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LOCAL HÖLDER CONDITIONS FOR THE LOCAL TIMES OF CERTAIN STATIONARY GAUSSIAN PROCESSES[1]

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A local Hölder condition is obtained for the local time of a stationary Gaussian process with spectral density function proportional to $(a^2 + \lambda^2)^{-(\alpha+\frac{1}{2})}$. A lower bound for the Hausdorff measure of the zero set of the process is also obtained.

1. Introduction. Let $X(t, \omega)$ be a stochastically continuous separable stationary Gaussian process defined on the probability space $(\Omega, \mathscr{F}, P)$ and with spectral density function $f(\lambda)$. The standard deviation $\sigma(h)$ of the process is given by

$$(1) \qquad \sigma^2(h) = E((X(t+h, \omega) - X(t, \omega))^2) = 4 \int_{-\infty}^{\infty} \sin^2(\tfrac{1}{2}\lambda h) f(\lambda)\, d\lambda .$$

We denote by $T(a, t, \omega)$ the amount of time spent by the process below the level a during the time interval $(0, t]$, i.e.

$$T(a, t, \omega) = \int_0^t D(a - X(s, \omega))\, ds$$

where $D(x) = 1$ for $x > 0$ and is zero otherwise.

In a recent series of papers [1, 2, 3, 4, 5] Berman proved that if $\sigma(h) \sim |h|^\alpha$, $0 < \alpha < 1$, for small h, plus other regularity conditions then there exists a stochastic process $\varphi(x, t, \omega)$, the local time of $X(t, \omega)$, jointly continuous in x and t, such that

$$(2) \qquad T(b, t, \omega) - T(a, t, \omega) = \int_a^b \varphi(x, t, \omega)\, dx .$$

Furthermore, for all $\gamma < 1 - \alpha$, there exist random variables η and Δ such that

$$(3) \qquad \sup_{0 \leqq x \leqq 1; 0 \leqq t, t+h \leqq 1} |\varphi(x, t+h) - \varphi(x, t)| \leqq \Delta |h|^\gamma$$

for all h satisfying $|h| \leqq \eta$.

The local time is, for fixed x, an increasing function of t whose points of increase coincide with the points of the set

$$Q_x = \{t : X(t, \omega) = x\} .$$

In [4] Berman used the uniform Hölder condition (3) to show that if, for example, $X(t, \omega)$ is ergodic and $f(\lambda) \sim |\lambda|^{-(1+2\alpha)}$ then $P(\forall\, x, \dim Q_x = 1 - \alpha) = 1$ where $\dim Q_x$ denotes the Hausdorff dimension of Q_x.

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This paper is devoted to a two-sided Hölder condition for the local time. Such a condition has a bearing on the exact measure function for the sets Q_x and in Section 6 we show that if $X(t, \omega)$ has spectral density function $f(\lambda)$ given by (4) then the Hausdorff measure of Q_x with respect to the function $\psi(h) = h^{1-\alpha}(\log(-\log h))^\alpha$ is almost surely strictly positive for each given x. This result gives a more precise lower bound for the Hausdorff measure of Q_x than that obtained by Berman although it should be emphasized that our estimate does not necessarily hold uniformly in x. The question of an upper bound seems to be more difficult but in view of the work of Taylor and Wendel [15] it would seem plausible that $\psi(h) = h^{1-\alpha}(\log(-\log h))^\alpha$ is the exact measure function of Q_x for stationary Gaussian processes with spectral density functions of the form (4).

As $\varphi(x, t)$ is an increasing function of t which increases only at the points of the set Q_x, it follows that if $x \neq X(t, \omega)$ then $\varphi(x, t + h) = \varphi(x, t)$ for all sufficiently small h. We therefore consider the behaviour of $\varphi(X(t, \omega), t + h) - \varphi(X(t, \omega), t)$ for small h. We prove the following theorem.

THEOREM 1. *If the process $X(t, \omega)$ has spectral density function $f(\lambda)$ given by*

$$f(\lambda) = a^{2\alpha} \frac{\Gamma(\alpha + \frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(\alpha)} (\lambda^2 + a^2)^{-(\alpha+\frac{1}{2})}, \qquad 0 < \alpha < \tfrac{1}{2}, \tag{4}$$

then there exist constants c_1 and c_2 with $0 < c_1 \leqq c_2 < \infty$ such that

$$c_1 \leqq \limsup_{h \downarrow 0} \frac{|\varphi(X(t, \omega), t + h) - \varphi(X(t, \omega), t)|}{h^{1-\alpha}(\log(-\log h))^\alpha} \leqq c_2 \tag{5}$$

for almost all ω. (The normalizing constants in (4) are chosen so that $E(X(t, \omega)^2) = 1$.)

The method of proof does in fact provide bounds for c_1 and c_2 in terms of a and α. However, as $c_1 \neq c_2$ except in the limit as $\alpha \to \frac{1}{2}$, we do not give them explicitly. We restrict ourselves to the spectral density function (4), although the result holds for certain other strictly nondeterministic stationary Gaussian processes and also for certain processes with stationary Gaussian increments. One reason for considering the spectral density (4) is that the upper bound in (5) can then be shown to hold for first passage times. The methods of Taylor and Wendel [15] can then be applied to obtain the aforementioned lower bound for the Hausdorff measure of the zero set of $X(t, \omega)$.

In Section 2 we introduce the necessary notation and state certain results in the theory of least squares prediction. Section 3 contains some preliminary lemmas, the upper bound is obtained in Section 4 and the lower in Section 5. The relationship with the Hausdorff measure of the zero set of the process is discussed in Section 6.

2. Notation.

2.1. A necessary and sufficient condition that a stationary Gaussian process be strictly nondeterministic is that the spectral distribution $F(\lambda)$ should be absolutely continuous with respect to Lebesgue measure and satisfy

$$\int_{-\infty}^{\infty} \frac{|\log f(\lambda)|}{1+\lambda^2}\,d\lambda < \infty$$

where $f(\lambda) = F'(\lambda)$. This guarantees the existence of the function $\hat{g}(x)$ defined below which plays an important role in the paper. For the justification of the definitions and results listed below see Karhunen [9].

$\mathscr{F}_t$: σ-algebra generated by $X(s)$, $s \leqq t$.

$g(w)$: $g(w) = \exp\left(-\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{1+\lambda w}{\lambda - w}\,\frac{\log f(\lambda)}{1+\lambda^2}\,d\lambda\right)$ $\mathscr{I}(w) < 0$, defined for real x by $g(x) = \lim_{y\uparrow 0} g(x - iy)$.

$\hat{g}(x)$: Fourier transform of $g(x)$ in $L^2(-\infty, \infty)$ $g(x) =$ l.i.m. $(2\pi)^{-\frac{1}{2}}\int_{-\infty}^{\infty} e^{iux} g(u)\,du$.

$X_p(s\,|\,t)$: Least squares predictor of $X(s)$ based on $\mathscr{F}_t$; $X_p(s\,|\,t) = E(X_s\,|\,\mathscr{F}_t)$.

$X_e(s\,|\,t)$: prediction error; $X_e(s\,|\,t) = X(s) - X_p(s\,|\,t)$.

$\xi(t, \omega)$: Brownian motion.

We have the following representations, the integrals being interpreted as integrals in quadratic mean:

$$X(t, \omega) = \int_{-\infty}^{t} \hat{g}(t-u)\,d\xi(u, \omega)\,, \tag{6}$$

$$X_e(s\,|\,t) = \int_t^s \hat{g}(s-u)\,d\xi(u, \omega)\,. \tag{7}$$

The function $\hat{g}(x)$ may be assumed real ($f(\lambda)$ symmetric) and the following hold:

$$\hat{g}(x) \in L^2(-\infty, \infty)\,, \tag{8}$$

$$\hat{g}(x) = 0\,, \quad x \leqq 0\,, \tag{9}$$

$$E(X_e(s\,|\,0)^2) = \int_{-s}^{0} \hat{g}(-u)^2\,du\,, \tag{10}$$

$$\sigma^2(s) = \int_{-\infty}^{\infty} (\hat{g}(s-u) - \hat{g}(-u))^2\,du\,. \tag{11}$$

In the present case with $f(\lambda)$ given by (4) we have

$$g(w) = a^{\alpha}\left(\frac{\Gamma(\alpha+\frac{1}{2})}{\Gamma(\alpha)\Gamma(\frac{1}{2})}\right)^{\frac{1}{2}}(w - ia)^{-(\alpha+\frac{1}{2})} \tag{12}$$

and

$$\hat{g}(x) = \frac{(2a)^{\alpha}}{(\Gamma(2\alpha))^{\frac{1}{2}}}\,x^{\alpha-\frac{1}{2}}e^{-ax}\,, \qquad x > 0\,. \tag{13}$$

These expressions were first given by Yaglom [16, 17]. We also note that for small h

(14) $$\sigma^2(h) \sim \frac{2\Gamma(1-\alpha)}{\alpha\Gamma(\alpha)} (\tfrac{1}{2}a|h|)^{2\alpha}$$

which follows from (4) by a theorem of Pitman [13].

