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ASYMPTOTIC MOMENTS OF RANDOM WALKS WITH APPLICATIONS TO LADDER VARIABLES AND RENEWAL THEORY[1]

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Let $X_1, X_2, \cdots$ be i.i.d. random variables with $EX_1 = 0$, $EX_1^2 = 1$ and let $S_n = X_1 + \cdots + X_n$. In this paper, we study the ladder variable S_N where $N = \inf\{n \geqq 1 : S_n > 0\}$. The well-known result of Spitzer concerning ES_N is extended to the higher moments $ES_N{}^k$. In this connection, we develop an asymptotic expansion of the one-sided moments $E[(n^{-\frac{1}{2}}S_n)^-]^\nu$ related to the central limit theorem. Using a truncation argument involving this asymptotic expansion, we obtain the absolute convergence of Spitzer's series of order $k-2$ under the condition $E|X_1|^k < \infty$, extending earlier results of Rosén, Baum and Katz in connection with ES_N. Some applications of these results to renewal theory are also given.

1. Introduction. Throughout this paper, we shall let $X_1, X_2, \cdots$ be i.i.d. random variables with $EX_1 = 0$, $EX_1^2 = 1$. Setting $S_n = X_1 + \cdots + X_n$, $S_0 = 0$, we define

$$N = \inf\{n \geqq 1 : S_n > 0\}. \tag{1.1}$$

In [10], Spitzer has proved that

$$ES_N = 2^{-\frac{1}{2}} \exp\{\sum_1^\infty n^{-1}(P[S_n \leqq 0] - \tfrac{1}{2})\}. \tag{1.2}$$

Spitzer [10] has shown that the series in (1.2) is indeed convergent, while Rosén [7] has proved that it is in fact absolutely convergent. Later, Baum and Katz [1] have obtained the following stronger result:

$$E|X_1|^{2+\delta} < \infty \quad \text{for some} \quad 0 \leqq \delta < 1 \Rightarrow \sum_1^\infty n^{\delta/2-1}|P[S_n \leqq 0] - \tfrac{1}{2}| < \infty. \tag{1.3}$$

We note that if X_1 is normal, then $ES_N = 2^{-\frac{1}{2}}$ and by using Wald's upper bound for the overshoot (cf. [12], page 172 for the case $k = 1$), it is easy to see that in this case $ES_N{}^k < \infty$ for $k = 1, 2, \cdots$. Letting μ_k denote $ES_N{}^k$ in the case where X_1 is normal, we can write (1.2) for the general case as

$$ES_N = \mu_1 e^{\sigma_0} \tag{1.4}$$

so that the additional factor involving Spitzer's series σ_0 can be regarded as some sort of nonnormal adjustment. In Section 3 below, we shall generalize (1.4) to obtain higher moments of the ladder variable S_N. For example, if $E|X_1|^3 < \infty$,

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then

(1.5) $$ES_N^2 = \{\mu_2 + (3(2)^{\frac{1}{2}})^{-1}EX_1^3 - 2^{\frac{1}{2}} \sum_1^\infty n^{-\frac{1}{2}}[E(n^{-\frac{1}{2}}S_n)^- - (2\pi)^{-\frac{1}{2}}]\}e^{\sigma_0}.$$

Hence the nonnormal adjustment involves EX_1^3, Spitzer's series σ_0 and the series $\sigma_1 = \sum_1^\infty n^{-\frac{1}{2}}[E(n^{-\frac{1}{2}}S_n)^- - (2\pi)^{-\frac{1}{2}}]$, which we shall call Spitzer's series of order one. Setting $\binom{x}{n} = x(x-1)\cdots(x-n+1)/(n!)$ for any real number x, μ_2 is given by

(1.6) $$\mu_2 = 1 - \pi^{-\frac{1}{2}} \sum_1^\infty \{n^{-\frac{1}{2}} - \pi^{\frac{1}{2}}\tbinom{-\frac{1}{2}}{n}(-1)^n\}.$$

Using the convergence rate of $E(n^{-\frac{1}{2}}S_n)^-$ to $(2\pi)^{-\frac{1}{2}}$, which is the mean of the negative part of a standard normal random variable, it will be shown in Section 2 that the series σ_1 is indeed absolutely convergent. In general, if $E|X_1|^k < \infty$, then letting $\alpha_j = EX_1^j$ for $j = 3, \cdots, k$, and letting f_n be the characteristic function of $n^{-\frac{1}{2}}S_n$, we have the following well-known asymptotic expansion of f_n (cf. [3]):

$$e^{t^2/2}f_n(t) = 1 + \sum_{j=1}^{k-2} n^{-j/2}P_j(it) + o(n^{-(k-2)/2})$$

where $P_j(z)$ is a sequence of polynomials in z of the form

(1.7) $$P_j(z) = \sum_{\nu=1}^{j} c_{\nu j} z^{j+2\nu}$$

and the constants $c_{\nu j}$ are polynomials in the moments α_m $(m = 3, \cdots, j+2)$. For example,

$$P_1(z) = (1/3!)\alpha_3 z^3,$$
$$P_2(z) = (1/4!)(\alpha_4 - 3)z^4 + (10/6!)\alpha_3^2 z^6, \quad \text{etc.}$$

Letting $\Phi(x)$ denote the standard normal distribution function, we now introduce the functions $P_j(-\Phi)(x)$ whose Fourier–Stieltjes transforms are $e^{-t^2/2}P_j(it)$:

(1.8) $$P_1(-\Phi)(x) = (-1/3!)\alpha_3\Phi^{(3)}(x)$$
$$P_2(-\Phi)(x) = (1/4!)(\alpha_4 - 3)\Phi^{(4)}(x) + (10/6!)\alpha_3^2\Phi^{(6)}(x), \quad \text{etc.}$$

It will be shown in Section 2 that under certain conditions, for any $\nu = 1, 2, \cdots, k$,

(1.9) $$E[(n^{-\frac{1}{2}}S_n)^-]^\nu = \int_{-\infty}^0 |x|^\nu \, d\Phi(x) + \sum_{j=1}^{k-2} n^{-j/2} \int_{-\infty}^0 |x|^\nu \, dP_j(-\Phi)(x) + o(n^{-(k-2)/2}).$$

As will be shown in Section 2, $\int_{-\infty}^0 |x|^\nu \, dP_\nu(-\Phi)(x) = 0$, and $E|X_1|^k < \infty$ implies that σ_{k-2} converges, where we define Spitzer's series of order ν $(\geqq 2)$ as:

(1.10) $$\sigma_\nu = \sum_{n=1}^\infty n^{\nu/2-1}\{E[(n^{-\frac{1}{2}}S_n)^-]^\nu - [\int_{-\infty}^0 |x|^\nu \, d\Phi(x) + \sum_{j=1}^{\nu-1} n^{-j/2} \int_{-\infty}^0 |x|^\nu \, dP_j(-\Phi)(x)]\}.$$

In fact, we shall prove a stronger result analogous to the Baum–Katz theorem referred to in (1.3).

In Section 3, under the assumption $E|X_1|^k < \infty$, we shall make use of the preceding result to find an analogue of (1.5) for ES_N^{k-1} involving $\sigma_0, \cdots, \sigma_{k-2}$, $\alpha_3, \cdots, \alpha_k$ together with μ_k which has an expression analogous to (1.6). Some applications of our results to renewal theory are given in Section 4.

2. Asymptotic moments and the absolute convergence of Spitzer's series of order h. Throughout this section, we shall set $\beta_r = E|X_1|^r$. We note that for $r \geqq 2$, $\beta_r \geqq (EX_1^2)^{r/2} = 1$. In [11], von Bahr has proved the following result on the asymptotic absolute moments of random walks: If $\beta_r < \infty$ for some $r \geqq 4$, then

$$(2.1) \quad \begin{aligned} |E|n^{-\frac{1}{2}}S_n|^\nu - \textstyle\int_{-\infty}^{\infty} |x|^\nu \, d\Phi(x) - \sum_{j=1}^{[r/2]-1} n^{-j} \int_{-\infty}^{\infty} |x|^\nu \, dP_{2j}(-\Phi)(x)| \\ \leqq C_r\{\beta_r{}^3 n^{-(r-2)/2} + \beta_r{}^{3(\nu+1)/r} n^{-(\nu+1)/2} + \beta_r{}^{3(\nu+r)/r} n^{-(\nu+r)/2}\} \end{aligned}$$

for every $0 < \nu \leqq r$, where C_r is a finite constant depending only on r. We shall need an expression analogous to (2.1) for $E[(n^{-\frac{1}{2}}S_n)^-]^\nu$. However, because of our truncation argument in the proof of the absolute convergence of Spitzer's series (see Theorem 1), we want the coefficient of $n^{-(r-2)/2}$ in (2.1) to be β_r instead of $\beta_r{}^3$. One such estimate for $E[(n^{-\frac{1}{2}}S_n)^-]^\nu$ is obtained in Lemma 3 below.

LEMMA 1. *Let $f_n(t)$ be the characteristic function of $n^{-\frac{1}{2}}S_n$. Let k be an integer $\geqq 3$ and let $k < r < k+1$. Assume $\beta_k < \infty$ and set*

$$\Delta = |f_n(t) - e^{-t^2/2}(1 + \textstyle\sum_{j=1}^{k-2} n^{-j/2}P_j(it))| \, .$$

(i) *For $|t| \leqq \beta_k{}^{-3/k} n^{\frac{1}{2}}/(8k)$,*

$$(2.2) \quad \Delta \leqq \delta(n, k)\beta_k{}^{3(k-2)/k} n^{-(k-2)/2}(|t|^k + |t|^{3(k-2)})e^{-t^2/4}$$

where $\lim_{n\to\infty} \delta(n, k) = 0$. Furthermore $\Delta = o(t^k)$ as $t \to 0$.

