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CENTRAL TERMS OF MARKOV WALKS

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A {0, 1}-valued discrete time stochastic process $\beta = \{\beta_n\}_{n=1}^{\infty}$ will be referred to simply as a *walk*. The notion of central (modal) term of a binomial distribution is generalized to the conditional-on-the-past distributions of Nth partial sums of walks. The emphasis here is placed on the smallest possible central term $V_A(N)$ within a given class A of walks. If A consists of (i) all walks, (ii) all stationary independent walks, (iii) all stationary Markov walks which are invariant under interchange of 0 and 1, then, respectively,

(i) $\{N \cdot V_A(N)\}_{N=1}^{\infty}$,
(ii) $\{N^{\frac{1}{2}} \cdot V_A(N)\}_{N=1}^{\infty}$,
(iii) $\{N \cdot V_A(N)/(\log N)^{\frac{1}{2}}\}_{N=2}^{\infty}$

are bounded sequences which are bounded away from zero.

1. Introduction. We say $\beta = \{\beta_n\}_{n=1}^{\infty}$ is a *walk* if β is a {0, 1}-valued discrete time stochastic process. We sometimes will write $\beta(s)$ in place of $P[\beta_1 \cdots \beta_k = s]$, for $s \in \{0, 1\}^k$. A walk β is *stationary* provided $\beta(s) = \beta(0s) + \beta(1s)$ for all finite binary sequences s. β is *invariant* (under interchange of 0 and 1) provided $\beta(s) = \beta(s')$ for each finite binary sequence s, where s' is the sequence obtained from s by interchanging 0 and 1. β is a *Markov walk* if $P[\beta_{k+1} = 1 \mid \beta_1 \cdots \beta_k = s_1 \cdots s_k] = P[\beta_{k+1} = 1 \mid \beta_k = s_k]$ for all possible k and $s_1 \cdots s_k$.

For N a positive integer, and β a walk, define the *N-central term* of β by

$$g_N(\beta) = \sup_{0 \leq k < \infty; s \in \{0,1\}^k} \max_{0 \leq r \leq N} P[\textstyle\sum_{i=1}^N \beta_{k+i} = r \mid \beta_1 \cdots \beta_k = s] ; \tag{1.1}$$

if $k = 0$ or $P[\beta_1 \cdots \beta_k = s] = 0$ we set

$$P[\textstyle\sum_{i=1}^N \beta_{k+i} = r \mid \beta_1 \cdots \beta_k = s] = P[\sum_{i=1}^N \beta_{k+i} = r] .$$

Of special interest is the smallest possible N-central term within a given class A of walks,

$$V_A(N) = \inf_{\beta \in A} g_N(\beta) . \tag{1.2}$$

It was shown in [6] that if W is the collection of all walks, then

$$1/(N+1) \leqq V_W(N) < 2e/(N+1) \qquad \text{for all } N \geqq 1 . \tag{1.3}$$

The lower bound is obvious and the upper bounds are obtained by considering the "fair die" processes (whereby β_1 is determined by tossing a fair coin and the sojourn times are determined by repeatedly rolling a fair $(N+1)$-sided die).

If SI is the set of all stationary independent walks, then

$$V_{\mathrm{SI}}(N) \sim (2/\pi N)^{\frac{1}{2}} , \tag{1.4}$$

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meaning the limit of the quotient tends to 1. (1.4) holds because the binomial distribution with smallest central term has success probability $p = \frac{1}{2}$ for N odd and $p = N/2(N+1)$ or $p = (N+2)/2(N+1)$ for N even. In all cases $p \sim \frac{1}{2}$, and (1.4) follows from Stirling's formula.

