x, F_m(x)) - L(x, F(x))| : x \in B_m(\delta)] \\ &\qquad \geqq \tfrac{1}{2}c_2 d_m^{k_2} \text{ for some } m \geqq n\} \\ &\quad \leqq \textstyle\sum_{m=n}^{\infty} [c_1 \exp\{-\tfrac{1}{4}c_2^2 d_m^{2k_2}\}] = o(\exp\{-\psi^2(n)\})\,, \qquad \text{as } n \to \infty\,, \end{aligned} \tag{4.8}$$

as $d_n^{2k_2} = [\max\{\log n, \psi^2(n)\}]^{2k_2} \geqq (\log n)^{k_2}[\psi^2(n)]^{k_2}$ and $k_2 \geqq 1$. Finally,

$$\begin{aligned} &P\{Z_m^{(1)} < \theta - \tfrac{1}{2}c_2 m^{-\frac{1}{2}} d_m^{k_2}, \text{ for some } m \geqq n\} \\ &\quad \leqq P\{Z_m^* < -\tfrac{1}{2}c_2 d_m^{k_2}, \text{ for some } m \geqq n\} \\ &\quad = o(\exp\{-\psi^2(n)\})\,, \qquad \text{as } n \to \infty\,, \end{aligned} \tag{4.9}$$

where the last step follows from Lemma 3.2 by noting that $d_n \geqq [\psi^2(n)]$ is large compared to $\psi(n)$. Hence, noting that $\psi^2(n)/n \log \rho^*(\delta) \to 0$, we have

$$\begin{aligned} &P\{Z_m \neq Z_m^{(1)} \text{ for some } m \geqq n\} \\ &\quad \leqq C^*[\rho^*(\varepsilon)]^n + 2[o(\exp\{-\psi^2(n)\})] = o(\exp\{-\psi^2(n)\})\,. \end{aligned} \tag{4.10}$$

Further,

$$\begin{aligned} P_n^+(\psi) \leqq\ & P\{m^{\frac{1}{2}}(Z_m^{(1)} - \theta) > \sigma\psi(m), \text{ for some } m \geqq n\} \\ &+ P\{Z_m \neq Z_m^{(1)}, \text{ for some } m \geqq n\}\,, \end{aligned} \tag{4.11}$$

and hence, by Lemma 3.4 and (4.10), we have

$$\limsup_n [\{\nu_1(n)\}^{-1} \log P_n^+(\psi)] \leqq -1\,, \tag{4.12}$$

which completes the proof of (2.10). (2.11) follows from (2.10) by noting that $\lim_{n\to\infty} (\log\log n)/\psi^2(n) = 0 \Rightarrow \lim_{n\to\infty} \{\nu_1(n)/\psi^2(n)\} = \frac{1}{2}$. □

5. A few applications and concluding remarks. (i) *The law of iterated logarithm.* We let $\psi^2(t) = 2(1+\varepsilon)\log\log t$ for $t \geqq 3$; $\psi^2(t) = 1$, $0 \leqq t < 3$, where $\varepsilon > 0$. Then, by (2.10),

$$\begin{aligned} &\lim_{n\to\infty} P\{m^{\frac{1}{2}}(Z_m - \theta) > \sigma(2(1+\varepsilon)\log\log m)^{\frac{1}{2}} \\ &\qquad \text{for some } m \geqq n\} = 0\,, \quad \forall\, \varepsilon > 0\,. \end{aligned} \tag{5.1}$$

On the other hand, $Z_n \geqq L(x_0, F_n(x_0))$, $\forall\, n \geqq 1$, so that every $\varepsilon > 0$,

$$\begin{aligned} &P\{m^{\frac{1}{2}}(Z_m - \theta) > \sigma(2(1-\varepsilon)\log\log m)^{\frac{1}{2}} \text{ for some } m \geqq n\} \\ &\quad \geqq P\{Z_m^* > (2(1-\varepsilon)\log\log m)^{\frac{1}{2}} \\ &\qquad \text{for some } m \geqq n\} \to 1\,, \qquad \text{as } n \to \infty\,, \end{aligned} \tag{5.2}$$

where the last step follows from the fact that $Z_n^* = \gamma_n U_n$ where $\gamma_n \to 1$ a.s. and $U_n = n^{\frac{1}{2}}[F_n(x_0) - F(x_0)]$ is attracted by the usual law of iterated logarithm.

