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CONVERGENCE OF THE AGE DISTRIBUTION IN THE ONE-DIMENSIONAL SUPERCRITICAL AGE-DEPENDENT BRANCHING PROCESS[1]

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The age distribution for a supercritical Bellman–Harris process is proven to converge in probability to a deterministic distribution under assumptions slightly more than finite first moment. If the usual "$j \log j$" condition holds, then the convergence can be strengthened to hold w.p. 1. As a corollary to this result, the population size, properly normalized is shown to converge w.p. 1 to a nondegenerate random variable under the "$j \log j$" assumption.

1. Introduction. An important and useful aspect of age-dependent branching processes is the limiting behavior of the age distribution. That is, if for any family tree ω, $Z(x, t, \omega) =$ number of objects living at time t with age $\leqq x$ and $A(x, t, \omega) \equiv (Z(x, t, \omega))/(Z(\infty, t, \omega))$, then the asymptotic behavior of the random distribution function $A(\cdot, t, \omega)$ as $t \to \infty$ is of practical interest. Of course this is well defined only if $Z(t, \omega) \equiv Z(\infty, t, \omega)$ does not go to zero as $t \to \infty$. We study this question for the supercritical one-dimensional age-dependent process (sometimes called the Bellman–Harris process). Our results are that: (i) with assumptions slightly more than finite mean for the offspring distribution, the age distribution at time t converges vaguely to a deterministic distribution $A(\cdot)$ in probability; and (ii) under the usual "$j \log j$" assumption this convergence can be strengthened to hold with probability one.

The only known result on this problem is due to Harris ([7], page 154), who showed that if the offspring distribution $\{p_j\}$ has a second moment and the lifetime distribution $G(\cdot)$ satisfies certain regularity conditions, then $A(\cdot, t, \omega)$ converges vaguely to a deterministic distribution $A(\cdot)$ with probability one. Jagers [8] improved this somewhat by dropping all of the regularity assumptions on $G(\cdot)$ but still requiring a second moment on $\{p_j\}$. Their proofs rely on L_2 theory which requires second moments in an essential way.

The approach here is somewhat different. As explained in [2] a natural way to view an age-dependent branching process is as a multitype process (necessarily of infinite type). In the finite case, it is well known ([4]) that on the set of explosion the vector of proportions of the number of particles of various types converges to a fixed nonrandom vector in probability under the assumption of finiteness of the first moments and with probability one under the "$j \log j$"

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condition. This leads us to suspect that the analog of the proportions vector which is the age distribution in the age-dependent case should converge vaguely to a nonrandom distribution in probability assuming only the finiteness of the first moment and with probability one under the "$j \log j$" condition. Indeed this essentially turns out to be the case here.

The first moment hypothesis is indispensable in the sense that the limiting age distribution involves the so-called Malthusian parameter α which is finite iff $m = \sum_{j=1}^{\infty} jp_j < \infty$. When $m = \infty$, our preliminary investigation indicates that the age distribution converges to a delta distribution at 0. The problem of determining a proper normalization for the age-distribution at time t to obtain a nondegenerate limit is under investigation.

An important corollary to our result is that for any bounded measurable function continuous almost everywhere with respect to Lebesgue measure on the support of G, $\int_0^\infty f(x)\, dA(x, t, \omega)$ converges in probability to $\int_0^\infty f(x)\, dA(x)$ under mild assumptions on G and with probability one under "$j \log j$." In particular, if we take f to be the reproductive age value $V(\cdot)$ defined by

$$V(x) = me^{\alpha x}[\textstyle\int_x^\infty e^{-\alpha u}\, dG(u)][1 - G(x)]^{-1},$$

then

$$\int_0^\infty V(x)\, dA(x, t, \omega)$$

converges with probability one under "$j \log j$." Further,

$$e^{-\alpha t} Z(t, \omega) \int V(x)\, dA(x, t, \omega),$$

being a nonnegative martingale ([7], page 153), converges with probability one. Combining these two, we see that under "$j \log j$" $Z(t, \omega)e^{-\alpha t}$ converges w.p. 1 to a nonnegative limit, thus extending the Kesten–Stigum theorem fully to the age-dependent case. In [2] the convergence in law of $Z(t, \omega)e^{-\alpha t}$ had been established and it was conjectured there that the above method could be used to prove almost sure convergence.

Our technique consists in writing $A(x, t + s, \omega)$ in terms of the age chart at time t and using the law of large numbers. This is similar to the idea employed in [3]. It is spelled out in detail in Section 3. It is quite powerful and yields the results under minimal hypothesis. In particular, we feel that it would yield limit theorems for the various types of processes studied by Jagers [9] and Crump and Mode [6] with very few assumptions.

We now outline the rest of the paper.