Our main objective now is to get information about the asymptotic behavior of $V_{\mathrm{SM}}(N)$, where SM is the set of all stationary Markov walks. While we have not been able to come up with a statement as strong as (1.3) or (1.4) for this class, we find that by restricting attention further to the class SIM of stationary invariant Markov walks,

$$\{N \cdot V_{\mathrm{SIM}}(N)/(\log N)^{\frac{1}{2}}\}_{N=2}^{\infty} \tag{1.5}$$

is a bounded sequence which is bounded away from 0. In particular,

$$V_{\mathrm{SIM}}(N) \leqq 4(\log N)^{\frac{1}{2}}/N \qquad \text{for all} \quad N \geqq 12\,, \tag{1.6}$$

whence $V_{\mathrm{SM}}(N)$ satisfies the same inequality.

At this point we cannot give compelling reasons for the invariance restriction, though the stationarity condition seems natural enough. In [1] Blackwell defines a stationarity operator which preserves optimality, whence $V_W(N) = V_S(N)$ for all $N \geqq 1$, where S is the set of all stationary walks. It is easy to see that the invariance operator $\bar{I}$ defined by $(\bar{I}\beta)(s) = \frac{1}{2}[\beta(s) + \beta(s')]$ preserves stationarity and optimality, so that, letting I denote the set of all invariant walks,

$$V_W(N) = V_S(N) = V_I(N) = V_{S\cap I}(N) \qquad \text{for all} \quad N \geqq 1\,. \tag{1.7}$$

(1.7) in no way implies the likes of $V_{\mathrm{SM}}(N) = V_{\mathrm{SIM}}(N)$. Unfortunately, both the stationarity and invariance operators can lengthen the memory of a process, and we may have lost some generality. In this connection, it is known that $V_W(2) = V_{\mathrm{SM}}(2) = V_{\mathrm{SIM}}(2)$, while $V_W(3) < V_{\mathrm{SIM}}(3)$; see [3] and [5].

Henceforth we let $B_1 = \mathrm{SIM}$ and $V_1(N) = V_{\mathrm{SIM}}(N)$. Note that a stationary invariant Markov walk is just a Markov chain on $\{0, 1\}$ with initial distribution $(\frac{1}{2}, \frac{1}{2})$ and transition matrix of the form $\begin{pmatrix} 1-a & a \\ a & 1-a \end{pmatrix}$.

REMARKS. The following statistical game is discussed in [6].

Let N be a strictly positive integer. Player 2 selects a $\{0, 1\}$-valued discrete time process $\beta = \{\beta_n\}_{n=1}^{\infty}$ from the collection W of all such processes. Player 1 (the statistician) observes realizations of as many terms of β as he wants, and then predicts the sum of the next N terms. 1 wins one dollar if his prediction turns out to be correct, and wins nothing otherwise.

Of course, $1/(N+1)$ is what the statistician expects to get in this game by waiving his right to observe part of 2's process and arbitrarily predicting at the outset. The inequalities (1.3) then suggest the game to be "of a hopeless order" for the statistician; for the purpose of prediction, there are processes so "random" that he can learn very little about them by observing them. Asymptotically speaking, it is as if he has been asked to produce a procedure so "robust" that he can do little better than use a mindless procedure.

The natural thing to do in attempting to make this game more interesting (or at least more profitable) for the statistician is to restrict further the class of processes from which 2 may choose his strategy. In this way we get a new class of games whenever we alter the rules by replacing W with some subset A of W. The value of the Nth game will always exist and equal $V_A(N)$ because W is compact [1].

2. An upper bound for $V_1(N)$. Let $\beta(r\mid 1) = P[\sum_{i=2}^{N+1} \beta_i = r \mid \beta_1 = 1]$; the dependence on N is suppressed.

Using the Markov property, stationarity and invariance, one can show that for all $\beta \in B_1$ and $0 \leqq r \leqq N$,

$$P[\textstyle\sum_{i=1}^{N} \beta_i = r] \leqq \max_{0 \leqq r \leqq N} \beta(r\mid 1)\,,$$

whence by (1.1)

$$g_N(\beta) = \max_{0\leqq r\leqq N} \beta(r\mid 1)\,. \tag{2.1}$$

It follows that

$$V_1(N) = \inf_{\beta\in B_1} \max_{0\leqq r\leqq N} \beta(r\mid 1)\,. \tag{2.2}$$