(ii) *If $\beta_r < \infty$, then for $|t| \leqq b_1(r)\beta_r{}^{-3/r} n^{\frac{1}{2}}$,*

$$(2.3) \quad \Delta \leqq c_1(r)\beta_r{}^{3(k-1)/r} n^{-(r-2)/2}(|t|^r + |t|^{3(k-1)})e^{-t^2/4}$$

where $b_1(r)$ and $c_1(r)$ are positive constants depending only on r.

(iii) *If $\beta_r < \infty$, then for $|t| \leqq T_{rn} = b(r)n^{\frac{1}{2}} \min\{\beta_k{}^{-3/k}, \beta_r{}^{-1/(r-2)}\}$,*

$$(2.4) \quad \begin{aligned} \Delta \leqq c(r)e^{-t^2/4}\{\beta_k{}^{3(k-1)/k} n^{-(k-1)/2}(|t|^{k+1} + |t|^{3(k-1)}) \\ + \beta_r n^{-(r-2)/2}(|t|^r + |t|^{r+2(k-2)})\} \, , \end{aligned}$$

where $b(r)$ and $c(r)$ are positive constants depending only on r.

PROOF. (i) and (ii) are well known (cf. [3], [11]). To prove (iii), we shall modify the proof of (i) given in [3] (pages 204–208). First letting $f(t) = Ee^{itX_1}$ and choosing $b(r)$ sufficiently small, we can write for $|t| \leqq T_{rn}$,

$$(2.5) \quad \log f(n^{-\frac{1}{2}}t) = -t^2/(2n) + \textstyle\sum_{j=3}^{k} \lambda_j(itn^{-\frac{1}{2}})^j/(j!) + \theta \, d_r\beta_r(tn^{-\frac{1}{2}})^r \, ,$$

where $|\theta| \leqq 1$, $\lambda_3, \cdots, \lambda_k$ are the semi-invariants of X_1 and $d_r \geqq 1$ is a constant depending only on r (cf. [6], page 199 and [3], page 205). For $|z| \leqq 1$, put

$$(2.6) \quad \begin{aligned} V = \log\{e^{t^2/2}(f_n(tz))^{z^{-2}}\} = \textstyle\sum_{j=1}^{k-2} \lambda_{j+2}(it)^{j+2}(n^{-\frac{1}{2}}z)^j/((j+2)!) \\ + \theta \, d_r\beta_r(n^{-\frac{1}{2}}z)^{r-2}t^r = A + B \, , \quad \text{say.} \end{aligned}$$

By choosing $b(r)$ sufficiently small, we have for $|z| \leqq 1$ and $|t| \leqq T_{rn}$,

$$(2.7) \quad d_r\beta_r(n^{-\frac{1}{2}}|tz|)^{r-2} \leqq \tfrac{1}{8} \qquad \text{and} \qquad k\beta_k{}^{3/k}n^{-\frac{1}{2}}|tz| \leqq \tfrac{1}{8}e^{-\frac{1}{8}} \, .$$

As shown in [3] (page 206), for $\nu = 1, 2, \cdots$,

$$|A|^{\nu} \leqq 3^{\nu}(\beta_k^{1/k}|t|)^{3\nu}|n^{-\frac{1}{2}}z|^{\nu} \sum_{j=0}^{\infty} (n^{-\frac{1}{2}}\nu k\beta_k^{1/k}|tz|)^j/(j!) . \tag{2.8}$$

From (2.7) and (2.8), it then follows that

$$|A| \leqq 3(8k)^{-1}|t|^2 , \qquad |B| \leqq \tfrac{1}{8}|t|^2 . \tag{2.9}$$

Using (2.7), (2.8) and (2.9), we obtain that

$$\begin{aligned} |V|^{k-1}e^{|V|} &\leqq c_1(k)(|A|^{k-1} + |B|^{k-1})e^{|A|+|B|} \\ &\leqq c_2(r)\{(\beta_k^{1/k}|t|)^{3(k-1)}n^{-(k-1)/2} + \beta_r n^{-(r-2)/2}|t|^{r+2(k-2)}\}e^{t^2/4} . \end{aligned} \tag{2.10}$$

We now write

$$\sum_{j=0}^{k-2} V^j/(j!) = 1 + \sum_{\nu=1}^{k-2} P_\nu(it)(n^{-\frac{1}{2}}z)^\nu + \omega(z) . \tag{2.11}$$

By making use of (2.7), (2.8) and (2.9), we find that

$$\begin{aligned} |\omega(z)| &\leqq \sum_{\nu=1}^{k-2} \{3^\nu(\beta_k^{1/k}|t|)^{3\nu}|n^{-\frac{1}{2}}z|^\nu \sum_{j=k-1-\nu}^{\infty} (n^{-\frac{1}{2}}\nu k\beta_k^{1/k}|tz|)^j/(j!) \\ &\qquad + \tbinom{\nu}{1}|B|\,|A|^{\nu-1} + \tbinom{\nu}{2}|B|^2|A|^{\nu-2} + \cdots + |B|^\nu\} \\ &\hspace{20em}\text{(cf. [3], page 207)} \\ &\leqq c_3(r)(n^{-\frac{1}{2}}\beta_k^{1/k}|t|)^{k-1} \sum_{\nu=1}^{k-2} (\beta_k^{1/k}|t|)^{2\nu} \\ &\qquad + c_4(r)\beta_r n^{-(r-2)/2}|t|^r \sum_{\nu=1}^{k-2} |t|^{2(\nu-1)} \\ &\leqq c_5(r)\{n^{-(k-1)/2}\beta_k^{3(k-1)/k}(|t|^{k+1} + |t|^{3(k-1)}) \\ &\qquad + \beta_r n^{-(r-2)/2}(|t|^r + |t|^{r+2(k-2)})\} . \end{aligned} \tag{2.12}$$

The last inequality above follows from $\sum_1^p a^j \leqq p(a + a^{p+1})$. We note that

$$e^V = \sum_{j=0}^{k-2} V^j/(j!) + \theta_1 V^{k-1}e^{|V|} \qquad (|\theta_1| \leqq 1/(k-1)!) . \tag{2.13}$$

Setting $z = 1$ in (2.6), we obtain that $f_n(t) = e^{-t^2/2}e^V$ and the desired conclusion (2.4) follows from (2.10), (2.11), (2.12) and (2.13).

LEMMA 2. *Let $H(x)$ be a function of bounded variation on $(-\infty, \infty)$ such that $\int_{-\infty}^{\infty} |x|^\nu|dH(x)| < \infty$ for some positive even integer ν. Define $h(t) = \int_{-\infty}^{\infty} e^{itx}\,dH(x)$ and $\gamma_j = \int_{-\infty}^{\infty} x^j\,dH(x)$ for $j = 1, \cdots, \nu$. Then*

$$\begin{aligned} &\int_0^\infty x^\nu\,dH(x) - \int_{-\infty}^0 x^\nu\,dH(x) \\ &\quad = (-1)^{\nu/2}(2/\pi)(\nu!) \int_0^\infty (\{\mathscr{I}h(t) + \textstyle\sum_{j=1}^{\nu/2} [(-1)^j\gamma_{2j-1}t^{2j-1}/(2j-1)!]\}/t^{\nu+1})\,dt \end{aligned}$$

where $\mathscr{I}$ and $\mathscr{R}$ below denote the imaginary and real parts respectively.

PROOF. Integration by parts gives $A_\nu = (-1)^{\nu/2}(\pi/2)/(\nu!)$, where

$$A_\nu = \int_0^\infty (\{\sin u + \textstyle\sum_{j=1}^{\nu/2} [(-1)^j u^{2j-1}/(2j-1)!]\}/u^{\nu+1})\,du . \tag{2.14}$$

Letting $\Psi(x) = x^\nu$ if $x > 0$ and $\Psi(x) = -x^\nu$ if $x \leqq 0$, we obtain by using a change of variable $u = xt$ in (2.14) that

$$A_\nu\Psi(x) = \int_0^\infty (\{\sin xt + \textstyle\sum_{j=1}^{\nu/2} [(-1)^j(xt)^{2j-1}/(2j-1)!]\}/t^{\nu+1})\,dt . \tag{2.15}$$

Since $\int_{-\infty}^{\infty} \Psi(x)\,dH(x) = \int_0^\infty x^\nu\,dH(x) - \int_{-\infty}^0 x^\nu\,dH(x)$, the desired conclusion follows easily from (2.15).