In Section 2 we describe the basic setup, terminology and notation, and state the results. Section 3 gives the outline of the proof while Section 4 gives the details. Section 5 discusses the V_t martingale and proves the Kesten–Stigum theorem for the age-dependent case.

2. Statement of results. We shall consider an age-dependent branching process with offspring distribution $\{p_j\}$ and lifetime distribution $G(\cdot)$. We make the following assumptions throughout. Sometimes they will appear in lemmas and theorems explicitly and sometimes not, but they will always be in force.

(i) $p_0 = 0$.
(ii) $1 < m = \sum_{j=1}^{\infty} jp_j < \infty$.
(iii) $G(0+) = 0$.

The assumption $p_0 = 0$ is primarily for convenience of exposition. Otherwise one has to keep qualifying "on the set of explosion." However, in some of the proofs, we will explicitly consider the case $p_0 \neq 0$, since slightly different arguments are required. The assumption $G(0+) = 0$ is standard. Also without any loss of generality we may assume that G is not lattice with finite support; since this is a multi-type Galton–Watson process in disguise for which our results are already available ([3]). We shall also exclude the case of lattice G with non-compact support for which our proofs here could easily be adapted.

Since we want to be able to talk about the age chart at various times, we need to describe the state of the system quite adequately. We shall, in fact, overdo it a bit by assuming that our sample space Ω is the space of all family histories and our probability measure P is defined on a sufficiently big σ-algebra $\mathscr{B}$ on Ω ([7]). Introduce the following notation. For any family history ω let:

$Z(t, \omega) =$ the number of particles living at time t.

$Z(x, t, \omega) =$ the number of particles living at time t whose age $\leqq x$.

(Clearly $\lim_{x\to\infty} Z(x, t, \omega) = Z(t, \omega)$).

$\{x_i(t, \omega); i = 1, 2, \cdots, Z(t, \omega)\} =$ the age chart at time t.

$Z_{x_i(t,\omega)}(x, s, \omega) =$ the number of particles living at time $t + s$ with age $\leqq x$ in a line of descent initiated by a particle of age $x_i(t, \omega)$ living at time t.

$$M(t) = E\{Z(t, \omega)\}\,.$$

$$M(x, t) = E\{Z(x, t, \omega)\}\,.$$

We may occasionally write $M(\infty, t)$ for $M(t)$.

$$A(x, t, \omega) = Z(x, t, \omega)/Z(t, \omega)\,, \qquad \text{if} \quad Z(t, \omega) > 0\,.$$

(Since we assume $p_0 = 0$, extinction occurs with zero probability and thus $A(x, t, \omega)$ is well defined a.e.)

We add a subscript y to all the previous random variables and their expectations to indicate the case when P is supported by those ω's which start with one particle of age $y > 0$. Thus we write

$$F_y(\theta, t) = E\{e^{-\theta Z_y(t,\omega)}\} \qquad \theta > 0$$

$$M_y(t) = E\{Z_y(t, \omega)\}$$

$$F_y(\theta, x, t) = E\{e^{-\theta Z_y(x,t,\omega)}\} \qquad \theta > 0$$

$$M_y(x, t) = E\{Z_y(x, t, \omega)\}\,.$$

We also put:

$$f(s) = \sum_{j=0}^{\infty} p_j s^j\,.$$

The Malthusian parameter α is the root of the equation

$$m \int_0^\infty e^{-\alpha t}\, dG(t) = 1 .$$

Put

$$G_y(x) = \frac{G(x+y) - G(y)}{1 - G(y)} \qquad x > 0, y > 0$$

$$V(x) = m \int_0^\infty e^{-\alpha u}\, dG_x(u)$$

$$A(x) = \frac{\int_0^x e^{-\alpha u}[1 - G(u)]\, du}{\int_0^\infty e^{-\alpha u}[1 - G(u)]\, du}$$

$$V_t = \int_0^\infty V(x)\, dZ(x, t, \omega)$$

$$= \sum_{i=1}^{Z(t,\omega)} V(x_i)$$

where $x_1, \cdots, x_{Z(t)}$ are the ages of the particles alive at t. It will always be assumed that the probability measure P satisfies

$$P\{\omega: Z(0, \omega) < \infty\} = 1 .$$

We are ready to state our results.

THEOREM A. *Let* $\{p_j\}$ *and* $G(\cdot)$ *satisfy the assumptions*: $1 < m = \sum jp_j < \infty$, $p_0 = 0$, $G(0+) = 0$ *and* $G(\cdot)$ *nonlattice. Assume either of the following two additional conditions hold.*

(a) $\inf_{y \in \text{supp}\, G} V(y) > 0$ (supp G = *support of* G)

or

(b) $\sum p_j j \log j < \infty$.