As a special case of (1.2) in [2] (taking $q = \bar{q} = a$), or directly, using (3) of [6] and the method of Lemmas 1 and 2 in [6], it may be shown that if $\beta \in B_1$ has transition probability (t.p.) $a = 1 - b$, then

$$\begin{aligned}\beta(r\mid 1) &= b^N \qquad \text{for} \quad r = N \\ &= \textstyle\sum_{i\geqq 0} [\binom{r}{i}\binom{N-r-1}{i-1} a^{2i} b^{N-2i} + \binom{r}{i}\binom{N-r-1}{i} a^{2i+1} b^{N-2i-1}] \qquad \text{for} \quad r < N\,.\end{aligned} \tag{2.3}$$

Combining this with (2.2) yields

$$\begin{aligned}V_1(N) = \inf_{0\leqq a\leqq 1,\, b=1-a} \max\, [b^N, \max_{0\leqq r<N} \textstyle\sum_{i\geqq 0} [\binom{r}{i}\binom{N-r-1}{i-1} a^{2i} b^{N-2i} \\ + \binom{r}{i}\binom{N-r-1}{i} a^{2i+1} b^{N-2i-1}]]\,.\end{aligned} \tag{2.4}$$

It is clear from (2.4) that the infimum is attained by some a.

LEMMA 1. *Let $\beta \in B_1$ have* t.p. $a = 1 - b$. *Then*

$$g_N(\beta) \leqq \max\, [b^N, 2a \max_i [\tbinom{[N/2]}{i} a^i b^{[N/2]-i}]]\,,$$

where $[N/2]$ denotes the greatest integer in $N/2$.

PROOF. For $r < N$, $\beta(r\mid 1)$

$$\begin{aligned}&= \textstyle\sum_{i\geqq 0} [\binom{r}{i}\binom{N-r-1}{i-1} a^{2i} b^{N-2i} + \binom{r}{i}\binom{N-r-1}{i} a^{2i+1} b^{N-2i-1}] \\ &\leqq a\{\min\, [\max_i \tbinom{r}{i} a^i b^{r-i}, \max_i \tbinom{N-r-1}{i-1} a^{i-1} b^{N-r-i}] \\ &\qquad + \min\, [\max_i \tbinom{r}{i} a^i b^{r-i}, \max_i \tbinom{N-r-1}{i} a^i b^{N-r-i-1}]\} \\ &\leqq 2a \max_i [\tbinom{[N/2]}{i} a^i b^{[N/2]-i}]\,,\end{aligned}$$

the last inequality following from the fact that for $m < n$ and fixed success probability p, the central (modal) term of the binomial distribution for m trials is at least as large as the central term for n trials.

LEMMA 2. *$\binom{n}{m}p^m q^{n-m} < (npq)^{-\frac{1}{2}}$ for any $1 \leqq m \leqq n-1$, as long as $2/(n+1) < p = 1 - q < (n-1)/(n+1)$.*

PROOF. The central term of the binomial distribution with parameters n and $p = 1 - q$ is defined ([4], page 151) to be $\binom{n}{m}p^m q^{n-m}$, where m is $[(n+1)p]$, the greatest integer in $(n+1)p$. The (not necessarily unique) mode of the binomial distribution occurs at m. Hence $\binom{n}{m}p^m q^{n-m}$ is the central term for any binomial distribution with $m \leqq (n+1)p < m+1$. Using the Stirling bounds (9.15) on page 54 of [4], together with the fact that $\binom{n}{m}p^m q^{n-m}$ (m and n fixed) is a maximum for $p = m/n$, we have

$$\binom{n}{m}p^m q^{n-m} < \tfrac{1}{2}(n(m/n)((n-m)/n))^{-\frac{1}{2}} < (npq)^{-\frac{1}{2}} .$$