LEMMA 3. *Let k be an integer $\geqq 3$ and let $k < r < k+1$, $\nu \in \{1, 2, \cdots, k\}$.*

(i) *There exists a positive constant C_r depending only on r such that if $\beta_r < \infty$, then*

$$(2.16)\qquad \begin{aligned} &|E[(n^{-\frac{1}{2}}S_n)^-]^\nu - \textstyle\int_{-\infty}^0 |x|^\nu\, d\Phi(x) - \sum_{j=1}^{k-2} n^{-j/2} \int_{-\infty}^0 |x|^\nu\, dP_j(-\Phi)(x)| \\ &\quad \leqq C_r\{\beta_k^{3(k-1)/k} n^{-(k-1)/2} + \beta_r n^{-(r-2)/2} + \beta_r^{3(\nu+1)/r} n^{-(\nu+1)/2} \\ &\qquad + \beta_r^{3(\nu+r)/r} n^{-(\nu+r)/2}\}\,. \end{aligned}$$

(ii) *Suppose $\beta_k < \infty$. Then* (1.9) *holds for $\nu = k-2, k-1, k$. If furthermore, $\limsup_{|t|\to\infty} E|e^{itX_1}| < 1$, then* (1.9) *also holds for $\nu = 1, \cdots, k-3$.*

PROOF. We use Lemma 1 (iii) to prove (i). Define

$$(2.17)\qquad \begin{aligned} &F_n(x) = P[n^{-\frac{1}{2}}S_n \leqq x]\,, \qquad F(x) = P[X_1 \leqq x]\,, \\ &G_n(x) = \Phi(x) + \textstyle\sum_{j=1}^{k-2} n^{-j/2} P_j(-\Phi)(x)\,, \qquad g_n(t) = \int_{-\infty}^{\infty} e^{itx}\, dG_n(x)\,, \\ &H_n(x) = F_n(x) - G_n(x)\,, \qquad h_n(t) = f_n(t) - g_n(t)\,. \end{aligned}$$

First consider the case where ν is odd. Then as shown in [11] (page 812),

$$(2.18)\qquad \int_{-\infty}^{\infty} |x|^\nu\, dH_n(x) = (\nu!/\pi)(-1)^{(\nu+1)/2} \int_{-\infty}^{\infty} (\mathscr{R}h_n(t)/|t|^{\nu+1})\, dt\,.$$

Let $T = b(r)\beta_r^{-3/r}\, n^{\frac{1}{2}}$, where $b(r)$ and T_{rn} are as defined in Lemma 1 (iii). Since $\beta_r \geqq 1$ and $3/r > 1/(r-2)$, we have $T_{rn} \geqq T$ and by (2.4),

$$(2.19)\qquad \int_{|t|\leqq T} (|\mathscr{R}h_n(t)|/|t|^{\nu+1})\, dt \leqq c_1(r)\{\beta_k^{3(k-1)/k} n^{-(k-1)/2} + \beta_r n^{-(r-2)/2}\}\,,$$

noting that $r - (\nu+1) > -1$. As shown in [11] (page 815),

$$(2.20)\qquad \int_{|t|>T} (|\mathscr{R}h_n(t)|/|t|^{\nu+1})\, dt \leqq c_2(r)\{\beta_r^{3(\nu+1)/r} n^{-(\nu+1)/2} + \beta_r^{3(\nu+r)/r} n^{-(\nu+r)/2}\}\,.$$

From (2.18), (2.19) and (2.20), it follows that

$$(2.21)\qquad \begin{aligned} &|E|n^{-\frac{1}{2}}S_n|^\nu - \textstyle\int_{-\infty}^{\infty} |x|^\nu\, d\Phi(x) - \sum_{j=1}^{k-2} n^{-j/2} \int_{-\infty}^{\infty} |x|^\nu\, dP_j(-\Phi)(x)| \\ &\quad \leqq C_r\{\beta_k^{3(k-1)/k} n^{-(k-1)/2} + \beta_r n^{-(r-2)/2} + \beta_r^{3(\nu+1)/r} n^{-(\nu+1)/2} \\ &\qquad + \beta_r^{3(\nu+r)/r} n^{-(\nu+r)/2}\}\,. \end{aligned}$$

It is well known (cf. [11], Theorem 1) that

$$(2.22)\qquad E(n^{-\frac{1}{2}}S_n)^\nu = \int_{-\infty}^{\infty} x^\nu\, d\Phi(x) + \textstyle\sum_{j=1}^{k-2} n^{-j/2} \int_{-\infty}^{\infty} x^\nu\, dP_j(-\Phi)(x)\,.$$

Since $(a^-)^\nu = \frac{1}{2}(|a|^\nu - a^\nu)$, (2.16) follows from (2.21) and (2.22).

Now consider the case where ν is even. From Lemma 1, $h_n(t) = o(t^k)$ as $t \to 0$ and so for $j = 0, 1, \cdots, k$,

$$(2.23)\qquad \int_{-\infty}^{\infty} x^j\, dH_n(x) = 0$$

(cf. Lemma 1 (b) of [11]). Hence using Lemma 2 and a similar argument as before, we obtain that

$$(2.24)\qquad \begin{aligned} &|\textstyle\int_0^\infty x^\nu\, dH_n(x) - \int_{-\infty}^0 x^\nu\, dH_n(x)| \\ &\quad = \nu!(2/\pi)|\textstyle\int_0^\infty (\mathscr{I}h_n(t)/t^{\nu+1})\, dt| \\ &\quad \leqq C_r\{\beta_k^{3(k-1)/k} n^{-(k-1)/2} + \beta_r n^{-(r-2)/2} \\ &\qquad + \beta_r^{3(r+1)/r} n^{-(\nu+1)/2} + \beta_r^{3(\nu+r)/r} n^{-(\nu+r)/2}\}\,. \end{aligned}$$

From (2.23) and (2.24), the desired conclusion (2.16) follows easily.

To prove (ii), let $T^* = \beta_k^{-3/k} n^{\frac{1}{2}}/(8k)$, $\theta > 0$ and write

$$\int_{-\infty}^{\infty} (|\mathscr{R} h_n(t)|/|t|^{\nu+1})\, dt = \int_{|t|\leq\theta} + \int_{\theta<|t|<T^*} + \int_{|t|\geq T^*} = I_1 + I_2 + I_3 . \tag{2.25}$$

By Lemma 1 (i), we have $I_2 = o(n^{-(k-2)/2})$. Using a similar argument as in [11] (pages 812–814), it can be shown that $I_1 \leqq \rho(\theta) n^{-(k-2)/2}$ where $\rho(\theta) \to 0$ as $\theta \downarrow 0$. It is also shown in [11] (page 815) that $I_3 = O(n^{-(\nu+1)/2})$ and so $I_3 = o(n^{-(k-2)/2})$ if $\nu = k-2, k-1, k$. In the case where Cramér's condition $\limsup_{|t|\to\infty} E|e^{itX_1}| < 1$ is assumed, it can be shown by a standard argument that $I_3 = o(n^{-(k-2)/2})$ for $\nu = 1, \cdots, k$. Hence we obtain (ii) using (2.25) when ν is odd. When ν is even, a similar analysis of $\int_0^\infty (\mathscr{I} h_n(t)/t^{\nu+1})\, dt$ again gives (ii).

LEMMA 4. *If i, j are positive integers such that $i < j$ and $j - i$ is even, then*

$$\int_{-\infty}^{0} x^i \, d\Phi^{(j)}(x) = 0 . \tag{2.26}$$

Consequently if $\beta_k < \infty$ for some integer $k \geqq 3$, then for $\nu = 1, \cdots, k-2$,

$$\int_{-\infty}^{0} x^\nu \, dP_\nu(-\Phi)(x) = 0 . \tag{2.27}$$

PROOF. Integration by parts shows that the left-hand side of (2.26) is equal to $(-1)^i i!\, \Phi^{(j-i)}(0)$. It is easy to see by induction that $\Phi^{(\nu)}(0) = 0$ if ν is a positive even integer. Hence we obtain (2.26), and (2.27) is an immediate corollary of (1.7) and (2.26).

THEOREM 1. *Let k be an integer $\geqq 3$ and let $0 \leqq \delta < 1$. If $E|X_1|^{k+\delta} < \infty$, then*

$$\sum_1^\infty n^{(k-2+\delta)/2-1} |(\int_{-\infty}^{0} |x|^{k-2} d\Phi(x) + \sum_{j=1}^{k-3} n^{-j/2} \int_{-\infty}^{0} |x|^{k-2} dP_j(-\Phi)(x)) - E[(n^{-\frac{1}{2}} S_n)^-]^{k-2}| < \infty . \tag{2.28}$$

Consequently if $E|X_1|^k < \infty$, then the series σ_{k-2} defined by (1.10) *is absolutely convergent.*

PROOF. Define $Y_i(n) = X_i I_{[|X_i| \leq n^{\frac{1}{2}}]}$, $v_n^2 = \operatorname{Var} Y_1(n)$ and let $X_i(n) = v_n^{-1}(Y_i(n) - EY_i(n))$, $S_n(n) = X_1(n) + \cdots + X_n(n)$. Then $X_1(n), \cdots, X_n(n)$ are i.i.d. with mean 0 and variance 1. Let $\alpha_j(n) = EX_1^j(n)$ for $j = 3, \cdots, k$ and let $\beta_p(n) = E|X_1(n)|^p$ for any $p > 0$. Let $P_j^*(z)$ be defined as in (1.7) with the moments $\alpha_m(n)$ replacing the moments $\alpha_m = EX_1^m$ $(m = 2, \cdots, j+2)$. Set $r = k + \delta + \theta$ where $\theta > 0$ satisfies

$$3\theta(k-2+r)/r < (1-\delta)/2. \tag{2.29}$$

By Lemma 3 (i),

$$\begin{aligned} &|E[(n^{-\frac{1}{2}} S_n(n))^-]^{k-2} - \int_{-\infty}^{0} |x|^{k-2} d\Phi(x) - \sum_{j=1}^{k-2} n^{-j/2} \int_{-\infty}^{0} |x|^{k-2} dP_j^*(-\Phi)(x)| \\ &\quad \leqq C_r\{(\beta_k(n))^{3(k-1)/k} n^{-(k-1)/2} + \beta_r(n) n^{-(r-2)/2} \\ &\qquad + (\beta_r(n))^{3(k-1)/r} n^{-(k-1)/2} + (\beta_r(n))^{3(k-2+r)/r} n^{-(k-2+r)/2}\} . \end{aligned} \tag{2.30}$$

Since $\beta_r(n) \leqq 2^r v_n^{-r} E|Y_1(n)|^r = O(n^{\theta/2})$, it follows from (2.29) that

$$1 = (\beta_2(n))^{\frac{1}{2}} \leqq (\beta_r(n))^{3(k-1)/r} \leqq (\beta_r(n))^{3(k-2+r)/r} = O(n^{(1-\delta)/4}) . \tag{2.31}$$