Then

$$\sup_x |A(x, t, \omega) - A(x)| \to_p 0$$

as $t \to \infty$.

The reader should note that condition (a) holds for example if G has bounded support or if G is negative exponential. We conjecture that Theorem A holds assuming only that $m < \infty$.

THEOREM B. *When* $\sum p_j j \log j < \infty$, *the convergence on Theorem* A *can be strengthened to hold* w.p. 1, i.e.

$$\sup_x |A(x, t, \omega) - A(x)| \to 0 \quad \text{w.p. } 1$$

as $t \to \infty$.

THEOREM C. *Under the hypothesis of Theorem* B

$$\lim_{t\to\infty} Z(t, \omega)e^{-\alpha t} = W(\omega)$$

exists w.p. 1 *and* $P(W(\omega) > 0) = 1$.

Since $A(\cdot)$ is continuous and $A(\infty) = 1$ we can always find (for any given $\varepsilon > 0$) $\delta > 0$ and a positive integer N such that

$$\sup_x |A(x, t, \omega) - A(x)| \leqq \varepsilon + \sup_{1 \leqq j \leqq N} |A(j\delta, t, \omega) - A(j\delta)| .$$

Thus Theorems A and B are consequences of the following:

THEOREM 1. *Under the hypothesis of Theorem* A

$$A(x, t, \omega) \to_p A(x) \qquad as \quad t \to \infty$$

for each fixed $0 < x < \infty$.

THEOREM 2. *Under the hypothesis of Theorem* B

$$A(x, t, \omega) \to A(x) \qquad as \quad t \to \infty$$

w.p. 1 *for each fixed* $0 < x < \infty$.

3. Plan of the proof. By the additive property of branching processes we can write

$$Z(x, t + s, \omega) = \textstyle\sum_{i=1}^{Z(\infty,t,\omega)} Z_{x_i(t,\omega)}(x, s, \omega) \tag{3.1}$$

where $\{x_i(t, \omega); i = 1, 2, \cdots, Z(\infty, t, \omega)\}$ is the age chart at time t and $Z_{x_i(t,\omega)}(x, s, \omega)$ denotes the number of objects of age $\leqq x$ at time $(t + s)$ in the line of descent initiated by the particle, of age $x_i(t, \omega)$ at time t. The σ-algebra $\mathscr{B}$ is assumed to be big enough to make these measurable. It is well known that conditioned on the age chart at time t, $\{Z_{x_i(t,\omega)}(x, s, \omega); i = 1, 2, \cdots, Z(\infty, t, \omega)\}$ are independently distributed and further if $x_i(t, \omega) = y$ then the conditional distribution of $Z_{x_i(t,\omega)}(x, s, \omega)$ is the same as $Z_y(x, s, \omega)$ defined in Section 2. By an abuse of notation we shall rewrite (3.1) as

$$Z(x, t + s) = \textstyle\sum_{i=1}^{Z(t)} Z_{x_i}(x, s) \tag{3.2}$$

suppressing ω, and (t, ω).

Starting from (3.2) we have the identity (in the notation defined in Section 2):

$$\begin{aligned} e^{-\alpha s}\frac{1}{Z(t)} Z(x, t + s) &= \frac{1}{Z(t)} \textstyle\sum_{i=1}^{Z(t)} [Z_{x_i}(x, s) - M_{x_i}(x, s)]e^{-\alpha s} \\ &\quad + \frac{1}{Z(t)} \textstyle\sum_{i=1}^{Z(t)} [M_{x_i}(x, s)e^{-\alpha s} - n_1 V(x_i)A(x)] \\ &\quad + (V_t/Z(t))n_1 A(x) \\ &= a_t(x, s) + b_t(x, s) + c_t A(x)\,, \quad \text{say}, \end{aligned} \tag{3.3}$$

where n_1 is a suitable constant. Trivially,

$$A(x, t + s) = \frac{(a_t(x, s) + b_t(x, s) + c_t A(x))}{(a_t(\infty, s) + b_t(\infty, s) + c_t)}\,. \tag{3.4}$$

We first show that $|b_t(x, s)|$ and $|b_t(\infty, s)|$ can be made small uniformly with respect to t and ω by choosing s large (Lemma 1). Next we show that for fixed s, both $a_t(x, s)$, $a_t(\infty, s)$ go to zero in probability as $t \to \infty$ and that c_t is bounded below in probability (Lemmas 2 and 3). When we assume $\sum j \log j p_j < \infty$, we show using a proposition of Kurtz [11] that the above convergences can be strengthened to hold with probability one if t and s are restricted to lattices of the form $\{n\delta;\ n = 0, 1, 2, \cdots, \delta$ a positive rational$\}$ (Lemmas 2′ and 3′). Finally, some technical arguments are needed to push the almost sure convergence on the lattice to the whole continuum.