2.2. In the remainder of the paper bold face letters will denote row vectors, e.g.

$$\mathbf{s}_m = (s_1, \cdots, s_m)$$

and integrals of the form

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(s_1, \cdots, s_m)\, ds_1 \cdots ds_m$$

will be written $\int f(\mathbf{s}_m)\, d\mathbf{s}_m$. If the range of integration is not explicitly stated it will be taken to be R^m, an integral over a subset B_m of R^m being written as $\int_{B_m} f(\mathbf{s}_m)\, d\mathbf{s}_m$. Finally, for $0 < \delta \leqq 1$ we define the sets $\mathscr{S}_m(\delta)$ and $\mathscr{J}_m(\delta)$ by

(15) $$\mathscr{S}_m(\delta) = \{\mathbf{s}_m : 0 \leqq 1 - \delta < s_1 < \cdots s_m < 1\}$$

and

(16) $$\mathscr{J}_m(\delta) = \{\mathbf{s}_m : 0 \leqq 1 - \delta < s_j < 1, j = 1, \cdots, m\}\,.$$

3. Preliminaries.

3.1. We first remark that because of the stationarity of the process it is sufficient to prove (5) for $t = 0$. We shall write $X_p(s)$ and $X_e(s)$ for $X_p(s \mid 0)$ and $X_e(s \mid 0)$ respectively. Given $\varepsilon_0 > 0$ we write

(17) $$h_n = \exp(-n^{1+\varepsilon_0})\,, \qquad h_n' = \exp(-\varepsilon_0 n)\,, \qquad n = 1, 2, \cdots\,.$$

If $\mathbf{s}_m \in \mathscr{S}_m(1)$ we denote the variance-covariance matrix of

$$X_e(s_1 h_n'),\ X_e(s_2 h_n') - X_e(s_1 h_n'),\ \cdots,\ X_e(s_m h_n') - X_e(s_{m-1} h_n')$$

by

$$\boldsymbol{\lambda}(m) = \boldsymbol{\lambda}(s_1, \cdots, s_m, h_n')\,.$$

The variance-covariance matrix of $X_e(s_1 h_n \mid h_{n+1})$, $X_e(s_2 h_n \mid h_{n+1}) - X_e(s_1 h_n \mid h_{n+1}), \cdots$, $X_e(s_m h_n \mid h_{n+1}) - X_e(s_{m-1} h_n \mid h_{n+1})$ will be denoted by $\boldsymbol{\mu}(m) = \boldsymbol{\mu}(s_1, \cdots s_m, h_n)$. Most of this section will be devoted to various inequalities involving the matrices $\boldsymbol{\lambda}(m)$ and $\boldsymbol{\mu}(m)$. It will also be convenient to take $s_0 \equiv 0$.

LEMMA 1. *Suppose that $\mathbf{s}_m \in \mathscr{S}_m(\delta)$ where $0 < \delta < 1$. Then, given $\varepsilon > 0$ there exists a $\delta_1 = \delta_1(\varepsilon_0, \varepsilon)$, $0 < \delta_1 < 1$, and an $n_1 = n_1(\varepsilon_0, \varepsilon)$ such that for all δ, $0 < \delta < \delta_1$, and all $n \geqq n_1$,*

(18) $$\mu_{ij}(m) < 0\,, \qquad 1 \leqq i < j \leqq m\,,$$

(19) $$\sum_{\nu=1;\nu\neq j}^{m} |\mu_{\nu j}(m)| \leqq \mu_{jj}(m) - \frac{(1-\varepsilon)(2ah_n)^{2\alpha}}{\Gamma(2\alpha)}(s_j - s_{j-1})\,, \qquad 2 \leqq j \leqq m\,,$$

and

(20) $$\sum_{\nu=2}^{m} |\mu_{1\nu}(m)| \leqq \mu_{11}(m) - \frac{(1-\varepsilon)(2as_1 h_n)^{2\alpha}}{2\alpha\Gamma(2\alpha)}\,.$$

The same inequalities hold for the matrix $\boldsymbol{\lambda}(m)$ with h_n' replacing h_n.

PROOF. Suppose that $i < j$. Then $s_i < s_{i+1} \leqq s_j < s_{j+1}$ and from (7) and (9) we obtain

$$-E((X_e(s_{i+1}h_n \mid h_{n+1}) - X_e(s_i h_n \mid h_{n+1}))(X_e(s_{j+1}h_n \mid h_{n+1}) - X_e(s_j h_n \mid h_{n+1})))$$
$$= \int_{h_{n+1}}^{h_{n+1}+(s_{i+1}-s_i)h_n} \hat{g}(s_{i+1}h_n - u)(\hat{g}(s_j h_n - u) - \hat{g}(s_{j+1}h_n - u))\, du$$
$$+ \int_{h_{n+1}+(s_{i+1}-s_i)h_n}^{s_{i+1}h_n} \hat{g}(s_{i+1}h_n - u)(\hat{g}(s_j h_n - u) - \hat{g}(s_{j+1}h_n - u)$$
$$- \hat{g}(s_j h_n + (s_{i+1} - s_i)h_n - u) + \hat{g}(s_{j+1}h_n + (s_{i+1} - s_i)h_n - u))\, du\,.$$

As s_i is bounded away from zero it follows that $s_{i+1}h_n > h_{n+1} + (s_{i+1} - s_i)h_n$ for sufficiently large n. Using this together with (13) and the fact that as $0 < \alpha < \frac{1}{2}$, $\hat{g}(x)$ is convex and decreasing for $x > 0$, it follows that both integrals are positive, proving (18).

From (18) we have

$$\sum_{\nu=1;\nu\neq j}^{m} |\mu_{j\nu}(m)| = \mu_{jj}(m) - E(X_e(s_m h_n \mid h_{n+1})(X_e(s_j h_n \mid h_{n+1}) - X_e(s_{j-1}h_n \mid h_{n+1})))\,.$$

Now

$$E(X_e(s_m h_n \mid h_{n+1})(X_e(s_j h_n \mid h_{n+1}) - X_e(s_{j-1}h_n \mid h_{n+1})))$$
$$= \int_{h_{n+1}}^{h_{n+1}+h_n(s_j-s_{j-1})} \hat{g}(h_n s_m - u)\hat{g}(h_n s_j - u)\, du$$
$$+ \int_{h_{n+1}+h_n(s_j-s_{j-1})}^{h_n s_j} \hat{g}(h_n s_j - u)(\hat{g}(h_n s_m - u)$$
$$- \hat{g}(h_n s_m + h_n(s_j - s_{j-1}) - u))\, du$$
$$\geqq \int_0^{h_n(s_j-s_{j-1})} \hat{g}(h_n s_m - h_{n+1} - u)\hat{g}(h_n s_j - h_{n+1} - u)\, du\,.$$

From (13) we have $g(x) \geqq (2a)^\alpha x^{\alpha-\frac{1}{2}}/\Gamma(2\alpha)^{\frac{1}{2}}$ for $x > 0$ which implies $\hat{g}(h_n s_j - h_{n+1} - u) > (2a)^\alpha h_n^{\alpha-\frac{1}{2}}(s_j - h_{n+1}/h_n)^{\alpha-\frac{1}{2}}/\Gamma(2\alpha)^{\frac{1}{2}}$ for $0 \leqq u \leqq h_n(s_j - s_{j-1})$. Now $\lim_{n\to\infty} h_{n+1}/h_n = 0$ and as $s_m^{\alpha-\frac{1}{2}} > 1$ ($0 < \alpha < \frac{1}{2}$ and $\mathbf{s}_m \in \mathscr{S}_m(\delta)$) we obtain

$$\hat{g}(h_n s_m - h_{n+1} - u)\hat{g}((h_n s_j - h_{n+1} - u) \geqq (1 - \varepsilon)(2ah_n)^{2\alpha}h_n^{-1}/\Gamma(2\alpha))$$

for sufficiently large n. On applying this inequality in the last integral it follows that this integral is greater than

$$(1 - \varepsilon)(2ah_n)^{2\alpha}(s_j - s_{j-1})/\Gamma(2\alpha)$$

for sufficiently large n. This proves (19).

To prove (20) we argue as before to obtain

$$\sum_{\nu=2}^{m} |\mu_{1\nu}(m)| \leqq \mu_{11}(m) - \frac{(2a)^{2\alpha}}{\Gamma(2\alpha)} \int_{h_{n+1}}^{h_n s_1} (h_n s_1 - u)^{\alpha-\frac{1}{2}}(h_n s_m - u)^{\alpha-\frac{1}{2}}\, du$$
$$\leqq \mu_{11}(m) - \frac{(2ah_n s_1)^{2\alpha}}{\Gamma(2\alpha)} \int_{h_{n+1}s_1/h_n}^{1} (1 - v)^{\alpha-\frac{1}{2}}(s_m/s_1 - v)^{\alpha-\frac{1}{2}}\, dv$$
$$\leqq \mu_{11}(m) - \frac{(2ah_n s_1)^{2\alpha}}{\Gamma(2\alpha)} \int_{h_{n+1}s_1/h_n}^{1} (s_m/s_1 - v)^{2\alpha-1}\, dv$$
$$= \mu_{11}(m) - \frac{(2ah_n s_1)^{2\alpha}}{2\alpha\Gamma(2\alpha)} \left(\left(\frac{s_m}{s_1} - \frac{h_{n+1}s_1}{h_n}\right)^{2\alpha} - \left(\frac{s_m - s_1}{s_1}\right)^{2\alpha}\right).$$

Now as $\lim_{n\to\infty} h_{n+1}/h_n = 0$ and $1 - \delta < s_1 < s_m < 1$ we may choose $\delta_1(\varepsilon_0, \varepsilon)$ and

$n_1(\varepsilon_0, \varepsilon)$ such that for all δ, $0 < \delta < \delta_1$, and all $n \geqq n_1$ the last expression is greater than $(1-\varepsilon)(2ah_n s_1)^{2\alpha}/(2\alpha\Gamma(2\alpha))$. This proves (20). The proofs for the matrix $\boldsymbol{\lambda}(m)$ are similar and are consequently omitted.

LEMMA 2. *Denote the inverse of* $\boldsymbol{\mu}(m)$ *by* $\boldsymbol{\mu}^{-1}(m)$ *and suppose that* $\mathbf{s}_m \in \mathscr{F}_m(\delta)$ *where* $0 < \delta < 1$. *Then given* $\varepsilon > 0$ *there exists a* $\delta_2 = \delta_2(\varepsilon_0, \varepsilon)$, $0 < \delta_2 < 1$, *and an* $n_2 = n_2(\varepsilon_0, \varepsilon)$ *such that for all* δ, $0 < \delta < \delta_2$, *all* $n \geqq n_2$ *and all* $x_1, \cdots, x_m$ *we have*

$$\begin{aligned} &(1-\varepsilon)c_3(s_1 h_n)^{-2\alpha}x_1^2 \\ &\leqq \textstyle\sum_{1\leqq\nu,\nu'\leqq m} x_\nu x_{\nu'} \mu^{-1}_{\nu\nu'}(m) \\ &\leqq (1+\varepsilon)c_3(s_1 h_n)^{-2\alpha}x_1^2 + c_4 h_n^{-2\alpha} \textstyle\sum_{\nu=2}^m x_\nu^2(s_\nu - s_{\nu-1})^{-1} \end{aligned} \tag{21}$$

where $c_3 = 2\alpha\Gamma(2\alpha)/(2a)^{2\alpha}$.