We note that

$$(2.32)\qquad \begin{aligned}\textstyle\sum_1^\infty \beta_r(n)n^{-(\theta/2)-1} &\leqq c \textstyle\sum_1^\infty n^{-(\theta/2)-1}E|Y_1(n)|^r \\ &= c \int_{|X_1|\geqq 1} |X_1|^r(\textstyle\sum_{|X_1|\leqq n^{\frac12}} n^{-(\theta/2)-1})\, dP \\ &\leqq c'E|X_1|^{k+\delta} < \infty\,.\end{aligned}$$

Since $\int_{-\infty}^0 |x|^{k-2}\, dP_{k-2}^*(-\Phi)(x) = 0$ by Lemma 4 and $\beta_k(n) = O(1)$, we obtain from (2.30), (2.31) and (2.32) that

$$(2.33)\qquad \begin{aligned}\textstyle\sum_1^\infty n^{(k-2+\delta)/2-1}|(\int_{-\infty}^0 |x|^{k-2}\, d\Phi(x) + \sum_{j=1}^{k-3} n^{-j/2} \int_{-\infty}^0 |x|^{k-2}\, dP_j^*(-\Phi)(x)) \\ - E[(n^{-\frac12}S_n(n))^-]^{k-2}| < \infty\,.\end{aligned}$$

Let $q_i(n) = \alpha_i(n) - \alpha_i$ for $i = 3, \cdots, k-1$. We note that

$$(2.34)\qquad \begin{aligned}|q_i(n)| &\leqq |EY_1^i(n) - \alpha_i| + O(EY_1(n)) + O(1 - v_n^2) \\ &\leqq \int_{|X_1|>n^{\frac12}} |X_1|^i\, dP + O(n^{-(k-2+\delta)/2})\,.\end{aligned}$$

Hence for $j = 1, \cdots, k-3$ and $i = 3, \cdots, j+2$, we have

$$(2.35)\qquad \begin{aligned}\textstyle\sum_1^\infty n^{(k-2+\delta)/2-1-j/2}|q_i(n)| &\leqq \int_{|X_1|\geqq 1} |X_1|^i(\textstyle\sum_{|X_1|>n^{\frac12}} n^{(k-2+\delta)/2-1-j/2})\, dP + c_1 \\ &\leqq c_2 E|X_1|^{k+\delta-(j+2-i)} + c_1\,.\end{aligned}$$

By (1.7), we can write for $j = 1, \cdots, k-3$,

$$\int_{-\infty}^0 |x|^{k-2}\, dP_j^*(-\Phi)(x) - \int_{-\infty}^0 |x|^{k-2}\, dP_j(-\Phi)(x) = h_j(q_3(n), \cdots, q_{j+2}(n))$$

where $h_j\,(x_1, \cdots, x_j)$ is a polynomial of degree j. Since $q_i(n) = o(1)$ for $i = 3, \cdots, k-1$, it follows from (2.35) that for $j = 1, \cdots, k-3$,

$$(2.36)\qquad \begin{aligned}\textstyle\sum_1^\infty n^{(k-2+\delta)/2-1-j/2}|\int_{-\infty}^0 |x|^{k-2}\, dP_j^*(-\Phi)(x) - \int_{-\infty}^0 |x|^{k-2}\, dP_j(-\Phi)(x)| \\ < \infty\,.\end{aligned}$$

Let $Z_i(n) = X_i - Y_i(n)$, $S_n' = Y_1(n) + \cdots + Y_n(n)$, $S_n'' = S_n - S_n'$. Using the inequality $a^- - |b| \leqq (a-b)^- \leqq a^- + |b|$ and noting that $n^{\frac12}EY_1(n) = o(1)$ and therefore $E|n^{-\frac12}S_n'|^j = O(1)$ for $j \leqq k$, we obtain that

$$(2.37)\qquad \begin{aligned}\textstyle\sum_1^\infty n^{(k-2+\delta)/2-1}|E[(n^{-\frac12}S_n(n))^-]^{k-2} - E[(n^{-\frac12}S_n')^-]^{k-2}| \\ \leqq c \textstyle\sum_1^\infty n^{(k-2+\delta)/2-1}\{|n^{\frac12}EY_1(n)| + (1 - v_n^2)\} \leqq c'E|X_1|^{k+\delta}\,.\end{aligned}$$

We also observe that

$$(2.38)\qquad |E[(n^{-\frac12}S_n)^-]^{k-2} - E[(n^{-\frac12}S_n')^-]^{k-2}| \leqq Cn^{-(k-2)/2} \textstyle\sum_{j=1}^{k-2} E(|S_n'|^{k-2-j}|S_n''|^j)\,.$$

For $j = 1, \cdots, k-2$, expanding $|S_n''|^{j} = |Z_1(n) + \cdots + Z_n(n)|^j$ and noting that for $\nu = 1, \cdots, j$ and $i_1 \geqq 1, \cdots, i_\nu \geqq 1$ such that $i_1 + \cdots + i_\nu = j$,

$$\begin{aligned}E(|S_n'|^{k-2-j}|Z_1(n)|^{i_1} \cdots |Z_\nu(n)|^{i_\nu}) &= E|S'_{n-\nu}|^{k-2-j}E|Z_1(n)|^{i_1} \cdots E|Z_1(n)|^{i_\nu} \\ &= O(n^{(k-2-j)/2})O(n^{-(k+\delta-i_1)/2}) \cdots O(n^{-(k+\delta-i_\nu)/2}) \\ &= O(n^{-((k+\delta)\nu+2-k)/2})\,,\end{aligned}$$

we obtain that

$$(2.39)\qquad \begin{aligned}E(|S_n'|^{k-2-j}|S_n''|^j) &= nE|S'_{n-1}|^{k-2-j}E|Z_1(n)|^j + \textstyle\sum_{\nu=2}^j O(n^{\nu-\frac12((k+\delta)\nu+2-k)}) \\ &= O(n^{(k-j)/2})E|Z_1(n)|^j + O(n^{2-\frac12(2(k+\delta)+2-k)})\,.\end{aligned}$$

It then follows from (2.38) and (2.39) that

$$\begin{aligned}(2.40)\quad &\sum_1^\infty n^{(k-2+\delta)/2-1}|E[(n^{-\frac{1}{2}}S_n)^-]^{k-2} - E[(n^{-\frac{1}{2}}S_n')^-]^{k-2}| \\ &\leqq c_1 \sum_{n=1}^\infty n^{\delta/2-1} \sum_{j=1}^{k-2} n^{(k-j)/2} \int_{|X_1|>n^{\frac{1}{2}}} |X_1|^j \, dP + c_2 \\ &\leqq c_3 E|X_1|^{k+\delta} + c_2 .\end{aligned}$$

From (2.33), (2.36), (2.37) and (2.40), the desired conclusion (2.28) follows.

3. Moments of the ladder variable. In this section, we shall find the kth moment of the ladder variable S_N defined in Section 1. We shall need an asymptotic expansion of the function $\sum_{n=1}^\infty n^{k+\frac{1}{2}} t^n$ as $t \uparrow 1$ which is given in the following lemma.

LEMMA 5. *Let $0 < \alpha < 1$. For $i = 0, 1, 2, \cdots$, the following asymptotic expansion holds as $n \to \infty$:*

$$\begin{aligned}(3.1)\quad &\Gamma(\alpha + i + n)/(n!) \\ &= n^{\alpha+i-1}\{1 + C_1^{(i)} n^{-1} + C_2^{(i)} n^{-2} + \cdots + C_i^{(i)} n^{-i} + O(n^{-i-1})\}\end{aligned}$$

where $C_1^{(i)}, \cdots, C_i^{(i)}$ are constants depending only on i and α. Consequently, given $h = -1, 0, 1, 2, \cdots$, if we set $\xi_{h,\alpha}(h+1) = \Gamma(\alpha + h + 1)$ and define $\xi_{h,\alpha}(i)$ for $i = h, h-1, \cdots, 0$ inductively by

$$\begin{aligned}(3.2)\quad &\xi_{h,\alpha}(h+1) C_{h+1-i}^{(h+1)}/\Gamma(\alpha + h + 1) + \xi_{h,\alpha}(h) C_{h-i}^{(h)}/\Gamma(\alpha + h) + \cdots \\ &+ \xi_{h,\alpha}(i)/\Gamma(\alpha + i) = 0 ,\end{aligned}$$

then the series in the expression

$$\begin{aligned}(3.3)\quad \zeta_{h,\alpha} = &-\sum_{i=0}^{h+1} \xi_{h,\alpha}(i) \\ &+ \sum_{n=1}^\infty \{n^{h+\alpha} - (\sum_{i=0}^{h+1} \xi_{h,\alpha}(i)\Gamma(\alpha + i + n)/(n!\,\Gamma(\alpha + i)))\}\end{aligned}$$

converges absolutely, and as $t \uparrow 1$,

$$(3.4)\quad \sum_{n=1}^\infty n^{h+\alpha} t^n = \sum_{i=0}^{h+1} \xi_{h,\alpha}(i)(1-t)^{-\alpha-i} + \zeta_{h,\alpha} + o(1) .$$

PROOF. The relation (3.1) can be seen from the well-known asymptotic expansion

$$\Gamma(y) = (2\pi)^{\frac{1}{2}} e^{-y} y^{y-\frac{1}{2}} \exp\{y^{-1}B_2/(1.2) + y^{-3}B_4/(3.4) + \cdots + O(y^{-(2i+1)})\}$$

as $y \to \infty$, where $B_2, B_4, \cdots$ are the Bernoulli numbers (cf. [4], page 530). From (3.1) and (3.2), it follows that

$$n^{h+\alpha} - \sum_{i=0}^{h+1} \xi_{h,\alpha}(i)\Gamma(\alpha + i + n)/(n!\,\Gamma(\alpha + i)) = O(n^{-2+\alpha}) .$$