THEOREM 1. *$V_1(N) < 4(\log N)^{\frac{1}{2}}/N$ for $N \geqq 12$.*

PROOF. Take $b = 1 - a = ((\log N)^{\frac{1}{2}}/N)^{1/N}$ in Lemma 1 and note that a satisfies the conditions imposed on p in Lemma 2; also, $b > \frac{3}{4}$ and $[N/2] > N/3$. We have

$$\begin{aligned} 2a \max_i [\tbinom{[N/2]}{i}a^i b^{[N/2]-i}] &\leqq 2a/([N/2]ab)^{\frac{1}{2}} < 4(a/N)^{\frac{1}{2}} \\ &\leqq 4(\log N)^{\frac{1}{2}}/N , \qquad \text{since} \quad 1 - e^{-x} \leqq x . \end{aligned}$$

COROLLARY 1. *If SM is the set of all stationary Markov walks, then*

$$V_{\mathrm{SM}}(N) < 4(\log N)^{\frac{1}{2}}/N \qquad \textit{for} \quad N \geqq 12 .$$

COROLLARY 2. *The minimizing a in (2.4) satisfies (for $N \geqq 12$)*

$$1 - (4(\log N)^{\frac{1}{2}}/N)^{1/N} < a < (4(\log N)^{\frac{1}{2}}/N)^{1/N} .$$

PROOF. Neither a^N nor b^N may exceed $V_1(N)$.

3. A lower bound for $V_1(N)$.

THEOREM 2. *There exists $M > 0$ such that $V_1(N) > (\log N)^{\frac{1}{2}}/4N$ for all $N \geqq M$.*

PROOF. Assume N is even; this assumption will be removed at the end of the proof. Let $n = N/2$ and $C_N = 1 - (4(\log N)^{\frac{1}{2}}/N)^{1/N}$. Then Corollary 2 and the definition of $V_1(N)$ imply

$$V_1(N) \geqq \inf_{C_N \leqq a} \beta(n \mid 1) ; \tag{3.1}$$

as before, a is the generic t.p. for β. We will show that for all even N sufficiently large

$$\inf_{C_N \leqq a < \frac{1}{2}} \beta(n \mid 1) > (\log N)^{\frac{1}{2}}/(4\pi)^{\frac{1}{2}}N \tag{3.2}$$

and

$$\inf_{\frac{1}{2} \leqq a \leqq 1} \beta(n \mid 1) > (\pi N)^{-\frac{1}{2}} . \tag{3.3}$$

To show (3.2), assume $C_N \leqq a < \frac{1}{2}$. From (2.3) we have

$$\beta(n \mid 1) \geqq \textstyle\sum_{i \geqq 1} \binom{n}{i}\binom{n-1}{i-1} a^{2i} b^{N-2i} . \tag{3.4}$$

The Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ is given by [7] (page 68, 4.3.2):

$$(3.5) \qquad P_n^{(\alpha,\beta)}(x) = \sum_{i=0}^{n} \binom{n+\alpha}{n-i}\binom{n+\beta}{i}((x-1)/2)^i((x+1)/2)^{n-i} .$$

Taking $x = (b^2 + a^2)/(b^2 - a^2)$, we see that

$$(3.6) \qquad \sum_{i\geqq 1} \binom{n}{i}\binom{n-1}{i-1} a^{2i}b^{N-2i} = (b^2 - a^2)^n P_n^{(-1,0)}(x) .$$

We will apply the following result, due to Darboux ([7], page 194, Theorem 8.21.7):

Let α and β be arbitrary real numbers. Then

$$(3.7) \qquad \begin{aligned} P_n^{(\alpha,\beta)}(x) \sim (x-1)^{-\alpha/2}(x+1)^{-\beta/2}\{(x+1)^{\frac{1}{2}} + (x-1)^{\frac{1}{2}}\}^{\alpha+\beta} \\ \times (2\pi n)^{-\frac{1}{2}}(x^2-1)^{-\frac{1}{4}}\{x + (x^2-1)^{\frac{1}{2}}\}^{n+\frac{1}{2}} \end{aligned}$$

for $|x| > 1$. This formula holds *uniformly* for $|x| > 1$ in the sense that the ratio tends uniformly to 1. For $\alpha = -1$ and $\beta = 0$ the r.h.s. of (3.7) equals $(4\pi n)^{-\frac{1}{2}}(a/b)^{\frac{1}{2}}(b^2 - a^2)^{-n}$. From (3.6) and (3.7) we then deduce the existence of $M > 0$ such that $N = 2n > M$ implies (for *all* $C_N \leqq a < \frac{1}{2}$)

$$(3.8) \qquad \sum_{i\geqq 1} \binom{n}{i}\binom{n-1}{i-1} a^{2i}b^{N-2i} > (a/4\pi n)^{\frac{1}{2}} .$$