PROOF. Define the matrix $\boldsymbol{\eta}$ by

$$\eta_{\nu\nu'} = \mu_{\nu\nu'}(m)\,, \qquad 1 \leqq \nu,\ \nu' \leqq m,\ \nu \neq \nu'\,,$$
$$\eta_{\nu\nu} = \mu_{\nu\nu}(m) - \frac{(2ah_n)^{2\alpha}(s_\nu - s_{\nu-1})}{(1+\varepsilon)\Gamma(2\alpha)}\,, \qquad 2 \leqq \nu \leqq m\,,$$

and

$$\eta_{11} = \mu_{11}(m) - \frac{(2as_1 h_n)^{2\alpha}}{(1+\varepsilon)2\alpha\Gamma(2\alpha)}$$

where $\varepsilon > 0$. Then for large enough n Lemma 1 implies that $\boldsymbol{\eta}$ has a dominant main diagonal and is therefore positive definite [8]. From this one deduces

$$\begin{aligned} &\textstyle\sum_{1\leqq\nu,\nu'\leqq m} x_\nu x_{\nu'} \mu_{\nu\nu'}(m) \\ &\geqq \frac{(2as_1 h_n)^{2\alpha}}{(1+\varepsilon)2\alpha\Gamma(2\alpha)} x_1^2 + \frac{(2ah_n)^{2\alpha}}{(1+\varepsilon)\Gamma(2\alpha)} \left(\textstyle\sum_{\nu=2}^m x_\nu^2(s_\nu - s_{\nu-1})\right). \end{aligned}$$

Now if $\mathbf{A}$ and $\mathbf{B}$ are positive definite matrices such that $\mathbf{x}'\mathbf{A}\mathbf{x} \geqq \mathbf{x}'\mathbf{B}\mathbf{x}$ for all $\mathbf{x}$ it follows that $\mathbf{x}'\mathbf{A}^{-1}\mathbf{x} \leqq \mathbf{x}'\mathbf{B}^{-1}\mathbf{x}$ for all $\mathbf{x}$ (see [8]). On applying this we obtain the second inequality of (21).

To prove the first inequality we this time define $\boldsymbol{\eta}$ by

$$\eta_{\nu\nu'} = -\mu_{\nu\nu'}(m)\,, \qquad 1 \leqq \nu,\ \nu' \leqq m,\ \nu \neq \nu'\,,$$
$$\eta_{\nu\nu} = \mu_{\nu\nu}(m)\,, \qquad 2 \leqq \nu \leqq m\,,$$

and

$$\eta_{11} = \frac{\varepsilon}{1-\varepsilon}\mu_{11}(m)$$

where $0 < \varepsilon < 1$. This gives

$$\begin{aligned} \textstyle\sum_{1\leqq\nu,\nu'\leqq m} x_\nu x_{\nu'} \mu_{\nu\nu'}(m) &= (1-\varepsilon)^{-1}\mu_{11}(m)x_1^2 + 2\textstyle\sum_{\nu=2}^m x_\nu^2 \mu_{\nu\nu}(m) \\ &\quad - \textstyle\sum_{1\leqq\nu,\nu'\leqq m} x_\nu x_{\nu'} \eta_{\nu\nu'}\,. \end{aligned}$$

Using (11) and (13) we have for large $\mathbf{n}$

$$\mu_{11}(m) \sim (2as_1 h_n)^{2\alpha}/(2\alpha\Gamma(2\alpha)) \tag{22}$$

which, together with (20), implies that η has a dominant main diagonal for sufficiently large n and is therefore positive definite. As before we obtain

$$(1-\varepsilon)(\mu_{11}(m))^{-1}x_1^2 \leqq (1-\varepsilon)(\mu_{11}(m))^{-1}x_1^2 + \tfrac{1}{2}\sum_{\nu=2}^m x_\nu^2(\mu_{\nu\nu}(m))^{-1} \leqq \sum_{1\leqq\nu,\nu'\leqq m} x_\nu x_{\nu'}\mu_{\nu\nu'}^{-1}(m)\,.$$

The first inequality of (21) now follows from this and and (22) completing the proof of the lemma.

LEMMA 3. *Suppose that* $\mathbf{s}_m \in \mathscr{S}_m(\delta)$ *where* $0<\delta<1$. *Then given any* ε, $0<\varepsilon<1$, *there exists an* $n_3 = n_3(\varepsilon_0, \varepsilon, \delta)$ *such that for all* $n \geqq n_3$

$$(23)\qquad \begin{aligned} &(1-\varepsilon)^m\left(\frac{\alpha\Gamma(\alpha)}{2\Gamma(1-\alpha)}\right)^{m/2}(\tfrac{1}{2}ah_n)^{-\alpha m}\left(\prod_{\nu=1}^m (s_\nu - s_{\nu-1})^{-\alpha}\right)\\ &\leqq |\boldsymbol{\mu}(m)|^{-\frac{1}{2}}\\ &\leqq (1+\varepsilon)^m(2\alpha\Gamma(2\alpha))^{m/2}(2ah_n)^{-\alpha m}\left(\prod_{\nu=1}^m (s_\nu - s_{\nu-1})^{-\alpha}\right)\end{aligned}$$

where $|\boldsymbol{\mu}(m)|$ *denotes the determinant of* $\boldsymbol{\mu}(m)$ *and* $s_0 \equiv 0$. *Furthermore, the same inequalities hold for* $|\boldsymbol{\lambda}(m)|^{-\frac{1}{2}}$ *with* h_n' *in place of* h_n.

PROOF. We first note that

$$(24)\qquad \begin{aligned} &|\boldsymbol{\mu}(m)|^{-\frac{1}{2}}\\ &= (\pi/2)^{m/2}\lim_{y\downarrow 0}\frac{P(|X_e(h_n s_\nu \,|\, h_{n+1}) - X_e(h_n s_{\nu-1}\,|\, h_{n+1})| \leqq y,\ \nu=1,\cdots,m)}{y^m}\,.\end{aligned}$$

We set $Y_1 = X_e(h_n s_1 \,|\, h_{n+1})$ and define Y_ν recursively as $Y_\nu = X_e(h_n s_\nu \,|\, h_{n+1}) - E(X_e(h_n s_\nu \,|\, h_{n+1})|Y_1, \cdots, Y_{\nu-1})$.

The Y_ν, $1 \leqq \nu \leqq m$, are then independent and following the line of argument on pages 150–151 of Marcus [10] we obtain

$$(25)\qquad \begin{aligned} &P(|X_e(h_n s_\nu \,|\, h_{n+1}) - X_e(h_n s_{\nu-1}\,|\, h_{n+1})| \leqq y,\ \nu = 1,\cdots,m)\\ &\leqq \prod_{\nu=1}^m (P(|y_\nu| \leqq y))\\ &\leqq y^m((\pi/2)^{m/2}\prod_{\nu=1}^m \sigma_\nu)^{-1}\end{aligned}$$

where σ_ν is the standard deviation of Y_ν. As $Y_1, \cdots, Y_\nu$ are $\mathscr{F}_{h_n s_\nu}$-measurable it follows that

$$\sigma_\nu^2 \geqq E((X_e(h_n s_\nu \,|\, h_{n+1}) - X_{ep}(h_n s_\nu \,|\, h_n s_{\nu-1}))^2)\,, \qquad \nu = 2,\cdots,m$$

where $X_{ep}(h_n s_\nu \,|\, h_n s_{\nu-1})$ is the least squares predictor of $X_e(h_n s_\nu \,|\, h_{n+1})$ based on $\mathscr{F}_{h_n s_{\nu-1}}$. For sufficiently large n, $h_{n+1} < h_n(1-\delta) < h_n s_{\nu-1}$, $\nu = 2,\cdots,m$ and hence

$$X_e(h_n s_\nu \,|\, h_{n+1}) - X_{ep}(h_n s_\nu \,|\, h_n s_{\nu-1}) = X_e(h_n s_\nu \,|\, h_n s_{\nu-1})\,.$$

We therefore have

$$\begin{aligned} \sigma_\nu^2 &\geqq E((X_e(h_n s_\nu \,|\, h_n s_{\nu-1}))^2)\,, & \nu = 2,\cdots,m\\ &= \int_{h_n s_{\nu-1}}^{h_n s_\nu} \hat{g}(h_n s_\nu - u)^2\,du\\ &\geqq (1-\varepsilon)(2ah_n)^{2\alpha}(s_\nu - s_{\nu-1})^{2\alpha}/(2\alpha\Gamma(2\alpha))\,, & \nu = 2,\cdots,m\,,\end{aligned}$$

for large enough n. This inequality also holds for $\nu = 1$ and the second inequality of (23) now follows from (24) and (25).

To prove the first inequality we note that

$$\begin{aligned} E((X_e(h_n s_\nu \mid h_{n+1}) - X_e(h_n s_{\nu-1} \mid h_{n+1}))^2) &\leqq E((X(h_n s_\nu) - X(h_n s_{\nu-1}))^2) \\ &= \sigma^2(h_n(s_\nu - s_{\nu-1})) , \\ |\boldsymbol{\mu}(m)| \leqq \prod_{\nu=1}^m E((X_e(h_n s_\nu \mid h_{n+1}) &- X_e(h_n s_{\nu-1} \mid h_{n+1}))^2) \\ \leqq \prod_{\nu=1}^m \sigma^2(h_n(s_\nu &- s_{\nu-1})) \end{aligned}$$

and the desired inequality now follows from (14).

The proofs for the matrix $\boldsymbol{\lambda}(m)$ are similar and are therefore omitted.