Hence, on letting $\varepsilon \to 0$, we have

$$(5.3)\qquad P\{\limsup_n n^{\frac{1}{2}}(Z_n - \theta)/[2\sigma^2 \log\log n]^{\frac{1}{2}} = 1\} = 1\,.$$

Similarly, it follows that $P\{\liminf_n n^{\frac{1}{2}}(Z_n - \theta)/[2\sigma^2 \log\log n]^{\frac{1}{2}} = -1\} = 1$.

(ii) *Probability of moderate deviations* (PMD). Here, we let $\psi^2(n) = c^2 \log n$, for some positive c. Then, from (2.10), we have

$$(5.4)\qquad \lim_{n\to\infty} [(\log n)^{-1} \log P\{m^{\frac{1}{2}}(Z_m - \theta)/\sigma > c \log m \ \text{ for some } \ m \geqq n\}] = -\tfrac{1}{2}c^2\,,$$

which is a stronger version of the usual PMD result

$$(5.5)\qquad \lim_{n\to\infty} [(\log n)^{-1} \log P\{n^{\frac{1}{2}}(Z_n - \theta)/\sigma > c \log n\}] = -\tfrac{1}{2}c^2\,.$$

(iii) *Embedding of Wiener processes.* Let us now impose another condition on L, namely, that the first partial derivative L_{01} satisfies a local Lipschitz condition for all (x, y): $|x - x_0| < \delta_1$, $|y - F(x_0)| < \delta_2$, where δ_1, δ_2 are sufficiently small. Specifically, we assume that

$$(5.6)\qquad |L_{01}(x, y) - L_{01}(x_0, p_0)| \leqq K_1|x - x_0|^{d_1} + K_2|y - p_0|^{d_2}\,, \quad \forall\, |x - x_0| < \delta_1,\ |y - p_0| < \delta_2\,,$$

where d_1 and d_2 are positive numbers. Then, by virtue of Lemma 3.1, Lemma 3.2, (3.8), (3.9) and (5.6) that the second term on the rhs of (3.11) is

$$(5.7)\qquad O(n^{-d} \log n) \ \text{ a.s.} \quad \text{as } n \to \infty, \ \text{where } d > 0\,,$$

and by Lemma 1 of Bahadur (1966) along with Theorem 3.2 of Sen (1973b), the first term on the rhs of (3.11) is

$$(5.8)\qquad O(n^{-(4k_2)^{-1}} \log n) \ \text{ a.s.} \qquad \text{as } n \to \infty\,.$$

Thus, under (5.6), for some $0 < d \leqq (4k_2)^{-1}$, $G_n^* = O(n^{-d} \log n)$ a.s. as $n \to \infty$, and as a result,

$$(5.9)\qquad n^{\frac{1}{2}}(Z_n - \theta)/\sigma = Z_n^* + O(n^{-d} \log n) \ \text{ a.s.} \qquad \text{as } n \to \infty\,.$$

Let now $W = \{W(t), t \geqq 0\}$ be a standard Wiener process on $[0, \infty)$. Then, on using Theorem 4.4 of Strassen (1967) along with our Lemma 3.1, (5.6) and following some standard steps, we claim that as $n \to \infty$,

$$(5.10)\qquad n^{\frac{1}{2}} Z_n^* = W(n) + O(n^a \log n) \ \text{ a.s.} \quad \text{for some } \ 0 < a < \tfrac{1}{2}\,.$$

From (5.9) and (5.10), we obtain that as $n \to \infty$,

$$(5.11)\qquad n^{\frac{1}{2}}(Z_n - \theta)/\sigma = n^{-\frac{1}{2}} W(n) + O(n^{-d} \log n) \ \text{ a.s.,}$$

for some $d > 0$. Thus, the main results in Sen (1973a, c), deduced for the particular case of bundle strength of filaments, also hold for general $L(x, F(x))$, $x \in A$, provided (5.6) holds.

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