Since $(a/4\pi n)^{\frac{1}{2}}$ is an increasing function of a for $a > 0$, its infimum over the interval $C_N \leqq a < \frac{1}{2}$ is attained at C_N. Furthermore, $C_N = 1 - (4(\log N)^{\frac{1}{2}}/N)^{1/N} > \log N/2N$ for all N sufficiently large, because $1 - e^{-x} > x/(x+1)$ whenever $x > -1$. (3.2) follows from this, (3.4) and (3.8).

We now verify (3.3). Assume $\frac{1}{2} < a < 1$. Using the facts that $a/b > 1$ and $\binom{n-1}{i-1} + \binom{n-1}{i} = \binom{n}{i}$, we have

$$\begin{aligned} \beta(n\,|\,1) &= \sum_{i\geqq 0} \binom{n}{i} a^{2i}b^{N-2i}[\binom{n-1}{i-1} + \binom{n-1}{i}a/b] \\ &> \sum_{i\geqq 0} \binom{n}{i}^2 a^{2i}b^{N-2i} \\ &= \sum_{i\geqq 0} \binom{n}{i}^2 b^{2i}a^{N-2i} \\ &= (a^2 - b^2)^n P_n^{(0,0)}((a^2 + b^2)/(a^2 - b^2)) \\ &\geqq (2\pi n)^{-\frac{1}{2}} = (\pi N)^{-\frac{1}{2}} \end{aligned}$$

for all even N sufficiently large, uniformly in $\frac{1}{2} \leqq a \leqq 1$, by (3.7) and the continuity of $\beta(n\,|\,1)$ in a.

This proves (3.3), which together with (3.2) implies

$$(3.9) \qquad V_1(N) > (\log N)^{\frac{1}{2}}/(4\pi)^{\frac{1}{2}}N$$

for all even N sufficiently large.

Since $V_1(N) \geqq V_1(N+1)$ for all $N \geqq 1$, we can remove the assumption that N is even, provided we replace the lower bound of (3.9) with that of Theorem 2.

This concludes the proof of Theorem 2.

REMARK 3.1. The proofs of Theorems 1 and 2 show that one of $\beta(N\,|\,1)$, $\beta(N/2\,|\,1)$ must nearly attain the order of the N-central term if β is an invariant Markov walk. This need not be true if β has unequal t.p.'s, and as strong as the Darboux approximation seems (holding uniformly in $|x| > 1$), it is not strong enough to deal with the asymmetric case, which requires an approximation of $P_n^{(\alpha,\beta)}(x)$ that permits α and β to depend on n and x.

REMARK 3.2. R. L. Dobrusin [2] has determined all possible limiting distributions of $S_N = \sum_{i=1}^N \beta_i^{(N)}$, where $\{\beta^{(N)}\}_{N=1}^\infty = \{\{\beta_n^{(N)}\}_{n=1}^\infty\}_{N=1}^\infty$ is a sequence of stationary Markov walks. We have seen that the smallest possible N-central term for a stationary invariant Markov walk is of order $(\log N)^{\frac{1}{2}}/N$, and the proof of Theorem 1 shows this order is attained by a t.p. of $a = a_N = 1 - ((\log N)^{\frac{1}{2}}/N)^{1/N}$. Since $a_N \to 0$ and $N \cdot a_N \to \infty$, Dobrusin's results ([2], page 98, (1.3)) imply that $(S_N - N/2)/(N(1 - a_N)/4a_N)^{\frac{1}{2}}$ has a limiting standard normal distribution.

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