LEMMA 4. *Given ε and δ, $0 < \varepsilon, \delta < 1$, there exists an $n_4 = n_4(\varepsilon_0, \varepsilon, \delta)$ such that for all $n \geqq n_4$ and $m \geqq 1$,*

$$\begin{aligned} c_3 \left(\frac{(1-\varepsilon)\delta^{2(1-\alpha)} 2^{2\alpha-1} \alpha \Gamma(\alpha) \Gamma(1-\alpha)}{(1-\alpha)^{2(1-\alpha)} (ah_n)^{2\alpha}} \right)^{m/2} \Gamma(m+1)^{-(1-\alpha)} \\ \leqq \int_{\mathscr{T}_m(\delta)} |\boldsymbol{\mu}(2m)|^{-\frac{1}{2}} \, d\mathbf{s}_m \\ \leqq c_4 \left(\frac{(1+\varepsilon)\delta^{2(1-\alpha)} 2\alpha \Gamma(2\alpha) \Gamma(1-\alpha)^2}{(1-\alpha)^{2(1-\alpha)} (2ah_n)^{2\alpha}} \right)^{m/2} \Gamma(m+1)^{-(1-\alpha)} \end{aligned} \tag{26}$$

where $c_3 > 0$ and $c_4 > 0$ are constants depending on α, δ and ε. The same inequalities also hold for the matrix $\boldsymbol{\lambda}(m)$ with h_n' in place of h_n.

PROOF. We have, with $s_0 \equiv 0$,

$$\int_{\mathscr{T}_m(\delta)} \prod_{\nu=1}^m (s_\nu - s_{\nu-1})^{-\alpha} \, d\mathbf{s}_m = \frac{\Gamma(1-\alpha)^{m-1}}{\Gamma(m(1-\alpha)+\alpha)} \int_{1-\delta}^1 s^{-\alpha}(1-s)^{(m-1)(1-\alpha)} \, ds ,$$

and using Stirling's approximation we obtain

$$\begin{aligned} c_5 \left(\frac{\delta^{1-\alpha}\Gamma(1-\alpha)}{(1-\alpha)^{1-\alpha}} \right)^m m^{-(1+\alpha)/2} \Gamma(m+1)^{-(1-\alpha)} \\ \leqq \int_{\mathscr{T}_m(\delta)} \prod_{\nu=1}^m (s_\nu - s_{\nu-1})^{-\alpha} \, d\mathbf{s}_m \\ \leqq c_6 \left(\frac{\delta^{1-\alpha}\Gamma(1-\alpha)}{(1-\alpha)^{1-\alpha}} \right)^m \Gamma(m+1)^{-(1-\alpha)} \end{aligned}$$

where $c_5 > 0$ and $c_6 > 0$ depend only on α and δ. The desired inequalities now follow from Lemma 3.

To ease the notation we shall in future write $\varphi(h)$ for $\varphi(X(0), h)$. The next lemma gives a bound for $E(\varphi(t)^{2m})$ and also an expression for the conditional expectation $E(\varphi(t)^{2m} \mid \mathscr{F}_s)$.

LEMMA 5. *For all $m \geqq 1$ and $n \geqq 1$ we have*

$$E(\varphi(h_n')^{2m}) \leqq \frac{(2m)!\,(h_n')^{2m}}{(2\pi)^m} \int_{\mathscr{T}_{2m}(1)} |\boldsymbol{\lambda}(2m)|^{-\frac{1}{2}} \, d\mathbf{s}_{2m} . \tag{27}$$

Furthermore, given δ, $0 < \delta < 1$, *there exists an* $n_5 = n_5(\varepsilon_0, \delta)$ *such that for all* $n \geqq n_5$

$$E((\varphi(h_n) - \varphi((1-\delta)h_n))^{2m} \mid \mathscr{F}_{h_{n+1}}) \tag{28}$$
$$= (2m)! \left(\frac{h_n^2}{2\pi}\right)^m \int_{\mathscr{I}_{2m}(\delta)} |\boldsymbol{\mu}(2m)|^{-\frac{1}{2}}$$
$$\times \exp(-\tfrac{1}{2} \textstyle\sum_{1 \leq \nu, \nu' \leq 2m} Z_\nu Z_{\nu'} \mu_{\nu\nu'}^{-1}(2m)) \, d\mathbf{s}_{2m}$$

where

$$Z_\nu = X_p(h_n s_\nu \mid h_{n+1}) - X_p(h_n s_{\nu-1} \mid h_{n+1}), \qquad \nu = 1, \cdots, 2m.$$

PROOF. In [4] Berman showed (under weaker conditions than (4)) that for each m, $m = 1, 2, \cdots$,

$$\varphi_\nu(x, t) = \frac{1}{2\pi} \int_{-\nu}^{\nu} \left(\int_0^t \exp(iu(X(s) - x)) \, ds\right) du$$

converges in $2m$th mean to $\varphi(x, t)$. Berman's argument requires little change for the present case (with $X(t)$ in place of x) and elementary manipulations give

$$E(\varphi(h_n')^{2m}) = (2m)! \left(\frac{h_n'}{2\pi}\right)^{2m} \int_{\mathscr{I}_{2m}(1)} \left(\int E(\exp(i \textstyle\sum_{\nu=1}^{2m} v_\nu(X(h_n' s_{\nu-1}) \right. \tag{29}$$
$$\left. - X(h_n' s_{\nu-1})))) \, d\mathbf{v}_{2m}\right) d\mathbf{s}_{2m}.$$

As $X(h_n' s_\nu) = X_p(h_n' s_\nu) + X_e(h_n' s_\nu)$ and the $X_e(h_n' s_\nu)$ are independent of the $X_p(h_n' s_\nu)$ we have

$$|E(\exp(i \textstyle\sum_{\nu=1}^{2m} v_\nu(X(h_n' s_\nu) - X(h_n' s_{\nu-1}))))|$$
$$\leqq \exp(-\tfrac{1}{2} \textstyle\sum_{1 \leq \nu, \nu' \leq 2m} v_\nu v_{\nu'} \mu_{\nu\nu'}(2m)).$$

On using this in (29) and integrating with respect to $\mathbf{v}_{2m}$ we obtain (27).

We now turn to (28). For each m,

$$\varphi_\nu(h_n) - \varphi_\nu((1-\delta)h_n) = \frac{1}{2\pi} \int_{-\nu}^{\nu} \int_{(1-\delta)h_n}^{h_n} \exp(iu(X(s) - X(0))) \, ds \, du$$

converges in $2m$th mean to $\varphi(h_n) - \varphi((1-\delta)h_n)$ and hence

$(\varphi_\nu(h_n) - \varphi_\nu((1-\delta)h_n))^{2m}$ converges in mean to $(\varphi(h_n) - \varphi((1-\delta)h_n))^{2m}$.

This implies

$$E((\varphi_\nu(h_n) - \varphi_\nu((1-\delta)h_n))^{2m} \mid \mathscr{F}_{h_{n+1}}) \tag{30}$$
$$\to_{(1)} E((\varphi(h_n) - \varphi((1-\delta)h_n))^{2m} \mid \mathscr{F}_{h_{n+1}})$$

where (r) denotes covergence in rth mean.

As $X_e(s \mid t)$ is independent of $\mathscr{F}_t$ it follows that

$$E(\exp(i \textstyle\sum_{\nu=1}^{2m} u_\nu(X(h_n s_\nu) - X(0))) \mid \mathscr{F}_{h_{n+1}})$$
$$= \exp(-\tfrac{1}{2} \textstyle\sum_{1 \leq \nu, \nu' \leq 2m} u_\nu u_{\nu'} \rho_{\nu\nu'}(2m) + i \textstyle\sum_{\nu=1}^{2m} u_\nu(X_p(s_\nu h_n \mid h_{n+1}) - X(0)))$$

where $\boldsymbol{\rho}(2m)$ is the variance-covariance matrix of $X_e(s_1 h_n \mid h_{n+1}), \cdots, X_e(s_{2m} h_n \mid h_{n+1})$.

This yields

$$E((\varphi_\nu(h_n) - \varphi_\nu((1-\delta)h_n))^{2m} \mid \mathscr{F}_{h_{n+1}})$$
$$= (2m)!\left(\frac{h_n}{2\pi}\right)^{2m} \int_{\mathscr{I}_{2m}(\delta)} \Big(\int_{-\nu}^{\nu} \cdots \int_{-\nu}^{\nu} E(\exp(i \textstyle\sum_{\nu=1}^{2m} u_\nu (X(h_n s_\nu)$$
$$- X(0))) \mid \mathscr{F}_{h_{n+1}})\, d\mathbf{u}_{2m}\Big)\, d\mathbf{s}_{2m}$$
$$= (2m)!\left(\frac{h_n}{2\pi}\right)^{2m} \int_{\mathscr{I}_{2m}(\delta)} \Big(\int_{-\nu}^{\nu} \cdots \int_{-\nu}^{\nu} \exp(-\tfrac{1}{2} \textstyle\sum_{1\le\nu,\nu'\le 2m} u_\nu u_{\nu'} \rho_{\nu\nu'}(2m)$$
$$+ i \textstyle\sum_{\nu=1}^{2m} u_\nu (X_p(s_\nu h_n \mid h_{n+1}) - X(0)))\, d\mathbf{u}_{2m}\Big)\, d\mathbf{s}_{2m}\,.$$

On writing

$$Y_m = (2m)!\left(\frac{h_n}{2\pi}\right)^{2m} \int_{\mathscr{I}_{2m}(\delta)} \Big(\int E(\exp(i \textstyle\sum_{\nu=1}^{2m} u_\nu (X(h_n s_\nu)$$
$$- X(0))) \mid \mathscr{F}_{h_{n+1}})\, d\mathbf{u}_{2m}\Big)\, d\mathbf{s}_{2m}$$
$$= (2m)!\left(\frac{h_n}{2\pi}\right)^{2m} \int_{\mathscr{I}_{2m}(\delta)} \Big(\int \exp(-\tfrac{1}{2} \textstyle\sum_{1\le\nu,\nu'\le 2m} u_\nu u_{\nu'} \rho_{\nu\nu'}(2m)$$
$$+ i \textstyle\sum_{\nu=1}^{2m} u_\nu (X_p(s_\nu h_n \mid h_{n+1}) - X(0)))\, d\mathbf{u}_{2m}\Big)\, d\mathbf{s}_{2m}$$

we obtain

$$Y_m - E((\varphi_\nu(h_n) - \varphi_\nu((1-\delta)h_n))^{2m} \mid \mathscr{F}_{h_{n+1}})$$
$$= (2m)!\left(\frac{h_n}{2\pi}\right)^{2m} \int_{\mathscr{I}_{2m}(\delta)} \Big(\int_{\mathscr{C}_{2m}(\nu)} \exp(-\tfrac{1}{2} \textstyle\sum_{1\le\nu,\nu'\le 2m} u_\nu u_{\nu'} \rho_{\nu\nu'}(2m)$$
$$+ i \textstyle\sum_{\nu=1}^{2m} u_\nu (X_p(s_\nu h_n \mid h_{n+1}) - X(0)))\, d\mathbf{u}_{2m}\Big)\, d\mathbf{s}_{2m}$$

where $\mathscr{C}_{2m}(\nu) = \{u_{2m} : |u_j| > \nu$ for at least one j, $1 \leqq j \leqq 2m\}$. This implies

$$|Y_m - E((\varphi_\nu(h_n) - \varphi_\nu((1-\delta)h_n))^{2m} \mid \mathscr{F}_{h_{n+1}})|$$
$$\leqq (2m)!\left(\frac{h_n}{2\pi}\right)^{2m} \tag{31}$$
$$\times \int_{\mathscr{I}_{2m}(\delta)} \Big(\int_{\mathscr{C}_{2m}(\nu)} \exp(-\tfrac{1}{2} \textstyle\sum_{1\le\nu,\nu'\le 2m} u_\nu u_{\nu'} \rho_{\nu\nu'}(2m))\, d\mathbf{u}_{2m}\Big)\, d\mathbf{s}_{2m}\,.$$