4. The proofs.

LEMMA 1. *Let* $n_1 = \int_0^\infty e^{-\alpha t}(1 - G(t))\, dt/m \int_0^\infty te^{-\alpha t}\, dG(t)$. *Then*

$$\sup_y (|M_y(x, s)e^{-\alpha s} - n_1 V(y)A(x)|,\ |M_y(\infty, s)e^{-\alpha s} - n_1 V(y)|) \to 0$$

as $s \to \infty$.

PROOF. We use the integral equation satisfied by $M_y(x, s)$, namely,

$$M_y(x, s) = J(x - y - s)(1 - G_y(s)) + \int_0^s mM_0(x, s - u)\, dG_y(u) \tag{4.1}$$

where $J(u) = 0$ for $u < 0$ and 1 if $u \geqq 0$. Let $T > 0$.

Noting the definition of V and (4.1) we get for $s > 2T$

$$\begin{aligned} &|M_y(x, s)e^{-\alpha s} - n_1 V(y)A(x)| \\ &\qquad \leqq e^{-\alpha s} + \int_0^{s-T} |mM_0(x, s - u)e^{-\alpha(s-u)} - n_1 A(x)|e^{-\alpha u}\, dG_y(u) \\ &\qquad\quad + ce^{-\alpha(s-T)} \end{aligned} \tag{4.2}$$

where c is a constant independent of x, y, s, T.

The lemma now follows since $M_0(x, t)e^{-\alpha t} \to n_1 A(x)$, and $M_0(\infty, t)e^{-\alpha t} \to n_1$. □

The following is a trivial corollary:

COROLLARY 1. $\sup_{t,\omega} (|b_t(x, s)|, |b_t(\infty, s)|) \to 0$ *as* $s \to \infty$.

LEMMA 2. *Fix* $0 < s < \infty$. *Then*

$$Y_t \equiv \frac{1}{Z(t)} \sum_{i=1}^{Z(t)} [Z_{x_i}(x, s) - M_{x_i}(x, s)] \to 0 \tag{4.3}$$

in probability as $t \to \infty$.

PROOF. Since $Z_{x_i}(x, s)$ are nonnegative random variables and $\sup_{y,x} M_y(x, s) < \infty$ we can employ moment generating functions. It suffices to show that for each $0 < \theta < \infty$

$$E\{e^{-\theta Y_t}\} \to 1 \quad \text{as} \quad t \to \infty .$$

But

$$E\{e^{-\theta Y_t} | \mathscr{F}_t\} = \exp\left\{\sum_{i=1}^{Z(t)} \left[\frac{\theta}{Z(t)} M_{x_i}(x, s) + \log F_{x_i}\left(\frac{\theta}{Z(t)}, x, s\right)\right]\right\} \tag{4.4}$$

where $\mathscr{F}_t$ is the σ-algebra of family histories up to time t and $F_y(\theta, x, s) = E(e^{-\theta Z_y(x,s)})$.

Now observe

$$\begin{aligned} P(Z_y(x, s) \geqq k) &\leqq P(\textstyle\sum_{j=1}^N Z_0^j(\infty, s) > k) &&\text{if } p_0 = 0 \\ &\leqq P(\textstyle\sum_{j=1}^N \tilde{Z}_0^j(\infty, s) > k) &&\text{if } p_0 \neq 0 \end{aligned}$$

where

(i) N has p.g.f. $f(s)$.

(ii) $\{Z_0^j(\infty, s)\}$ are independent of N and independent copies of $Z_0(\infty, s)$.

(iii) $\{\tilde{Z}_0^j(\infty, s)\}$ are independent of N and independent copies of a Bellman–Harris process with lifetime distribution G and offspring p.g.f. $\tilde{f}(s) = (p_0 + p_1)s + \sum_{j=2}^{\infty} s^j p_j$.

This makes the family $\{Z_y(x, s); 0 \leqq x < \infty, 0 \leqq y < \infty\}$ uniformly integrable. Thus,

$$\sup_{y,x}\left|\frac{1 - F_y(\eta, x, s)}{\eta} - M_y(x, s)\right| \to 0 \qquad \text{as} \quad \eta \downarrow 0\,,$$

and

$$\sup_{y,x} M_y(x, s) < \infty\,.$$

Using the fact $\log(1 - h) = -h + o(h)$ as $h \to 0$ we conclude that

$$\sup_{y,x}\left|M_y(x, s) + \frac{Z(t)}{\theta}\log F_y\left(\frac{\theta}{Z(t)}, x, s\right)\right| \to 0 \qquad \text{as} \quad Z(t) \to \infty\,.$$

Now use (4.4) to finish the proof. □

LEMMA 3. *Assume either*

(a) $\inf_{y \in \operatorname{supp} G} V(y) > 0$

or

(b) $\sum p_j j \log j < \infty$.