Hence the series in (3.3) is absolutely convergent. From the relation

$$(1-t)^{-\alpha-i} = \sum_{n=0}^\infty \binom{-\alpha-i}{n}(-t)^n = \sum_{n=0}^\infty t^n \Gamma(\alpha + i + n)/(n!\,\Gamma(\alpha + i)) ,$$

$i = 0, 1, \cdots, h+1$, we obtain that as $t \uparrow 1$,

$$\sum_{n=1}^\infty n^{h+\alpha} t^n - \sum_{i=0}^{h+1} \xi_{h,\alpha}(i)(1-t)^{-\alpha-i} \to \zeta_{h,\alpha} .$$

In our subsequent applications of Lemma 5, we shall set $\alpha = \frac{1}{2}$. Let ν be a positive odd integer, say $\nu = 2k+1$. In this case, we set $\zeta_\nu = \zeta_{k-1,\frac{1}{2}}$; $a_\nu(2i+1) = \xi_{k-1,\frac{1}{2}}(i)$ for $i = 0, \cdots, k$ and $a_\nu(j) = 0$ for even j, where $\zeta_{h,\alpha}$ and $\xi_{h,\alpha}(i)$ are as defined in Lemma 5. Thus by Lemma 5, we have the following asymptotic expansion as $t \uparrow 1$:

$$(3.5) \quad \begin{aligned} \sum_{n=1}^\infty n^{(\nu/2)-1}t^n &= \sum_{n=1}^\infty n^{(k-1)+\frac{1}{2}}t^n = \sum_{i=0}^k \xi_{k-1,\frac{1}{2}}(i)(1-t)^{-\frac{1}{2}-i} + \zeta_\nu + o(1) \\ &= \sum_{j=1}^\nu a_\nu(j)(1-t)^{-j/2} + \zeta_\nu + o(1)\,. \end{aligned}$$

In particular, for $\nu = 1$,

$$\sum_{n=1}^\infty n^{-\frac{1}{2}}\, t^n = \pi^{\frac{1}{2}}(1-t)^{-\frac{1}{2}} - \pi^{\frac{1}{2}} + \sum_1^\infty (n^{-\frac{1}{2}} - \pi^{\frac{1}{2}}\tbinom{-\frac{1}{2}}{n}(-1)^n) + o(1)\,.$$

In the case where ν is a positive even integer, it is obvious that we can still write

$$(3.6) \quad \sum_{n=1}^\infty n^{(\nu/2)-1}t^n = \sum_{j=1}^\nu a_\nu(j)(1-t)^{-j/2} + \zeta_\nu$$

where $a_\nu(j)$ are constants such that $a_\nu(j) = 0$ for odd j. For example, $\sum_1^\infty t^n = (1-t)^{-1} - 1$, $\sum_1^\infty nt^n = (1-t)^{-2} - (1-t)^{-1}$, etc.

LEMMA 6. *Suppose* $E|X_1|^{k+1} < \infty$ *for some integer* $k \geqq 2$. *For* $\nu = 1, \cdots, k$ *and* $j = 1, \cdots, k-1$, *let* $b_\nu(0) = \int_{-\infty}^0 |x|^\nu\, d\Phi(x)$, $b_\nu(j) = \int_{-\infty}^0 |x|^\nu\, dP_j(-\Phi)(x)$ *and define* $a_\nu(j)$ *and* ζ_ν *as in* (3.5) *and* (3.6) *and* σ_ν *as in* (1.10). *Set*

$$(3.7) \quad \begin{aligned} g_\nu &= (-1)^\nu\{\sigma_\nu + b_\nu(0)\zeta_\nu + \textstyle\sum_{j=1}^{\nu-1} b_\nu(j)\zeta_{\nu-j}\}\,, & \nu &= 1, \cdots, k-1\,, \\ \lambda_\nu(\nu) &= (-1)^\nu b_\nu(0)a_\nu(\nu)\,, & \nu &= 1, \cdots, k\,, \\ \lambda_\nu(j) &= (-1)^\nu\{b_\nu(0)a_\nu(j) + b_\nu(1)a_{\nu-1}(j) + \cdots + b_\nu(\nu-j)a_j(j)\}\,, & & \\ & & j &= 1, \cdots, \nu-1\,. \end{aligned}$$

Then as $t \uparrow 1$, *we have the following asymptotic expansions*:

$$(3.8) \quad \begin{aligned} \sum_1^\infty (t^n/n)ES_n{}^k I_{[S_n \leqq 0]} &= \lambda_k(k)(1-t)^{-k/2} + \lambda_k(k-1)(1-t)^{-(k-1)/2} + \cdots \\ &\quad + \lambda_k(1)(1-t)^{-\frac{1}{2}} + o((1-t)^{-\frac{1}{2}})\,; \end{aligned}$$

$$(3.9) \quad \sum_1^\infty (t^n/n)ES_n I_{[S_n \leqq 0]} = \lambda_1(1)(1-t)^{-\frac{1}{2}} + g_1 + o(1)\,,$$

and in general, for $\nu = 2, \cdots, k-1$,

$$(3.10) \quad \begin{aligned} \sum_1^\infty (t^n/n)ES_n{}^\nu I_{[S_n \leqq 0]} &= \lambda_\nu(\nu)(1-t)^{-\nu/2} + \lambda_\nu(\nu-1)(1-t)^{-(\nu-1)/2} + \cdots \\ &\quad + \lambda_\nu(1)(1-t)^{-\frac{1}{2}} + g_\nu + o(1)\,. \end{aligned}$$

PROOF. For $\nu = 1, \cdots, k-1$, define $r_n(\nu)$ by

$$(3.11) \quad \begin{aligned} n^{(\nu/2)-1}&E[(n^{-\frac{1}{2}}S_n)^-]^\nu \\ &= n^{(\nu/2)-1} \int_{-\infty}^0 |x|^\nu\, d\Phi(x) \\ &\quad + \sum_{j=1}^{\nu-1} n^{(\nu-j)/2-1} \int_{-\infty}^0 |x|^\nu\, dP_j(-\Phi)(x) + n^{(\nu/2)-1}r_n(\nu)\,. \end{aligned}$$

By (3.5), (3.6) and (3.11), we have

$$\begin{aligned} \sum_1^\infty (t^n/n)ES_n{}^\nu I_{[S_n \leqq 0]} &= (-1)^\nu \sum_1^\infty n^{(\nu/2)-1}E[(n^{-\frac{1}{2}}S_n)^-]^\nu t^n \\ &= \lambda_\nu(\nu)(1-t)^{-\nu/2} + \cdots + \lambda_\nu(1)(1-t)^{-\frac{1}{2}} + b_\nu(0)\zeta_\nu \\ &\quad + \sum_{j=1}^{\nu-1} b_\nu(j)\zeta_{\nu-j} + \sum_1^\infty n^{(\nu/2)-1}r_n(\nu)t^n + o(1)\,. \end{aligned}$$

Since $E|X_1|^{\nu+2} < \infty$ for $\nu = 1, \cdots, k-1$, it follows from Theorem 1 that $\sum_1^\infty n^{(\nu/2)-1} r_n(\nu) t^n \to \sigma_\nu$ as $t \uparrow 1$. Thus we have proved (3.9) and (3.10). By Lemma 3 (ii), $E|X_1|^{k+1} < \infty$ implies that

$$E[(n^{-\frac{1}{2}}S_n)^-]^k = \int_{-\infty}^0 |x|^k \, d\Phi(x) + \sum_{j=1}^{(k+1)-2} n^{-j/2} \int_{-\infty}^0 |x|^k \, dP_j(-\Phi)(x) + o(n^{-((k+1)-2)/2}) . \tag{3.12}$$

Noting that $\sum_1^\infty r_n t^n = o((1-t)^{-\frac{1}{2}})$ as $t \uparrow 1$ if $r_n = o(n^{-\frac{1}{2}})$, we obtain (3.8) by using (3.5), (3.6) and (3.12).

LEMMA 7. *Let Z be a random variable such that $E|Z|^k < \infty$ for some positive integer k. Then there exists a random variable Y whose distribution has support consisting of at most $([k/2]+1)$ points and $EY^i = EZ^i$ for $i = 1, \cdots, k$.*

PROOF. We shall assume that the distribution of Z has more than $([k/2]+1)$ points for otherwise we can simply set $Y = Z$. Letting $m_0 = 1$, $m_i = EZ^i$ for $i = 1, \cdots, k$, it follows that the Hankel determinants $\Delta_\nu > 0$ for $\nu = 1, \cdots, [k/2]$, where

$$\Delta_\nu = |m_{i+j}|_{i=0,\cdots,\nu;j=0,\cdots,\nu} \tag{3.13}$$

(cf. [8], page 5). If k is odd, we take m_{k+1} as the unique number satisfying $\Delta_{[k/2]+1} = 0$, where Δ_ν is defined as in (3.8). If k is even, we take m_{k+1} and m_{k+2} such that $\Delta_{[k/2]+1} = 0$. Then the reduced moment problem

$$m_i = \int_{-\infty}^\infty y^i \, dF(y) , \qquad i = 0, 1, \cdots, 2([k/2]+1)$$

has a solution F whose support has $([k/2]+1)$ points (cf. [8], pages 5 and 28–32).