From the definition of Y_m we have

$$Y_m = (2m)!\left(\frac{h_n}{2\pi}\right)^{2m} \int_{\mathscr{I}_{2m}(\delta)} \Big(\int E(\exp(i \textstyle\sum_{\nu=1}^{2m} u_\nu (X_e(h_n s_\nu \mid h_{n+1})$$
$$+ X_p(h_n s_\nu \mid h_{n+1}) - X(0))) \mid \mathscr{F}_{h_{n+1}})\, d\mathbf{u}_{2m}\Big)\, d\mathbf{s}_{2m}$$
$$= (2m)!\left(\frac{h_n}{2\pi}\right)^{2m} \int_{\mathscr{I}_{2m}(\delta)} \Big(\int E(\exp(i \textstyle\sum_{\nu=1}^{2m} v_\nu (X_e(h_n s_\nu \mid h_{n+1}) \tag{32}$$
$$- X_e(h_n s_{\nu-1} \mid h_{n+1})) + i \textstyle\sum_{\nu=1}^{2m} v_\nu Z_\nu) \mid \mathscr{F}_{h_{n+1}})\, d\mathbf{v}_{2m}\Big)\, d\mathbf{s}_{2m}$$
$$= (2m)!\left(\frac{h_n}{2\pi}\right)^{2m} \int_{\mathscr{I}_{2m}(\delta)} \Big(\int \exp(-\tfrac{1}{2} \textstyle\sum_{1\le\nu,\nu'\le 2m} v_\nu v_{\nu'} \mu_{\nu\nu'}(2m)$$
$$+ i \textstyle\sum_{\nu=1}^{2m} v_\nu Z_\nu)\, d\mathbf{v}_{2m}\Big)\, d\mathbf{s}_{2m}$$
$$= (2m)!\left(\frac{h_n^2}{2\pi}\right)^{m} \int_{\mathscr{I}_{2m}(\delta)} |\boldsymbol{\mu}(2m)|^{-\frac{1}{2}}$$
$$\times \exp(-\tfrac{1}{2} \textstyle\sum_{1\le\nu,\nu'\le 2m} Z_\nu Z_{\nu'} \mu_{\nu\nu'}^{-1}(2m)\, d\mathbf{s}_{2m}\,.$$

Now

$$\int_{\mathscr{S}_{2m}(\delta)} \left(\int \exp\left(-\tfrac{1}{2}\sum_{1\leq\nu,\nu'\leq 2m} u_\nu u_{\nu'}\rho_{\nu\nu'}(2m)\right) d\mathbf{u}_{2m}\right) d\mathbf{s}_{2m}$$
$$= (2\pi)^m \int_{\mathscr{S}_{2m}(\delta)} |\boldsymbol{\rho}(2m)|^{-\frac{1}{2}}\, d\mathbf{s}_{2m}$$
$$= (2\pi)^m \int_{\mathscr{S}_{2m}(\delta)} |\boldsymbol{\mu}(2m)|^{-\frac{1}{2}}\, d\mathbf{s}_{2m}$$

as $\boldsymbol{\mu}(2m)$ may be obtained from $\boldsymbol{\rho}(2m)$ by row and column operations. The last integral is finite by Lemma 4 and hence

$$\lim_{\nu\to\infty} \int_{\mathscr{S}_{2m}(\delta)} \left(\int_{\mathscr{C}_{2m}(\nu)} \exp\left(-\tfrac{1}{2}\sum_{1\leq\nu,\nu'\leq 2m} u_\nu u_{\nu'}\rho_{\nu\nu'}(2m)\right) d\mathbf{u}_{2m}\right) d\mathbf{s}_{2m} = 0\,.$$

This combined with (31) implies that $E((\varphi(h_n) - \varphi((1-\delta)h_n))^{2m} \mid \mathscr{F}_{h_{n+1}})$ tends almost surely to Y_m for each m and hence from (30) we have

$$Y_m = E((\varphi(h_n) - \varphi((1-\delta)h_n))^{2m} \mid \mathscr{F}_{h_{n+1}}) \quad \text{a.s.}$$

The lemma now follows from (32).

3.2. The next two lemmas are not directly concerned with the stochastic process $X(t)$. The first is a generalized Borel–Cantelli lemma whilst the second gives estimates of tail probabilities derived from estimates involving the even moments of a random variable.

LEMMA 6. *Let $(\mathscr{F}_\nu)_{-\infty}^{\infty}$ be an increasing sequence of sub-σ-fields of a probability space $(\Omega, \mathscr{F}, P)$ and let $(A_\nu)_{-\infty}^{\infty}$ be a sequence of events such that $A_\nu \in \mathscr{F}_\nu$, $\nu = 0, \pm 1, \pm 2, \cdots$. Then almost surely*

$$\{\omega : \omega \in A \ \text{ infinitely often}\} = \{\omega : \textstyle\sum_{\nu=-\infty}^{\infty} P(A_{\nu+1} \mid \mathscr{F}_\nu) = \infty\}\,.$$

PROOF. This is a two-sided version of the extended Borel–Cantelli lemma given in Neveu [12]. The proof given there can easily be adapted to the present situation and is therefore omitted.

LEMMA 7. *Let c_7, c_8 and β be positive constants satisfying $0 < c_7 \leqq c_8 < \infty$ and $0 < \beta < 1$ and suppose that X is a real random variable such that for all ε, $0 < \varepsilon < 1$ the inequality*

$$(c_7(1-\varepsilon))^{2n}\Gamma(2n+1)^\beta \leqq E(X^{2n}) \leqq (c_8(1+\varepsilon))^{2n}\Gamma(2n+1)^\beta \tag{33}$$

holds for all $n \geqq n_5(\varepsilon)$. Then for all ε, $0 < \varepsilon < 1$, there exists an $x_1 = x_1(\varepsilon)$ such for all $x \geqq x_1$

$$\begin{aligned} -(1+\varepsilon)\beta(r_u/r_l)^{\beta^{-1}}(x/c_8)^{\beta^{-1}} &\leqq \log P(|X| \geqq x) \\ &\leqq -(1-\varepsilon)\beta(x/c_8)^{\beta^{-1}}\,, \end{aligned} \tag{34}$$

where $0 < r_l \leqq 1 \leqq r_u < \infty$ are the two positive roots of the equation

$$\beta(c_7/c_8)^{\beta^{-1}} z^{\beta^{-1}} - z + (1-\beta) = 0\,. \tag{35}$$

PROOF. The proof of this lemma is to be found in [7].

4. The upper bound. The upper bound can now be easily obtained. Combining Lemmas 4, 5 and 7 we obtain that for all $\varepsilon > 0$ there exists an $x_2 = x_2(\varepsilon_0, \varepsilon)$ such that for all $x \geqq x_2$ and $n \geqq n_6(\varepsilon_0)$

$$P(\varphi(h_n')/(h_n')^{1-\alpha} \geqq x) \leqq \exp(-(1-\varepsilon)(x/c_9)^{1/\alpha})$$

where

$$c_9 = \frac{\Gamma(1-\alpha)}{(1-\alpha)^{1-\alpha}(2a\alpha)^\alpha}\left(\frac{\alpha\Gamma(2\alpha)}{\pi}\right)^{\frac{1}{2}}.$$

If we now set $x = ((1+\varepsilon)/(1-\varepsilon))c_9(\log(-\log h_n'))^\alpha$ a simple application of the Borel–Cantelli lemma gives

$$\limsup_{n\to\infty} \frac{\varphi(h_n')}{(h_n')^{1-\alpha}(\log(-\log h_n'))^\alpha} \leqq c_9 .$$

On noting that $\varphi(h)$ is a nondecreasing function of h and that ε_0 in (17) may be taken to be arbitrarily small we obtain

$$\limsup_{h\downarrow 0} \frac{\varphi(h)}{h^{1-\alpha}(\log(-\log h))^\alpha} \leqq c_9$$

which proves the first part of the theorem.

5. The lower bound.

5.1. The lower bound is somewhat more difficult. The first problem is to show that the quadratic form occurring in (28) is small. This is then used to obtain a lower bound for

$$E((\varphi(h_n) - \varphi((1-\delta)h_n))^{2m} \,|\, \mathscr{F}_{h_{n+1}})$$

which together with Lemma 7 gives a lower bound for

$$P(\varphi(h_n) - \varphi((1-\delta)h_n) \geqq h_n^{1-\alpha}x \,|\, \mathscr{F}_{h_{n+1}}) .$$

An application of Lemma 6 then completes the proof of the theorem.