Then for every $\varepsilon > 0$ *there exists an* $\eta > 0$ *such that*

$$\liminf_{t\to\infty} P(V_t/Z(t) > \eta) > 1 - \varepsilon\,.$$

PROOF. If (a) holds the result is obvious. So suppose $\sum p_j j \log j < \infty$. It is proven in [2] that

$$\lim_{t\to\infty} Z(t)e^{-\alpha t} = W$$

in distribution and $P(W > 0) = 1$. Also one can repeat the arguments in [2] almost verbatim to prove

$$\lim_{t\to\infty} V_t e^{-\alpha t} = W'$$

in distribution and $P(W' > 0) = 1$. The lemma now follows easily. □

PROOF OF THEOREM 1. Use (3.4), and Lemmas 1, 2 and 3.

REMARK. The additional assumptions of Theorem 1 are needed only to establish Lemma 3. It is conjectured that the lemma is valid assuming only $m < \infty$. The reader should note that the lemma is true when G has infinite support if one could show that for some $K > 0$, $Z(K, t)/Z(t)$ is bounded below in probability. This follows since

$$\inf_{0 \leqq y \leqq K} V(y) = a(K) > 0 \qquad \text{and} \qquad V_t/Z(t) \geqq a(K)Z(K, t)/Z(t)\,.$$

The proof of Theorem 2 depends crucially on the following strengthening of Lemma 2.

LEMMA 2′. *Let* $\sum_{j=1}^{\infty} p_j j \log j < \infty$. *Then for each* $\delta > 0$ *and integer* m

$$(4.5) \qquad \frac{1}{Z(n\delta)} \sum_{i=1}^{Z(n\delta)} [Z_{x_i}(x, m\delta) - M_{x_i}(x, m\delta)] \to 0 \quad \text{w.p. } 1 \qquad \text{as} \quad n \to \infty .$$

The proof of this lemma is a consequence of the following proposition due to T. Kurtz [11].

PROPOSITION 1 (T. Kurtz). *Assume for each* $k \geqq 1$, $X_{k1}, X_{k2}, \cdots, X_{kn_k}$ *are independent random variables with finite means. Assume that for some* $0 < c < \infty$, $1 < r < \infty$, $n_k > cr^k$ *and for* y *sufficiently large,*

$$\sup_{1 \leqq i \leqq n_k; k \geqq 1} P\{|X_{ki} - EX_{ki}| \geqq y\} \leqq 1 - H(y)$$

where $H(\cdot)$ *is a distribution function such that* $\int y(\log y)^+ \, dH(y) < \infty$. *Then* $\forall \varepsilon > 0$,

$$\sum_{n=1}^{\infty} P\{|\sum_{i=1}^{k_n} (X_{ki} - EX_{ki})| > k_n \varepsilon\} < \infty .$$

PROOF. Let $\varepsilon > 0$. Define

$$\begin{aligned} \theta(u) &= u^2 \qquad \text{if} \quad 0 < u < 1 \\ &= u \qquad \text{if} \quad u \geqq 1 \end{aligned}$$

Lemma 2.2 of [10] states that

$$P\left\{\frac{1}{n_k} |\sum_{i=1}^{n_k} (X_{ki} - EX_{ki})| > \varepsilon\right\} \leqq \frac{4}{\theta(2\varepsilon)} \int_0^\infty \theta'(u/n_k) L_k(u) \, du$$

where $L_k(u) = \sup_{1 \leqq i \leqq n_k} P\{|X_{ki} - EX_{ki}| > u\}$.

Now observe that

$$\sum_{k=1}^{\infty} \theta'(u/cr^k) = O(\log u) \qquad \text{as} \quad u \to \infty$$

and

$$\int_0^\infty (\log u)^+(1 - H(u)) \, du < \infty \quad \text{iff} \quad \int_0^\infty u(\log u)^+ \, dH(u) < \infty . \qquad \square$$

PROOF OF LEMMA 2′. Since the process is supercritical we can always find (by truncating the offspring distribution if necessary) an $0 < \alpha' \leqq \alpha$ and a constant c such that $Z(n\delta, \omega) \geqq ce^{\alpha' n\delta}$.