LEMMA 8. *Let $Q_n = \gamma_n(n)x^n + \cdots + \gamma_n(1)x + y_n$ for $n = 1, 2, \cdots$. Suppose $u_k(\theta)$, $k \geqq 1$, is of class $C^k[0, \theta_0]$ for some $\theta_0 > 0$ such that $\lim_{\theta \downarrow 0} \theta^{-k} u_k(\theta) = 0$, and*

$$P_k = \frac{\partial^k}{\partial \theta^k} \exp\{u_k(\theta) + \textstyle\sum_{n=1}^k (\theta^n/n!) Q_n\}\Big|_{\theta=0}$$

($\partial^k/\partial\theta^k|_{\theta=0}$ above denotes the right-hand derivative evaluated at $\theta = 0$). Then

$$P_k = \textstyle\sum_{d,\omega} p(k; d, \boldsymbol{\omega}) x^d y_1^{\omega_1} \cdots y_k^{\omega_k} \tag{3.14}$$

where $\sum_{d,\omega}$ denotes summation over integers $d = 0, 1, \cdots, k$ and ordered k-tuples $\boldsymbol{\omega} = (\omega_1, \cdots, \omega_k)$ of nonnegative integers such that $d + \sum_{i=1}^k i\omega_i \leqq k$. For fixed d, $\boldsymbol{\omega}$, the coefficient $p(k; d, \boldsymbol{\omega})$ is given by

$$p(k; d, \boldsymbol{\omega}) = k!\left(\textstyle\prod_{i=1}^k (i!)^{\omega_i}(\omega_i!)\right)^{-1} \times \textstyle\sum^{(k;d,\omega)} \left\{\prod_{i=1}^k \prod_{j=1}^i (\gamma_i(j)/(i!))^{t_i(j)}/(t_i(j)!)\right\} \tag{3.15}$$

where $\sum^{(k;d,\omega)}$ denotes summation over all ordered tuples $(t_i(j))_{i=1,\cdots,k;j=1,\cdots,i}$ of nonnegative integers satisfying the following two conditions:

$$\textstyle\sum_{i=1}^k \sum_{j=1}^i j t_i(j) = d , \tag{3.16a}$$

$$\textstyle\sum_{i=1}^k \sum_{j=1}^i i t_i(j) + \sum_{i=1}^k i\omega_i = k . \tag{3.16b}$$

For any tuple $\mathbf{v} = (v_i, \cdots, v_h)$ *of nonnegative integers, we set* $G(\mathbf{v}) = \sum_{i=1}^{h} iv_i$. *Suppose* $\boldsymbol{\omega} = (\omega_1, \cdots, \omega_k)$ *and* $\boldsymbol{\omega}^* = (\omega_1^*, \cdots, \omega_{k-1}^*)$ *are two tuples of nonnegative integers with* $k \geqq G(\boldsymbol{\omega}) \geqq 1$ *and* $G(\boldsymbol{\omega}^*) = G(\boldsymbol{\omega}) - 1$. *Then for any integer* $d \geqq 0$ *such that* $d + G(\boldsymbol{\omega}) \leqq k$,

$$p(k; d, \boldsymbol{\omega}) = k\{\prod_{i=1}^{k-1} (i!)^{\omega_i^*}(\omega_i^*!)\}\{\prod_{i=1}^{k} (i!)^{\omega_i}(\omega_i!)\}^{-1} p(k-1; d, \boldsymbol{\omega}^*) . \tag{3.17}$$

PROOF. We first note that

$$\begin{aligned} &\exp\{u_k(\theta) + \textstyle\sum_{n=1}^{k} (\theta^n/n!)Q_n\} \\ &\quad = 1 + \textstyle\sum_{n=1}^{k} (\theta^n/n!)Q_n \\ &\qquad + \textstyle\sum_{j=2}^{k} (\theta Q_1 + \cdots + (\theta^k/k!)Q_k)^j/(j!) + o(\theta^k) \\ &\quad = 1 + \textstyle\sum_{s=1}^{k} \theta^s \sum_{\alpha_1+\cdots+k\alpha_k=s} \dfrac{(Q_1/1!)^{\alpha_1} \cdots (Q_k/k!)^{\alpha_k}}{(\alpha_1! \cdots \alpha_k!)} + o(\theta^k) \end{aligned} \tag{3.18}$$

where $\sum_{\alpha_1+\cdots+k\alpha_k=s}$ denotes summation over all ordered k-tuples $(\alpha_1, \cdots, \alpha_k)$ of nonnegative integers such that $\alpha_1 + \cdots + k\alpha_k = s$. Since

$$Q_i^{\alpha_i} = \sum_{t_i(1)+\cdots+t_i(i)+s_i=\alpha_i} \{[\alpha_i!/(s_i! \prod_{j=1}^{i} (t_i(j)!))][\prod_{j=1}^{i} (\gamma_i(j)x^j)^{t_i(j)}]y_i^{s_i}\} ,$$

it then follows from (3.18) that

$$\begin{aligned} P_k &= k! \textstyle\sum_{\alpha_1+\cdots+k\alpha_k=k} [(Q_1/1!)^{\alpha_1} \cdots (Q_k/k!)^{\alpha_k}]/(\alpha_1! \cdots \alpha_k!) \\ &= \textstyle\sum_{d,\boldsymbol{\omega}} p(k; d, \boldsymbol{\omega})x^d y_1^{\omega_1} \cdots y_k^{\omega_k} \end{aligned}$$

where the coefficients $p(k; d, \boldsymbol{\omega})$ are given by (3.15).

To prove (3.17), we note that since $G(\boldsymbol{\omega}) = \sum_{i=1}^{k} i\omega_i \geqq 1$, (3.16b) implies that we can set $t_k(j) = 0$ for $j = 1, \cdots, k$ in (3.15), which can then be written as

$$p(k; d, \boldsymbol{\omega}) = k!(\prod_{i=1}^{k} (i!)^{\omega_i}(\omega_i!))^{-1} \textstyle\sum' \{\prod_{i=1}^{k-1} \prod_{j=1}^{i} (\gamma_i(j)/(i!))^{t_i(j)}/(t_i(j)!)\} \tag{3.19}$$

where $\sum'$ denotes summation over all ordered tuples $(t_i(j))_{i=1,\cdots,k-1; j=1,\cdots,i}$ satisfying

$$\textstyle\sum_{i=1}^{k-1} \sum_{j=1}^{i} jt_i(j) = d \tag{3.20a}$$

$$\textstyle\sum_{i=1}^{k-1} \sum_{j=1}^{i} it_i(j) = k - G(\boldsymbol{\omega}) = (k-1) - G(\boldsymbol{\omega}^*) . \tag{3.20b}$$

Hence (3.17) follows easily from (3.19).

LEMMA 9. *With the same assumptions and notations as in Lemma* 6, *let* $x = (1-t)^{-\frac{1}{2}}$ *and for* $\nu = 1, \cdots, k$, *let*

$$\begin{aligned} y_\nu = &\textstyle\sum_{n=1}^{\infty} (t^n/n)ES_n^{\nu} I_{[S_n \leqq 0]} \\ &- \{\lambda_\nu(\nu)(1-t)^{-\nu/2} + \cdots + \lambda_\nu(1)(1-t)^{-\frac{1}{2}}\} . \end{aligned} \tag{3.21}$$

Let $Q_0 = \sum_{n=1}^{\infty} (t^n/n)P[S_n \leqq 0]$. *Then for* $\nu = 1, \cdots, k$,

$$\textstyle\sum_{n=0}^{\infty} t^n ES_n^{\nu} I_{[N>n]} = e^{Q_0} \sum_{d=0}^{k} \sum_{\boldsymbol{\omega} \in \Omega_k} p_k(\nu; d, \boldsymbol{\omega})x^d y_1^{\omega_1} \cdots y_k^{\omega_k} \tag{3.22}$$

where Ω_k *denotes the set of all ordered k-tuples* $\boldsymbol{\omega} = (\omega_1, \cdots, \omega_k)$ *of nonnegative* *integers such that* $\sum_{i=1}^{k} i\omega_i \leqq k$, *and*

(3.23) $$p_k(\nu; d, \boldsymbol{\omega}) = \nu!\left(\prod_{i=1}^{k} (i!)^{\omega_i}(\omega_i!)\right)^{-1} \times \sum^{(k,\nu,d,\boldsymbol{\omega})} \left\{\prod_{i=1}^{k} \prod_{j=1}^{i} (\lambda_i(j)/(i!))^{t_i(j)}/(t_i(j)!)\right\}.$$

The summation sign $\sum^{(k,\nu,d,\boldsymbol{\omega})}$ *in* (3.23) *denotes summation over the set of all ordered tuples* $(t_i(j))_{i=1,\cdots,k;j=1,\cdots,i}$ *of nonnegative integers satisfying*:

(3.24a) $$\sum_{i=1}^{k} \sum_{j=1}^{i} j t_i(j) = d$$

(3.24b) $$\sum_{i=1}^{k} \sum_{j=1}^{i} i t_i(j) + \sum_{i=1}^{k} i\omega_i = \nu,$$

and summation over the empty set is taken to be 0. *Hence* $p_k(\nu; d, \boldsymbol{\omega}) = 0$ *if* $d + \sum_{i=1}^{k} i\omega_i > \nu$.

PROOF. It is well known (cf. [10]) that for $\theta \geqq 0$ and $0 < t < 1$,

(3.25) $$\sum_{n=0}^{\infty} t^n E e^{\theta S_n} I_{[N>n]} = \exp\{\sum_1^{\infty} (t^n/n) E e^{\theta S_n} I_{[S_n \leqq 0]}\}.$$

For $j = 1, 2, \cdots, k$, let

$$Q_j = \sum_{n=1}^{\infty} (t^n/n) E S_n^{\,j} I_{[S_n \leqq 0]} = \lambda_j(j)x^j + \cdots + \lambda_j(1)x + y_j.$$

Hence for $\nu = 1, \cdots, k$, we can write

$$\sum_1^{\infty} (t^n/n) E e^{\theta S_n} I_{[S_n \leqq 0]} = \sum_{j=0}^{\nu} (\theta^j/j!) Q_j + u_\nu(\theta)$$

where $u_\nu(\theta)$ belongs to class $C^k[0, \theta_0]$ and $\lim_{\theta \downarrow 0} \theta^{-\nu} u_\nu(\theta) = 0$. Since by (3.25),

$$\sum_{n=0}^{\infty} t^n E S_n^{\,\nu} I_{[N>n]} = \frac{\partial^\nu}{\partial\theta^\nu} \left(\sum_{n=0}^{\infty} t^n E e^{\theta S_n} I_{[N>n]}\right)\Big|_{\theta=0}$$
$$= e^{Q_0} \frac{\partial^\nu}{\partial\theta^\nu} \exp\{\sum_{j=1}^{\nu} (\theta^j/j!) Q_j + u_\nu(\theta)\}\Big|_{\theta=0},$$

the desired conclusion follows from Lemma 8, noting that (3.24b) implies that $\omega_i = t_i(j) = 0$ for all $j = 1, \cdots, i$ when $i > \nu$.