LEMMA 8. *Suppose $\varepsilon > 0$ and δ, $0 < \delta < 1$, are given. Then for almost all ω there exists an $n_6 = n_6(\varepsilon_0, \varepsilon, \delta, \omega)$ such that for all $n \geqq n_6$*

(36) $$\sup_{1-\delta\leqq s\leqq 1}\left|\frac{d}{ds} X_p(sh_n \,|\, h_{n+1})\right| \leqq (1+\varepsilon)(1-\delta)^{\alpha-1}c_{10}h_n^\alpha(\log(-\log h_n))^{\frac{1}{2}}$$

where

$$c_{10} = \frac{2\pi(\frac{1}{2}a)^\alpha(\alpha+\frac{3}{2})}{\Gamma(\alpha+\frac{1}{2})\Gamma(\frac{1}{2}-\alpha)}\left(\frac{\Gamma(1-\alpha)}{\alpha\Gamma(\alpha)}\right)^{\frac{1}{2}}.$$

PROOF. We first note that (4) implies that

(37) $$\limsup_{h\downarrow 0} \frac{|X(-h) - X(0)|}{h^\alpha(\log(-\log h))^{\frac{1}{2}}} \leqq c_{11} \quad \text{a.s.},$$

and

(38) $$\limsup_{t\to\infty} \frac{|X(-t)|}{(2\log t)^{\frac{1}{2}}} \leqq 1 \quad \text{a.s.},$$

where

$$c_{11} = 2(\tfrac{1}{2}a)^{\alpha}\left(\frac{\Gamma(1-\alpha)}{\alpha\Gamma(\alpha)}\right)^{\frac{1}{2}}$$

(see [11, 14]).

According to Yaglom ([16, 17])

$$\begin{aligned} X_p(sh_n \mid h_{n+1}) &- X(0) \\ &= c_{12}(sh_n - h_{n+1})^{\alpha+\frac{1}{2}} \int_0^{\infty} \frac{\exp[-a(sh_n - h_{n+1} + p)]X(h_{n+1} - p) - X(0)}{p^{\alpha+\frac{1}{2}}(sh_n - h_{n+1} + p)}\, dp \end{aligned}$$

where $c_{12} = (\Gamma(\frac{1}{2}+\alpha)\Gamma(\frac{1}{2}-\alpha))^{-1}$.

Because of (38) we may differentiate inside the integral and a short calculation shows that for all $n \geqq n_7(\varepsilon_0, \varepsilon, \delta, \omega)$

$$(39) \qquad \begin{aligned} &\left|\frac{d}{ds} X_p(sh_n \mid h_{n+1})\right| \\ &\qquad \leqq (1+\varepsilon)c_{12}(\alpha + \tfrac{3}{2})s^{\alpha-\frac{1}{2}}h_n^{\alpha+\frac{1}{2}} \int_0^{\infty} \frac{|X(h_{n+1}-p) - X(0)|}{p^{\alpha+\frac{1}{2}}(sh_n + p)}\, dp\,. \end{aligned}$$

We now obtain an upper bound for the last integral. We have

$$\begin{aligned} \int_0^{\infty} &\frac{|X(h_{n+1}-p) - X(0)|}{p^{\alpha+\frac{1}{2}}(sh_n + p)}\, dp \\ &= \left(\int_0^{h_{n+1}} + \int_{h_{n+1}}^{h_n^{\frac{1}{2}}} + \int_{h_n^{\frac{1}{2}}}^{\infty}\right) \frac{|X(h_{n+1}-p) - X(0)|}{p^{\alpha+\frac{1}{2}}(sh_n + p)}\, dp \\ &= I_1 + I_2 + I_3\,. \end{aligned}$$

An application of (37) yields

$$\begin{aligned} I_1 &= O\left(\int_0^{h_{n+1}} \frac{h_{n+1}^{\alpha}(\log(-\log h_{n+1}))^{\frac{1}{2}}}{p^{\alpha+\frac{1}{2}}(sh_n + p)}\, dp\right) \\ &= O(h_{n+1}^{\alpha} h_n^{-1}(\log(-\log h_{n+1}))^{\frac{1}{2}} \int_0^{h_{n+1}} p^{-(\alpha+\frac{1}{2})}\, dp) \end{aligned}$$

as s is bounded away from zero. We therefore obtain

$$\begin{aligned} I_1 &= O(h_{n+1}^{\frac{1}{2}} h_n^{-1}(\log(-\log h_{n+1}))^{\frac{1}{2}}) \\ &= O(h_n^{-\frac{1}{2}}) \end{aligned}$$

on using (17).

We turn now to I_3. From (38) it follows that

$$\begin{aligned} I_3 &= O\left(\int_{h_n^{\frac{1}{2}}}^{\infty} \frac{|\log p|^{\frac{1}{2}}}{p^{\alpha+\frac{1}{2}}p}\, dp\right) \\ &= O\left(h_n^{-(\alpha+\frac{1}{2})/2} \int_1^{\infty} \frac{|\log q|^{\frac{1}{2}} + |\log h_n|^{\frac{1}{2}}}{q^{\alpha+\frac{3}{2}}}\, dq\right) \\ &= o(h_n^{-\frac{1}{2}}) \end{aligned}$$

as $0 < \alpha < \frac{1}{2}$.

Finally we consider I_2. We have

$$I_2 \leqq \int_0^{h_n^{\frac{1}{2}}} \frac{|X(-p) - X(0)|}{p^{\alpha+\frac{1}{2}}(sh_n + p)}\, dp$$
$$\leqq (1 + \varepsilon)c_{11} \int_0^{h_n^{\frac{1}{2}}} \frac{(\log(-\log p))^{\frac{1}{2}}}{p^{\frac{1}{2}}(sh_n + p)}\, dp$$
$$\leqq (1 + 2\varepsilon)c_{11}(sh_n)^{-\frac{1}{2}} \int_0^{s^{-1}h_n^{-\frac{1}{2}}} \frac{(\log(-\log h_n p))^{\frac{1}{2}}}{p^{\frac{1}{2}}(1 + p)}\, dp$$
$$\leqq (1 + 3\varepsilon)\pi c_{11}(sh_n)^{-\frac{1}{2}}(\log(-\log h_n))^{\frac{1}{2}}$$

for all $n \geqq n_8(\varepsilon_0, \varepsilon, \delta, \omega)$ where the second inequality follows from (37) and last inequality is obtained by an application of Lebesgue's dominated convergence theorem. On substituting these inequalities for I_1, I_2 and I_3 into (39) we obtain (36) and this proves the lemma.

LEMMA 9. *Suppose $\varepsilon > 0$ is given. Then for almost all ω there exists an $n_9 = n_9(\varepsilon_0, \varepsilon, \omega)$ such that for all $n \geqq n_9$*

$$|X_p(h_n \mid h_{n+1}) - X(0)| \leqq (1 + \varepsilon)c_{13}h_n^{\alpha}(\log(-\log h_n))^{\frac{1}{2}} \tag{40}$$

where

$$c_{13} = \left(\frac{4(\frac{1}{2}a)^{2\alpha}\Gamma(1 - \alpha)}{\alpha\Gamma(\alpha)} - \frac{(2a)^{2\alpha}}{\alpha\Gamma(2\alpha)}\right)^{\frac{1}{2}}.$$

PROOF. We have

$$E((X_p(h_n \mid h_{n+1}) - X(0))^2)$$
$$= E((X(h_n) - X(0))^2) - E(X_e(h_n \mid h_{n+1})^2)$$
$$\leqq (1 + \varepsilon)h_n^{2\alpha}\left(\frac{2a^{2\alpha}\Gamma(1 - \alpha)}{2^{2\alpha}\alpha\Gamma(\alpha)} - \frac{(2a)^{2\alpha}}{2\alpha\Gamma(2\alpha)}\right)$$

by (10), (13) and (14). The result now follows from the Borel–Cantelli lemma and the usual estimate for the tail of the normal distribution.

LEMMA 10. *For all $\varepsilon < 0$ there exists a $\delta_3 = \delta_3(\varepsilon_0, \varepsilon)$ such that if $\mathbf{s}_{2m} \in \mathscr{S}_{2m}(\delta_3)$ the following holds. For almost all ω there exists an $n_{10} = n_{10}(\varepsilon_0, \varepsilon, \omega)$ such that for all $n \geqq n_{10}$*

$$c_3 h_n^{-2\alpha}(X_p(h_n \mid h_{n+1}) - X(0))^2 - \varepsilon \log(-\log h_n)$$
$$\leqq \sum_{1 \leqq \nu, \nu' \leqq 2m} Z_\nu Z_{\nu'} \mu_{\nu\nu'}^{-1}(2m) \tag{41}$$
$$\leqq c_3 h_n^{-2\alpha}(X_p(h_n \mid h_{n+1}) - X(0))^2 + \varepsilon \log(-\log h_n)$$

where

$$Z_\nu = X_p(s_\nu h_n \mid h_{n+1}) - X_p(s_{\nu-1}h_n \mid h_{n+1}), \qquad \nu = 1, \cdots, 2m.$$

PROOF. Suppose $\mathbf{s}_{2m} \in \mathscr{S}_{2m}(\delta)$. Then by Lemma 8 we have for $n \geqq n_6(\varepsilon_0, \varepsilon, \delta, \omega)$

$$|Z_\nu| = |X_p(s_\nu h_n \mid h_{n+1}) - X_p(s_{\nu-1}h_n \mid h_{n+1})|$$
$$\leqq (1 + \varepsilon)(1 - \delta)^{\alpha-1}c_{10}h_n^{\alpha}(\log(-\log h_n))^{\frac{1}{2}}(s_\nu - s_{\nu-1})$$

for $\nu = 2, \cdots, 2m$. This combined with Lemma 2 gives

$$
\begin{aligned}
(42) \qquad & (1-\varepsilon)c_3(s_1h_n)^{-2\alpha}Z_1^2 \\
& \leqq \textstyle\sum_{1\leqq\nu,\nu'\leqq 2m} Z_\nu Z_{\nu'}\mu_{\nu\nu'}^{-1}(2m) \\
& \leqq (1+\varepsilon)c_3(s_1h_n)^{-2\alpha}Z_1^2 \\
& \qquad + (1+\varepsilon)(1-\delta)^{\alpha-1}c_4c_{10}(\log(-\log h_n))(\textstyle\sum_{\nu=2}^m(s_\nu - s_{\nu-1})) \\
& \leqq (1+\varepsilon)c_3(s_1h_n)^{-2\alpha}Z_1^2 + \delta(1+\varepsilon)(1-\delta)^{\alpha-1}c_{14}\log(-\log h_n)
\end{aligned}
$$

for all δ, $0 < \delta < \delta_2(\varepsilon_0, \varepsilon)$, and for all $n \geqq n_{11}(\varepsilon_0, \varepsilon, \delta, \omega)$ by Lemmas 8 and 9.