Also for fixed $m\delta$ we know $\eta = \sup_y M_y(x, m\delta) < \infty$ and for any x, y, $Z_y(x, m\delta)$ is stochastically smaller than $R = \sum_{j=1}^{N} \tilde{Z}_0^j(\infty, m\delta)$ where N has p.g.f. $f(s)$ and $\{\tilde{Z}_0^j(\infty, t)\}_{t \geqq 0}$ are i.i.d. age-dependent branching processes with the same lifetime distribution but with offspring p.g.f. $f(s) = (p_0 + p_1 + p_2)s^2 + p_3 s^3 + \cdots$, and independent of N. Thus for any $z > \eta$,

$$\begin{aligned} P\{|Z_y(x, m\delta) - M_y(x, m\delta)| \geqq z\} &\leqq P\{Z_y(x, m\delta) \geqq z\} \\ &\leqq P\{R \geqq z\} = 1 - H(z) . \end{aligned}$$

Also it is known [1] that $\sum p_j j \log j < \infty \Rightarrow E\{R(\log R)^+\} < \infty$.

We now conclude from the previous proposition that

$$\sum_n P\left\{\left|\frac{1}{Z(n\delta)}\sum_{i=1}^{Z(n\delta)}[Z_{x_i}(x, m\delta) - M_{x_i}(x, m\delta)]\right| > \varepsilon \,|\, \mathscr{F}_{n\delta}\right\} < \infty \quad \text{w.p. } 1 .$$

By the extended Borel–Cantelli lemma [5] this is enough to finish the proof. □

LEMMA 3′. *Let* $\sum p_j j \log j < \infty$. *Then for every* $\delta > 0$,

$$\lim_{n\to\infty} \frac{V_{n\delta}}{Z(n\delta)} > 0 \quad \text{w.p. } 1 .$$

PROOF. Let $\delta > 0$. It follows from the results of the next section that $\lim_{n\to\infty} e^{-\alpha n\delta} V_{n\delta} = W'$ exists w.p. 1. Also in Lemma 3 we noted that $P(W' > 0) = 1$. Hence it suffices to show

$$\limsup_{n\to\infty} e^{-\alpha n\delta} Z(n\delta) < \infty \quad \text{w.p. } 1 .$$

Let $0 < \varepsilon < \frac{1}{2}$. By Lemma 1, there exists an n_0 such that

$$\sup_x |M_x(\delta n_0)e^{-\alpha\delta n_0} - n_1 V(x)| < \varepsilon .$$

To simplify notation put $W_k = e^{-\alpha k\delta n_0} Z(k\delta n_0)$, $k \geqq 1$. Using (3.1) we can write

$$\begin{aligned} W_{k+1} = W_k &\left\{\frac{1}{Z(k\delta n_0)}\sum_{i=1}^{Z(k\delta n_0)} e^{-\alpha\delta n_0}(Z_{x_i}(\delta n_0) - M_{x_i}(\delta n_0))\right. \\ &\left. + \frac{1}{Z(k\delta n_0)}\sum_{i=1}^{Z(k\delta n_0)} (e^{-\alpha\delta n_0} M_{x_i}(\delta n_0) - n_1 V(x_i))\right\} \\ &+ n_1 e^{-\alpha k\delta n_0} V_{k\delta n_0} . \end{aligned}$$

Since $\lim_{t\to\infty} e^{-\alpha t} V_t$ converges w.p. 1, $A = \sup_t e^{-\alpha t} V_t < \infty$ w.p. 1. It follows then from Lemma 2′ that there exists a finite integer valued random variable I such that w.p. 1

$$W_{k+1} \leqq W_k 2\varepsilon + A \qquad k \geqq I .$$

Iterating the last inequality proves that $\limsup_{k\to\infty} W_k < \infty$ w.p. 1.

Arguing in exactly the same way as above one can show

$$\limsup_{k\to\infty} e^{-\alpha\delta(kn_0+j)} Z(\delta(kn_0 + j)) < \infty \quad \text{w.p. } 1$$

for $j = 1, 2, \cdots, n_0 - 1$.

The lemma now follows since for any k,

$$e^{-\alpha k\delta} Z(k\delta) \leqq \sum_{j=0}^{n_0-1} \exp(-\alpha\delta([k/n_0]n_0 + j)) Z(\delta([k/n_0]n_0 + j)) .$$ □

Combining (3.4), Lemma 1, Lemma 2′ and Lemma 3′ we arrive at

THEOREM 2′. *Let* $\sum_j p_j j \log j < \infty$. *Then fix* $0 < x < \infty$, $\delta > 0$. $\exists$ *a set* E *of probability one such that for every* $\omega \in E$

$$A(x, n\delta, \omega) \to A(x) \qquad \textit{as } n \to \infty . \tag{4.6}$$

Letting δ and x range over positive rationals and noting that $A(x)$ is continuous in x we get

THEOREM 2″. *Let* $\sum p_j j \log j < \infty$. *Then* $\exists$ *a set* E *of probability one such that for every* $\omega \in E$,

$$\rho(A(\cdot, n\delta, \omega), A(\cdot)) \to 0 \tag{4.7}$$

for every positive rational δ, *where* ρ *is the usual Lévy metric.*

COROLLARY 2. *Let* $\sum p_j j \log j < \infty$. *Let* $h(\cdot)$ *be any bounded real valued function on* $[0, \infty)$ *and continuous almost everywhere (with respect to the Lebesgue measure) on the support of* G. *Then* $\exists$ *a set* E *of probability one such that for* ω *in* E

$$\int_0^\infty h(x)\, dA(x, n\delta, \omega) \to \int_0^\infty h(x)\, dA(x)$$

for each positive rational δ.