THEOREM 2. *Suppose* $E|X_1|^{k+1} < \infty$ *for some integer* $k \geqq 2$. *Let* σ_0 *be Spitzer's series of order* 0 *as introduced in Section* 1 *and let* $\alpha_i = EX_1^{\,i}$, $i \leqq k + 1$ $(\alpha_2 = 1)$. *Define* $\lambda_\nu(j)$ *and* g_ν *as in* (3.7) (*see Lemma* 6) *and define* Ω_k *and* $p_k(\nu; d, \boldsymbol{\omega})$ *as in* (3.22) *and* (3.23) (*see Lemma* 9).

(i) *The ladder variable* S_N *has finite kth moment, which is given by*:

(3.26) $$ES_N^{\,2} = -\sum_{\boldsymbol{\omega}\in\Omega_2} p_2(2; 1, \boldsymbol{\omega}) g_1^{\omega_1} e^{\sigma_0} = -(\lambda_2(1) + 2g_1\lambda_1(1))e^{\sigma_0}$$
$$= \{(3(2)^{\frac{1}{2}})^{-1}\alpha_3 - 2^{\frac{1}{2}}\sigma_1 + 1 - \pi^{-\frac{1}{2}} \sum_1^{\infty} (n^{-\frac{1}{2}} - \pi^{\frac{1}{2}}\tbinom{-\frac{1}{2}}{n}(-1)^n)\}e^{\sigma_0}$$

and in general, for $k \geqq 3$,

(3.27) $$ES_N^{\,k} = -\sum_{\boldsymbol{\omega}\in\Omega_k} \{p_k(k; 1, \boldsymbol{\omega}) + \sum_{j=2}^{k-1} \tbinom{k}{j}\alpha_j p_k(k-j; 1, \boldsymbol{\omega})\} g_1^{\omega_1} \cdots g_{k-1}^{\omega_{k-1}} e^{\sigma_0}.$$

(ii) *Let* $\mathbf{0}$ *denote the k-tuple* $(0, \cdots, 0)$ *and define* $p_k(0; 0, \mathbf{0}) = 1$ *and* $p_k(0; d, \boldsymbol{\omega}) = 0$

if $d \neq 0$ *or* $\boldsymbol{\omega} \neq \mathbf{0}$. *Then for any* $k \geqq 2$, *the following identity holds*:

$$(3.28) \qquad p_k(k; d, \boldsymbol{\omega}) = \sum_{j=2}^{k} \alpha_j \tbinom{k}{j} (p_k(k-j; d-2, \boldsymbol{\omega}) - p_k(k-j; d, \boldsymbol{\omega})) \quad \textit{for all } \boldsymbol{\omega} \in \Omega_k \textit{ and } d = 2, \cdots, k .$$

In particular,

$$(3.29) \qquad p_k(k; k, \mathbf{0}) = \tbinom{k}{2} p_k(k-2; k-2, \mathbf{0}) .$$

PROOF. We first note that

$$ES_N{}^k = \lim_{n\to\infty} \sum_{j=0}^{n-1} ES_{j+1}^k I_{[N=j+1]} = \lim_{n\to\infty} A_n , \quad \text{say}.$$

Since $S_{j+1} > 0$ on $[N = j+1]$, A_n is nondecreasing and so the Tauberian theorem implies that

$$(3.30) \qquad ES_N{}^k = \lim_{n\to\infty} A_n = \lim_{t\uparrow 1} (1-t) \sum_{n=1}^{\infty} A_n t^n .$$

Using the fact that $EX_1 = 0$, we have

$$(3.31) \qquad \begin{aligned} A_n &= \sum_{j=0}^{n-1} \{E(S_j + X_{j+1})^k I_{[N>j]} - ES_{j+1}^k I_{[N>j+1]}\} \\ &= -ES_n{}^k I_{[N>n]} + \tbinom{k}{2} \sum_{j=0}^{n-1} ES_j{}^{k-2} I_{[N>j]} + \cdots \\ &\quad + \tbinom{k}{k}\alpha_k \sum_{j=0}^{n-1} P[N > j] . \end{aligned}$$

Let $x = (1-t)^{-\frac{1}{2}}$ and define y_ν ($\nu = 1, \cdots, k$) and Q_0 as in Lemma 9. Then by Lemma 9,

$$(3.32) \qquad \sum_{n=0}^{\infty} t^n ES_n{}^k I_{[N>n]} = e^{Q_0} \sum_{d=0}^{k} \sum_{\boldsymbol{\omega}\in\Omega_k} p_k(k; d, \boldsymbol{\omega}) x^d y_1{}^{\omega_1} \cdots y_k{}^{\omega_k} .$$

Also using Lemma 9, we have for $\nu = 1, \cdots, k-2$,

$$(3.33) \qquad \begin{aligned} &\sum_{n=1}^{\infty} t^n \sum_{j=0}^{n-1} ES_j{}^\nu I_{[N>j]} \\ &= (t/(1-t)) \sum_{j=0}^{\infty} t^j ES_j{}^\nu I_{[N>j]} \\ &= (x^2 - 1) \sum_{j=0}^{\infty} t^j ES_j{}^\nu I_{[N>j]} \\ &= e^{Q_0} \sum_{d=2}^{k} \sum_{\boldsymbol{\omega}\in\Omega_k} p_k(\nu; d-2, \boldsymbol{\omega}) x^d y_1{}^{\omega_1} \cdots y_k{}^{\omega_k} \\ &\quad - e^{Q_0} \sum_{d=0}^{k} \sum_{\boldsymbol{\omega}\in\Omega_k} p_k(\nu; d, \boldsymbol{\omega}) x^d y_1{}^{\omega_1} \cdots y_k{}^{\omega_k} , \end{aligned}$$

noting that $p_k(\nu; \delta, \boldsymbol{\omega}) = 0$ if $\delta > k - 2 \geqq \nu$. Likewise we also have

$$(3.34) \qquad \begin{aligned} &\sum_{n=1}^{\infty} t^n \sum_{j=0}^{n-1} P[N > j] \\ &= (x^2 - 1) \sum_{j=0}^{\infty} t^j P[N > j] = (x^2 - 1) e^{Q_0} \quad (\text{by } (3.25)) \\ &= e^{Q_0} \sum_{d=2}^{k} \sum_{\boldsymbol{\omega}\in\Omega_k} p_k(0; d-2, \boldsymbol{\omega}) x^d y_1{}^{\omega_1} \cdots y_k{}^{\omega_k} - e^{Q_0} . \end{aligned}$$

It is well known (cf. [10]) that as $t \uparrow 1$,

$$(3.35) \qquad e^{Q_0} \sim e^{\sigma_0} (1-t)^{-\frac{1}{2}} .$$

By Lemma 6, $y_\nu = g_\nu + o(1)$ for $\nu = 1, \cdots, k-1$ and $y_k = o((1-t)^{-\frac{1}{2}})$ as $t \uparrow 1$. Since for $\nu \leqq k$, $p_k(\nu; 1, \boldsymbol{\omega}) = 0$ if $\omega_k \geqq 1$, we obtain that

$$(3.36) \qquad \lim_{t\uparrow 1} e^{\sigma_0} (1-t)^{\frac{1}{2}} \{-1 + \sum_{d=0}^{1} \sum_{\boldsymbol{\omega}\in\Omega_k} x^d y_1{}^{\omega_1} \cdots y_k{}^{\omega_k} [-p_k(k; d, \boldsymbol{\omega}) - \sum_{j=2}^{k-1} \tbinom{k}{j} \alpha_j p_k(k-j; d, \boldsymbol{\omega})]\} = J ,$$

where J denotes the expression on the right-hand side of (3.27). From the relations (3.30) through (3.36), it then follows that

$$ES_N{}^k = J + e^{\sigma_0} \lim_{t\uparrow 1} \{\sum_{d=2}^k \sum_{\boldsymbol{\omega}\in\Omega_k} x^{d-1}y_1{}^{\omega_1}\cdots y_k{}^{\omega_k}[-p_k(k; d, \boldsymbol{\omega}) + \sum_{j=2}^k \binom{k}{j}\alpha_j(p_k(k-j; d-2, \boldsymbol{\omega}) - p_k(k-j; d, \boldsymbol{\omega}))]\}. \tag{3.37}$$