It remains to consider Z_1. We have

$$
\begin{aligned}
& |Z_1^2 - (X_p(h_n\,|\,h_{n+1}) - X(0))^2| \\
& \quad \leqq (X_p(h_n\,|\,h_{n+1}) - X_p(s_1h_n\,|\,h_{n+1}))^2 \\
& \qquad + 2|X_p(h_n\,|\,h_{n+1}) - X_p(s_1h_n\,|\,h_{n+1})|\,|X_p(h_n\,|\,h_{n+1}) - X(0)| \\
& \quad \leqq \delta(1+\varepsilon)^2(1-\delta)^{\alpha-1}h_n^{2\alpha}\log(-\log h_n)(\delta(1-\delta)^{\alpha-1}c_{10}^2 + 2c_{10}c_{13})
\end{aligned}
$$

for $n \geqq n_{12}(\varepsilon_0, \varepsilon, \omega)$ by Lemmas 8 and 9.

On using this inequality and the fact that $s_1^{-2\alpha} = 1 + O(\delta)$ we obtain from (42)

$$
\begin{aligned}
& (1-\varepsilon)c_3h_n^{-2\alpha}(X_p(h_n\,|\,h_{n+1}) - X(0))^2 - O(\delta)\log(-\log h_n) \\
& \qquad \leqq \textstyle\sum_{1\leqq\nu,\nu'\leqq 2m} Z_\nu Z_{\nu'}\mu_{\nu\nu'}^{-1}(2m) \\
& \qquad \leqq (1+\varepsilon)c_3h_n^{-2\alpha}(X_p(h_n\,|\,h_{n+1}) - X(0))^2 + O(\delta)\log(-\log h_n)
\end{aligned}
$$

and the lemma follows on choosing δ sufficiently small.

LEMMA 11. *Suppose $\varepsilon_1 > 0$ and ε_2, $0 < \varepsilon_2 < 1$, are given and let $\delta_3 = \delta_3(\varepsilon_1)$ be the δ_3 of Lemma 10 with ε_1 in place of ε. Then there exist constants c_{15} and c_{16} depending on α, ε_1 and ε_2 and also, for almost all ω, an integer $n_{13} = n_{13}(\varepsilon_0, \varepsilon_1, \varepsilon_2, \omega)$ such that for all $n \geqq n_{13}$ and $m \geqq 1$*

$$
\begin{aligned}
& c_{15}\exp(-\tfrac{1}{2}c_3h_n^{-2\alpha}(X_p(h_n\,|\,h_{n+1}) - X(0))^2 \\
& \qquad - \varepsilon_1\log(-\log h_n))((1-\varepsilon_2)c_{17})^{2m}\Gamma(2m+1)^\alpha \\
(43) \qquad & \qquad \leqq E\left(\left(\frac{\varphi(h_n) - \varphi((1-\delta_3)h_n)}{(\delta_3h_n)^{1-\alpha}}\right)^{2m} \,\middle|\, \mathscr{F}_{h_{n+1}}\right) \\
& \qquad \leqq c_{16}\exp(-\tfrac{1}{2}c_3h_n^{-2\alpha}(X_p(h_n\,|\,h_{n+1}) - X(0))^2 \\
& \qquad\qquad + \varepsilon_1\log(-\log h_n))((1+\varepsilon_2)c_{18})^{2m}\Gamma(2m+1)^\alpha
\end{aligned}
$$

where

$$c_{17} = \frac{2^{\alpha-1}}{(1-\alpha)^{1-\alpha}a^\alpha}\left(\frac{\alpha\Gamma(\alpha)\Gamma(1-\alpha)}{\pi}\right)^{\frac{1}{2}}$$

and

$$c_{18} = \frac{\Gamma(1-\alpha)}{(1-\alpha)^{1-\alpha}(2a)^\alpha}\left(\frac{\alpha\Gamma(2\alpha)}{2\pi}\right)^{\frac{1}{2}}.$$

PROOF. This lemma follows from Lemmas 4, 5 and 10.

LEMMA 12. *Suppose* $\varepsilon > 0$ *is given. Then there exists a* $\delta_4 = \delta_4(\varepsilon_0, \varepsilon)$ *and, for almost all* ω, *an integer* $n_{14} = n_{14}(\varepsilon_0, \varepsilon, \omega)$ *such that for all* $n \geqq n_{14}$ *and* $x \geqq \varepsilon(\log(-\log h_n))^\alpha$ *we have*

$$\begin{aligned} P(\varphi(h_n) - \varphi((1-\delta_4)h_n) &\geqq (\delta_4 h_n)^{1-\alpha} x \mid \mathscr{F}_{h_{n+1}}) \\ &\geqq \exp(-\tfrac{1}{2}c_3 h_n^{-2\alpha}(X_p(h_n \mid h_{n+1}) - X(0))^2 - \varepsilon \log(-\log h_n) \\ &\quad - (1+\varepsilon)\alpha(r_1/r_2)^{1/\alpha}(x/c_{18}))^{1/\alpha}) \end{aligned} \tag{44}$$

where $0 < r_2 \leqq 1 \leqq r_1 < \infty$ *are the two positive roots of the equation*

$$\alpha(c_{17}/c_{18})^{1/\alpha} z^{1/\alpha} - z + 1 - \alpha = 0 .$$

PROOF. We may assume that the conditional probability occurring in (44) is a regular conditional probability and we write

$$F_n(x, \omega) = P(\varphi(h_n) - \varphi((1-\delta_4)h_n) \geqq (\delta_4 h_n)^{1-\alpha} x \mid \mathscr{F}_{h_{n+1}}) .$$

The proof of the lemma is now similar to that of Lemma 7. As the expression

$$\exp(-\tfrac{1}{2}c_3 h_n^{-2\alpha}(X_p(h_n \mid h_{n+1}) - X(0))^2)$$

appears on both sides of (43) its presence causes no problems. The only problem is caused by the presence of the expressions $\exp(-\varepsilon_1 \log(-\log h_n))$ and $\exp(\varepsilon_1 \log(-\log h_n))$ on the left and right-hand sides of (43) respectively. However, as ε_1 may be taken to be arbitrarily small, the proof goes through with minor modifications if it is assumed that $x \geqq \varepsilon(\log(-\log h_n))^\alpha$.

5.2. We are now in a position to obtain the lower bound. It follows from Lemmas 9 and 12 that for all $n \geqq n_{15}(\varepsilon_0, \varepsilon, \omega)$

$$\begin{aligned} P(\varphi(h_n) - \varphi((1-\delta_4)h_n) &\geqq \varepsilon(\delta_4 h_n)^{1-\alpha}(\log(-\log h_n))^\alpha \mid \mathscr{F}_{h_{n+1}}) \\ &\geqq \exp(-(\tfrac{1}{2}c_3 c_{13}^2 + O(\varepsilon)) \log(-\log h_n)) . \end{aligned}$$

Now

$$\begin{aligned} \tfrac{1}{2}c_3 c_{13}^2 &= \frac{4\Gamma(1-\alpha)\Gamma(2\alpha)}{2^{4\alpha}\Gamma(\alpha)} - 1 \\ &= \frac{\Gamma(1-\alpha)\Gamma(\alpha+\frac{1}{2})}{2^{2\alpha-1}\Gamma(\frac{1}{2})} - 1 . \end{aligned}$$

As $\Gamma(1-\alpha)\Gamma(\alpha+\frac{1}{2}) < \Gamma(\frac{1}{2}) < 2^{2\alpha}\Gamma(\frac{1}{2})$ for $0 < \alpha < \frac{1}{2}$ it follows that $\frac{1}{2}c_3 c_{13}^2 < 1$. We may therefore choose ε sufficiently small so that $\frac{1}{2}c_3 c_{13}^2 + O(\varepsilon) < (1 - 2\varepsilon_1)$ for some $\varepsilon_1 > 0$. This gives

$$P(\varphi(h_n) - \varphi((1-\delta_4)h_n) \geqq \varepsilon(\delta_4 h_n)^{1-\alpha}(\log(-\log h_n))^\alpha \mid \mathscr{F}_{h_{n+1}}) \geqq n^{-(1-2\varepsilon_1)(1+\varepsilon_0)}$$

for all $n \geqq n_{16}(\varepsilon_0, \varepsilon, \omega)$. As ε_0 may also be taken to be arbitrarily small we have for some ε_0 and ε_1

$$P(\varphi(h_n) - \varphi((1-\delta_4)h_n) \geqq \varepsilon(\delta_4 h_n)^{1-\alpha}(\log(-\log h_n))^\alpha \mid \mathscr{F}_{h_{n+1}}) \geqq n^{-(1-\varepsilon_1)}$$

and hence

$$\textstyle\sum_{n=1}^\infty P(\varphi(h_n) - \varphi((1-\delta_4)h_n) \geqq \varepsilon(\delta_4 h_n)^{1-\alpha}(\log(-\log h_n))^\alpha \mid \mathscr{F}_{h_{n+1}}) = \infty$$

almost surely. Now $\varphi(h_n)$ is $\mathscr{F}_{h_n}$-measurable and we therefore have from Lemma 6

$$\limsup_{n\to\infty} \frac{\varphi(h_n) - \varphi((1-\delta_4)h_n)}{(\delta_4 h_n)^{1-\alpha}(\log(-\log h_n))^\alpha} \geqq \varepsilon > 0$$

almost surely and this completes the proof of the theorem.