PROOF. Just note that $A(\cdot)$ is absolutely continuous with respect to Lebesgue measure.

We now finish the proof of Theorem 2.

The following inequalities are easily checked. For $\delta > 0$, $n\delta \leqq t < (n+1)\delta$, $\delta < x \leqq \infty$

$$Z(x-\delta, n\delta) - \sum_{i=1}^{Z(n\delta)} \eta_i \leqq Z(x, t) \leqq Z(x+\delta, (n+1)\delta) + \sum_{i=1}^{Z(n\delta)} \eta_i \tag{4.8}$$

where η_i = number of objects that die by time $(n+1)\delta$ in the line of descent initiated by a particle of age x_i at time $n\delta$ ($x_1, x_2, \cdots, x_{Z(n\delta)}$, are the ages of the particles at time $n\delta$). Thus,

$$A(x, t) \leqq \frac{\dfrac{Z(x+\delta, (n+1)\delta)}{Z((n+1)\delta)} \dfrac{Z((n+1)\delta)}{Z(n\delta)} + \dfrac{1}{Z(n\delta)} \sum_{i=1}^{Z(n\delta)} \eta_i}{1 - \dfrac{1}{Z(n\delta)} \sum_{i=1}^{Z(n\delta)} \eta_i}\,. \tag{4.9}$$

Repeating the proof of Lemma 2′ we see that

$$(Z(n\delta))^{-1}[Z((n+1)\delta) - \sum_{i=1}^{Z(n\delta)} M_{x_i}(\infty, \delta)] \to 0 \quad \text{w.p. } 1$$

and

$$\frac{1}{Z(n\delta)} \sum_{i=1}^{Z(n\delta)} (\eta_i - E\eta_i) \to 0 \quad \text{w.p. } 1\,.$$

By Corollary 3′

$$\lim_{n\to\infty} \frac{1}{Z(n\delta)} \sum_{i=1}^{Z(n\delta)} M_{x_i}(\delta) = \int M_x(\delta)\, dA(x) \equiv (1 + r_1(\delta))$$

and since $E(\eta_i) \leqq CG_{x_i}(\delta)$ for some constant C independent of x_i

$$\limsup_{n\to\infty} \frac{1}{Z(n\delta)} \sum_{i=1}^{Z(n\delta)} E(\eta_i) \leqq C \int G_x(\delta)\, dA(x) \equiv r_2(\delta) \quad \text{w.p. } 1\,.$$

Thus w.p. 1

$$\limsup_{t\to\infty} A(x, t) \leqq \frac{A(x+\delta)(1 + r_1(\delta)) + r_2(\delta)}{1 - r_2(\delta)}\,.$$

Now letting $\delta \downarrow 0$ we get by noting that $r_i(\delta) \to 0$ as $\delta \downarrow 0$, $i = 1, 2$,

$$\limsup_{t\to\infty} A(x, t) \leqq A(x) .$$

A similar argument applies to the $\liminf_{t\to\infty} A(x, t)$.

5. Consequences. The following strengthening of Corollary 2 is an immediate consequence of Theorem B.

COROLLARY 3. *Under the hypothesis of Theorem* B

$$\int_0^\infty h(x)\, dA(x, t, \omega) \to \int_0^\infty h(x)\, dA(x)$$

w.p. 1 *for any* $h(\cdot)$ *bounded continuous almost everywhere* (w.r.t. *Lebesgue measure*) *on the support of* G.

Recall the definition of $V(x)$ *and* $V_t(\omega)$:

$$V(x) = m \int_0^\infty e^{-\alpha u}\, dG_x(u) , \qquad V_t(\omega) = \sum_{i=1}^{Z(t,\omega)} V(x_i(t, \omega)) .$$

PROPOSITION 2. *The family* $\{V_t(\omega)e^{-\alpha t}, t \geqq 0\}$ *is a martingale.*

PROOF. It suffices by additivity (3.1) to check that

$$e^{\alpha t} V(x) = E\{V_t(\omega) \mid \text{the initial particle was of age } x\} . \tag{5.1}$$

Denote the right side of (5.1) by $h(t, x)$. Then,

$$h(t, x) = V(x + t)(1 - G_x(t)) + m \int_0^t h(t - u, 0)\, dG_x(u) . \tag{5.2}$$

Specializing (5.2) to the case $x = 0$ and setting $\tilde{h}(t) = e^{-\alpha t} h(t, 0)$ we get

$$\tilde{h}(t) = V(t)(1 - G(t))e^{-\alpha t} + \int_0^t \tilde{h}(t - u)\, d\tilde{G}(u) \tag{5.3}$$

where $d\tilde{G}(u) = me^{-\alpha u}\, dG(u)$.