We now prove (3.28) by induction on k. First, when $k = 2$, (3.29) reduces to the identity $\lambda_2(2) + \lambda_1{}^2(1) = 1$, and this obviously holds since $\lambda_2(2) = \frac{1}{2}$ and $\lambda_1(1) = -1/2^{\frac{1}{2}}$. Hence (3.28) holds in the case $k = 2 = d$, noting that $p_2(2; 2, \boldsymbol{\omega}) = p_2(0; 0, \boldsymbol{\omega}) = 0$ if $\boldsymbol{\omega} \neq \mathbf{0}$. Let $2 \leqq h \leqq k-1$. Suppose

$$p_h(h; d, \boldsymbol{\omega}^*) = \sum_{j=2}^h \alpha_j\binom{h}{j}(p_h(h-j; d-2, \boldsymbol{\omega}^*) - p_h(h-j; d, \boldsymbol{\omega}^*)) \tag{3.38}$$

holds for all $h \geqq d \geqq 2$ and $\boldsymbol{\omega}^* \in \Omega_h$. Given any $\boldsymbol{\omega} \in \Omega_{h+1}$ such that $\boldsymbol{\omega} \neq \mathbf{0}$, i.e., $G(\boldsymbol{\omega}) = \sum_{i=1}^{h+1} i\omega_i \geqq 1$, we take $\boldsymbol{\omega}^* = (G(\boldsymbol{\omega}) - 1, 0, \cdots, 0) \in \Omega_h$. Then $G(\boldsymbol{\omega}^*) = G(\boldsymbol{\omega}) - 1$. It therefore follows from (3.17) that for $\delta = 0, 1, \cdots, h$,

$$\begin{aligned} p_{h+1}(h+1; \delta, \boldsymbol{\omega}) &= (h+1)c(\boldsymbol{\omega}, \boldsymbol{\omega}^*)p_h(h; \delta, \boldsymbol{\omega}^*) \\ p_{h+1}(h+1-j; \delta, \boldsymbol{\omega}) &= (h+1-j)c(\boldsymbol{\omega}, \boldsymbol{\omega}^*)p_h(h-j; \delta, \boldsymbol{\omega}^*) \\ &= (h+1)\binom{h+1}{j}^{-1}\binom{h}{j}c(\boldsymbol{\omega}, \boldsymbol{\omega}^*)p_h(h-j; \delta, \boldsymbol{\omega}^*), \\ & \qquad j = 2, \cdots, h, \end{aligned}$$

where $c(\boldsymbol{\omega}, \boldsymbol{\omega}^*) = \{\prod_{i=1}^h (i!)^{\omega_i^*}(\omega_i{}^*!)\}\{\prod_{i=1}^{h+1} (i!)^{\omega_i}(\omega_i!)\}^{-1}$. Hence (3.38) implies that for $h \geqq d \geqq 2$,

$$p_{h+1}(h+1; d, \boldsymbol{\omega}) = \sum_{j=2}^{h+1} \alpha_j\binom{h+1}{j}(p_{h+1}(h+1-j; d-2, \boldsymbol{\omega}) - p_{h+1}(h+1-j; d, \boldsymbol{\omega})), \tag{3.39}$$

noting that $p_{h+1}(0; \delta, \boldsymbol{\omega}) = 0$ since $\boldsymbol{\omega} \neq 0$. When $d = h+1$, (3.39) still holds since $G(\boldsymbol{\omega}) \geqq 1$ implies that $p_{h+1}(h+1; d, \boldsymbol{\omega}) = 0 = p_{h+1}(\nu; d, \boldsymbol{\omega}) = p_{h+1}(\nu; d-2, \boldsymbol{\omega})$ for $\nu \leqq (h+1) - 2$.

To complete the induction proof, we now show that $w(d; \mathbf{0}) = 0$ for $d = 2, \cdots, h+1$, where we set

$$\begin{aligned} w(d; \mathbf{0}) = &-p_{h+1}(h+1; d, \mathbf{0}) \\ &+ \sum_{j=2}^{h+1} \alpha_j\binom{h+1}{j}(p_{h+1}(h+1-j; d-2, \mathbf{0}) - p_{h+1}(h+1-j; d, \mathbf{0})). \end{aligned}$$

From (3.23), it is clear that all the coefficients $p_{h+1}(\nu; \delta, \boldsymbol{\omega})$ are polynomials involving only the moments $\alpha_3, \cdots, \alpha_{h+2}$ of X_1 and are otherwise independent of the distribution of X_1. By Lemma 7, we can choose i.i.d. bounded random variables $Y_1, Y_2, \cdots$ such that $EY_1{}^i = EX_1{}^i$ for $i = 1, \cdots, h+2$. Let $S_n{}' = Y_1 + \cdots + Y_n$, $T = \inf\{n \geqq 1 : S_n{}' > 0\}$ and $U = S_T{}'$. Since (3.39) holds for $\boldsymbol{\omega} \neq \mathbf{0}$, it follows from (3.37) that

$$EU^{h+1} = c_1 + c_2 \lim_{x\to\infty} \{\sum_{d=2}^{h+1} w(d; \mathbf{0})x^{d-1}\}, \tag{3.40}$$

where $\infty > c_2 = \exp\{\sum_1^\infty n^{-1}(P[S_n{}' \leqq 0] - \frac{1}{2})\} > 0$ and c_1 is a finite constant. Since Y_1 is bounded, it is obvious that $EU^{h+1} < \infty$ and so (3.40) implies that $w(d; \mathbf{0}) = 0$ for $d = 2, \cdots, h+1$. Hence we have proved (3.28) by induction. The relation (3.27) then follows immediately from (3.28) and (3.37).

4. Applications to renewal theory. For $a > 0$, define

(4.1) $$T(a) = \inf\{n \geqq 1 : S_n > a\}.$$

It is well known that $ET^\gamma(a) = \infty$ if $\gamma \geqq \frac{1}{2}$ and $ET^\gamma(a) < \infty$ if $\gamma < \frac{1}{2}$. As an analogue of the classical renewal theorem for the case of positive mean, we show in [5] that in the present case of zero mean and unit variance, we have for $0 < \gamma < \frac{1}{2}$,

(4.2) $$ET^\gamma(a) \sim \gamma a^{2\gamma} \int_0^\infty u^{\gamma-1}(2\Phi(u^{-\frac{1}{2}}) - 1)\, du \qquad \text{as } a \to \infty.$$

By using the results of the preceding section, we shall obtain below the limiting distribution and the limiting moments of the overshoot $S_{T(a)} - a$. In [9], Siegmund has shown how the limiting expected overshoot can be used to obtain asymptotic expansions in sequential analysis.

THEOREM 3. *Suppose X_1 is nonlattice and $EX_1 = 0$, $EX_1^2 = 1$. Let $R(a) = S_{T(a)} - a$, where $T(a)$ is defined by* (4.1), *and define the ladder epoch N as in* (1.1). *Then*

(4.3) $$\lim_{a\to\infty} P[R(a) \leqq \xi] = (1/ES_N) \int_0^\xi P[S_N > t]\, dt$$

for all $\xi > 0$, where ES_N is given by (1.2). *If $E|X_1|^{k+1} < \infty$ for some positive integer $k \geqq 2$, then $ER^k(a) < \infty$ and*

(4.4) $$\lim_{a\to\infty} ER^{k-1}(a) = (ES_N{}^k)/(kES_N)$$

where $ES_N{}^k$ is given by Theorem 2.

PROOF. Let $N_0 = 0$ and let $N_1, N_2, \cdots$ be the successive ladder indices (cf. [2], page 190) of the random walk $\{S_n\}_{n=1,2,\ldots}$. Let $Z_i = S_{N_i} - S_{N_{i-1}}$ and define $M(a) = \inf\{n \geqq 1 : Z_1 + \cdots + Z_n > a\}$. Then $R(a) = \sum_{i=1}^{M(a)} Z_i - a$, and $Z_1, Z_2, \cdots$ are i.i.d. and have the same distribution as S_N. Hence $Z_1 > 0$ a.e., $EZ_1 < \infty$ and by Theorem 2, $E|X_1|^{k+1} < \infty$ implies that $EZ_1{}^k < \infty$, so that in this case $E(\sum_1^{M(a)} Z_i)^k < \infty$. The relation (4.3) is therefore a well-known corollary of the renewal theorem (cf. [2], pages 354–355). Let F be the distribution function of Z_1 and set $U(x) = \sum_{n=0}^\infty P[Z_1 + \cdots + Z_n \leqq x]$. We note that

$$ER^{k-1}(a) = \int_0^a \{\int_{a-x}^\infty (x + t - a)^{k-1}\, dF(t)\}\, dU(x) = \int_0^a g(a - x)\, dU(x)$$

where $g(y) = \int_y^\infty (t - y)^{k-1}\, dF(t)$. The function $g(y)$ is nonincreasing, and if $EZ_1{}^k < \infty$, it is easy to see that g is directly Riemann integrable and so (4.4) follows immediately from the renewal theorem (cf. [2], pages 348–350).

REFERENCES

[1] BAUM, L. and KATZ, M. (1963). On the influence of moments on the asymptotic distribution of sums of random variables. *Ann. Math. Statist.* **34** 1042–1044.

[2] FELLER, W. (1966). *An Introduction to Probability Theory and Its Applications* **2**. Wiley, New York.

[3] GNEDENKO, B. V. and KOLMOGOROV, A. N. (1954). *Limit Distributions for Sums of Independent Random Variables.* Addison-Wesley, Reading, Mass.

[4] KNOPP, K. (1951). *Theory and Application of Infinite Series.* Blackie, London.
[5] LAI, T. L. (1975). Uniform Tauberian theorems and their applications to renewal theory and first passage problems. To appear in *Ann. Probability.*
[6] LOÈVE, M. (1960). *Probability Theory.* Van Nostrand, Princeton.
[7] ROSÉN, B. (1962). On the asymptotic distribution of sums of independent identically distributed random variables. *Ark. Mat.* **4** 323-332.
[8] SHOHAT, J. A. and TAMARKIN, J. D. (1943). *The Problem of Moments.* Amer. Math. Soc., Providence.
[9] SIEGMUND, D. (1974). A note on the error probabilities and average sample number of the sequential probability ratio test. To appear.
[10] SPITZER, F. (1960). A Tauberian theorem and its probability interpretation. *Trans. Amer. Math. Soc.* **94** 150-169.
[11] VON BAHR, B. (1965). On the convergence of moments in the central limit theorem. *Ann. Math. Statist.* **36** 808-818.
[12] WALD, A. (1947). *Sequential Analysis.* Wiley, New York.

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