Since $\tilde{h} \equiv 1$ is the only bounded solution of (5.3) (here the definition of $V(\cdot)$ is used) we conclude that $h(t, 0) \equiv e^{\alpha t}$ and this with (5.2) yields (5.1). □

PROOF OF THEOREM C. The function $V(x)$ has the same discontinuity set as $G(\cdot)$ and hence satisfies the hypothesis of Corollary 3. Thus under the hypothesis of Theorem B

$$V_t/Z_t \equiv \int V(x)\, dA(x, t, \omega) \to \int V(x)\, dA(x) = n_1^{-1} \quad \text{w.p. } 1 . \tag{5.4}$$

By the preceding proposition $V_t(\omega)e^{-\alpha t}$ is a nonnegative martingale and hence converges with probability one. Call this limit $W'(\omega)$. From (5.4) it is clear that under the hypothesis of Theorem B

$$\lim_{t\to\infty} Z(t, \omega)e^{-\alpha t} = n_1^{-1} W'(\omega) \equiv W(\omega) .$$

This proves Theorem C in view of the known result [2] that $W(\omega)$ is nondegenerate at zero iff $\sum p_j j \log j < \infty$. □

If $\sum p_j j \log j = \infty$ then $Z(t)e^{-\alpha t} \to 0$ in probability as shown in [2]. By Theorem A, V_t/Z_t converges in probability to $\alpha^{-1}(m - 1)$. By Proposition 2, $\lim_{t\to\infty} V_t(\omega)e^{-\alpha t} = W'(\omega)$ exists w.p. 1. Thus we get

COROLLARY 4. $\sum p_j j \log j = \infty$ *then*

$$\lim_{t\to\infty} V_t(\omega)e^{-\alpha t} = 0 \quad \text{w.p. } 1 . \tag{5.5}$$

If $V(x)$ is bounded below this implies

$$\lim_{t\to\infty} Z(t, \omega)e^{-\alpha t} = 0 \quad \text{w.p. } 1 . \tag{5.6}$$

At any rate for each $K < \infty$ in the support of G we do have

$$\lim_{t\to\infty} Z(K, t, \omega)e^{-\alpha t} = 0 \quad \text{w.p. } 1$$

since

$$\inf_{0\leq x\leq K} V(x) > 0 .$$

Summarizing this yields

COROLLARY 5. (a) *If* $\sum p_j j \log j = \infty$ *and* $V(\cdot)$ *is bounded below as the support of G then*

$$\lim_{t\to\infty} Z(t)e^{-\alpha t} = 0 . \tag{5.7}$$

(b) *If* $\sum p_j j \log j = \infty$ *then for any K in the support of G*

$$\lim_{t\to\infty} Z(K, t, \omega)e^{-\alpha t} = 0 \quad \text{w.p. } 1 . \tag{5.8}$$

Perhaps (5.7) is true without any hypothesis on V.

Let $N(t, \omega)$ be the number of splits up to time t. Then we may write

$$Z(t, \omega) = Z(0, \omega) + \sum_{i=1}^{N(t,\omega)} \xi_i \tag{5.9}$$

where ξ_i is the net addition to the process at the ith split. Clearly, ξ_i $i = 1, 2, \cdots$ are i.i.d. with distribution $P\{\xi_i = j\} = p_{j-1}$. Further $N(t, \omega) \to \infty$ w.p. 1. By the strong law of large numbers (5.9) yields

$$Z(t, \omega)/N(t, \omega) \to (m - 1) \quad \text{w.p. } 1 \tag{5.10}$$

and hence

COROLLARY 6. *If* $\sum p_j j \log j < \infty$ *then*

$$N(t, \omega)e^{-\alpha t} \to W(\omega)(m - 1)^{-1} \quad \text{w.p. } 1 .$$

If $Y(t, \omega)$ is the total number of progeny up to time t we may write $Y(t, \omega) = \sum_{i=1}^{N(t,\omega)} (\xi_i + 1)$ and hence we get again by the strong law

COROLLARY 7. *If* $\sum p_j j \log j < \infty$ *then*

$$Y(t, \omega)e^{-\alpha t} \to W(\omega)m(m - 1)^{-1} \quad \text{w.p. } 1 .$$

CONCLUDING REMARKS. The extension of the results of this paper to the more general branching mechanisms considered by Jagers [9], Crump and Mode [6] etc. seems straightforward enough not to need a separate publication.

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