6. The Hausdorff measure of the zero set. In [15] Taylor and Wendel obtained the exact measure function for the zero set of a stable process. Their proof involved using an iterated logarithm law similar to (5) for the local time of the process. However, their proof also made use of the strong Markov property and is consequently not directly applicable in the present situation. In spite of this the obvious conjecture to make, in the light of their result, is that $h(x) = x^{1-\alpha}(\log(-\log x))^\alpha$ is the correct measure function for the zero set of stationary Gaussian processes with spectral density function f given by (4). A closer examination of the proof given in [15] shows that full use of the strong Markov property is only made in order to obtain an upper bound for the Hausdorff measure. Its use in obtaining the lower bound is restricted to deducing that, in our notation, the Hölder condition

$$\limsup_{h\downarrow 0} \frac{\varphi(x, \tau+h) - \varphi(x, \tau)}{h^{1-\alpha}(\log(-\log h))^\alpha} \leqq c_{19} < \infty \tag{45}$$

also holds for stopping times τ. It turns out that for the stationary Gaussian processes of Theorem 1 we can prove (45) directly for stopping times. The deduction of the lower bound for the Hausdorff measure of the zero set is then the same as that given in Section 5 of Taylor and Wendel for stable processes. We first prove:

LEMMA 13. *Let τ be a stopping time for the process $X(t)$ and let $\mathscr{F}_\tau$ denote the σ-field associated with τ. Then*

$$\begin{aligned} &E(\exp(i \textstyle\sum_{\nu=1}^m u_\nu X(s_\nu)) \mid \mathscr{F}_\tau) \\ &\quad = \exp(-\tfrac{1}{2} \textstyle\sum_{1\leqq\nu,\nu'\leqq m} u_\nu u_{\nu'} \rho(s_\nu - \tau, s_{\nu'} - \tau) + i \textstyle\sum_{\nu=1}^m u_\nu X_p(s_\nu \mid \tau)) \end{aligned} \tag{46}$$

where $\rho(s, t) = E(X_e(s)X_e(t))$ and $X_p(s_\nu \mid \tau)$ is $X_p(s_\nu \mid t)$ evaluated at $t = \tau$.

PROOF. We have for $h > 0$

$$\begin{aligned} E((X_e(s \mid t+h) - X_e(s \mid t))^2) &= \textstyle\int_t^{t+h} \hat{g}(s-u)^2\, du \\ &\leqq \frac{(2a)^{2\alpha}}{\Gamma(2\alpha)} \textstyle\int_0^h u^{2\alpha-1}\, du \\ &= \frac{(2ah)^{2\alpha}}{2\alpha\Gamma(2\alpha)} \end{aligned}$$

which implies that for fixed s the separable version of $X_e(s \mid t)$ has continuous sample paths almost surely (see [12], page 98). As $X(s) = X_e(s \mid t) + X_p(s \mid t)$ it follows that $X_p(s \mid t)$ also has continuous sample paths for fixed s. This in turn implies (as $X_p(s\,|\,t)$ is $\mathscr{F}_t$-measurable for each t) that $X_p(s\,|\,\tau)$ is $\mathscr{F}_\tau$-measurable (see [12], page 101).

From the definition of $\rho(s, t)$ we have

$$\begin{aligned} &|\rho(s, t) - \rho(s', t')| \\ &\quad = |E(X_e(s)(X_e(t) - X_e(t'))) + E(X_e(t')(X_e(s) - X_e(s'))| \\ &\quad \leqq E(X_e(s)^2)^{\frac{1}{2}}E((X_e(t) - X_e(t'))^2)^{\frac{1}{2}} + E((X_e(t))^2)^{\frac{1}{2}}E((X_e(s) - X_e(s'))^2)^{\frac{1}{2}} . \end{aligned}$$

As $X(t) = X_p(t) + X_e(t)$ and $X_e(s)$ and $X_p(t)$ are independent for all s and t $E(X_e(s)^2) \leqq E(X(s)^2) = 1$ and hence $E((X_e(s) - X_e(s'))^2) \leqq E((X(s) - X(s'))^2) = \sigma^2(|s - s'|)$. We therefore obtain

$$|\rho(s, t) - \rho(s', t')| \leqq \sigma(|s - s'|) + \sigma(|t - t'|)$$

and (14) now implies that $\rho(s, t)$ is jointly continuous in s and t.

Let F be any set in $\mathscr{F}_\tau$. For each positive integer N we define the random variables Z_N by

$$\begin{aligned} Z_N = &\textstyle\sum_{k=-\infty}^{\infty} \chi(F \cap \{k2^{-N} \leqq \tau < (k + 1)2^{-N}\}) \\ &\times \exp(-\tfrac{1}{2}(\textstyle\sum_{1\leqq\nu,\nu'\leqq m} u_\nu u_{\nu'} \rho(s_\nu - (k + 1)2^{-N}, s_{\nu'} - (k + 1)2^{-N})) \\ &+ i \textstyle\sum_{\nu=1}^{m} u_\nu X_p(s_\nu\,|\,(k + 1)2^{-N}))) \end{aligned} \tag{47}$$

where $\chi(E)$ denotes the indicator function of the set E. The continuity of $\rho(s, t)$ and $X_p(s\,|\,t)$ implies

$$\begin{aligned} \lim_{N\to\infty} Z_N = \chi(F) \exp(-\tfrac{1}{2}(&\textstyle\sum_{1\leqq\nu,\nu'\leqq m} u_\nu u_{\nu'} \rho(s_\nu - \tau, s_{\nu'} - \tau)) \\ &+ i \textstyle\sum_{\nu=1}^{m} u_\nu X_p(s_\nu\,|\,\tau)) \end{aligned}$$

and as $|Z_N| \leqq 1$ we may apply dominated convergence to obtain

$$\begin{aligned} \lim_{N\to\infty} E(Z_N) = \textstyle\int_F \exp(-\tfrac{1}{2}(&\textstyle\sum_{1\leqq\nu,\nu'\leqq m} u_\nu u_{\nu'} \rho(s_\nu - \tau, s_{\nu'} - \tau)) \\ &+ i \textstyle\sum_{\nu=1}^{m} u_\nu X_p(s_\nu\,|\,\tau))\, dP . \end{aligned} \tag{48}$$

Now,

$$\begin{aligned} &\textstyle\int_F \exp(i \sum_{\nu=1}^{m} u_\nu X(s_\nu))\, dP \\ &\quad = \textstyle\sum_{k=-\infty}^{\infty} \int_{F\cap\{k2^{-N}\leqq\tau<(k+1)2^{-N}\}} \exp(i \sum_{\nu=1}^{m} u_\nu(X_e(s_\nu\,|\,(k + 1)2^{-N}) \\ &\qquad + X_p(s_\nu\,|\,(k + 1)2^{-N})))\, dP \\ &\quad = \textstyle\sum_{k=-\infty}^{\infty} \exp(-\frac{1}{2} \sum_{1\leqq\nu,\nu'\leqq m} u_\nu u_{\nu'} \rho(s_\nu - (k + 1)2^{-N}, s_{\nu'} - (k + 1)2^{-N})) \\ &\qquad \times \textstyle\int_{F\cap\{k2^{-N}\leqq\tau<(k+1)2^{-N}\}} \exp(i \sum_{\nu=1}^{m} u_\nu X_p(s_\nu\,|\,(k + 1)2^{-N}))\, dP \\ &\quad = E(Z_N). \end{aligned}$$

The next to last step follows from the stationarity of the process $X(t)$ $(X_e(s_1\,|\,t), \cdots, X_e(s_n\,|\,t)$ have the same joint distribution as $X_e(s_1 - t), \cdots, X_e(s_n - t))$, the fact that $F \cap \{k2^{-N} \leqq \tau < (k + 1)2^{-N}\}$ belongs to $\mathscr{F}_{(k+1)2^{-N}}$ (F is in $\mathscr{F}_\tau$) and finally the fact that $X_e(s\,|\,t)$ is independent of $\mathscr{F}_t$.

On combining this with (48) we obtain

$$\int_F \exp(i \sum_{\nu=1}^m u_\nu X(s_\nu))\, dP = \int_F \exp(-\tfrac{1}{2} \sum_{1\leq \nu, \nu' \leq m} u_\nu u_{\nu'} \rho(s_\nu - \tau, s_{\nu'} - \tau) + i \sum_{\nu=1}^m u_\nu X_p(s_\nu \mid \tau))\, dP$$

for all $F \in F_\tau$. As $\rho(s_\nu - \tau, s_{\nu'} - \tau)$, $1 \leq \nu, \nu' \leq m$ and $X_p(s_\nu \mid \tau)$, $1 \leq \nu \leq m$ are $\mathscr{F}_\tau$-measurable because of the continuity properties, this implies the statement of the lemma.

LEMMA 14. *If $\tau > 0$ is a stopping time for the process $X(t)$ then*

$$E((\varphi(0, \tau + h_n') - \varphi(0, \tau))^{2m}) \leq (2m)!\, \frac{(h_n')^{2m}}{(2\pi)^m} \int_{\mathscr{I}_{2m}(1)} |\boldsymbol{\lambda}(2m)|^{-\frac{1}{2}}\, d\mathbf{s}_{2m} \tag{49}$$

where $(h_n')_1^\infty$ and $\boldsymbol{\lambda}(2m)$ are as in Section 3.1.

PROOF. Writing $\varphi(t)$ for $\varphi(0, t)$, an argument similar to that in the previous lemma yields, as $\varphi(t)$ has continuous sample paths,

$$E((\varphi(\tau + h_n') - \varphi(\tau))^{2m}) = \lim_{k\to\infty} \sum_{j=1}^\infty E(\chi((j-1)2^{-k} \leq \tau < j2^{-k}) \times E((\varphi(j2^{-k} + h_n') - \varphi(j2^{-k}))^{2m} \mid \mathscr{F}_\tau))\,. \tag{50}$$

Now $(\varphi_\nu(j2^{-k} + h_n') - \varphi_\nu(j2^{-k}))^{2m}$ tends in mean to $(\varphi(j2^{-k} + h_n') - \varphi(j2^{-k}))^{2m}$ and hence $E((\varphi_\nu(j2^{-k} + h_n') - \varphi_\nu(j2^{-k}))^{2m} \mid \mathscr{F}_\tau)$ tends in mean to $E((\varphi(j2^{-k} + h_n') - \varphi(j2^{-k}))^{2m} \mid \mathscr{F}_\tau)$. We therefore have

$$E(\chi((j-1)2^{-k} \leq \tau < j2^{-k}) E((\varphi(j2^{-k} + h_n') - \varphi(j2^{-k}))^{2m} \mid \mathscr{F}_\tau)) = \lim_{\nu\to\infty} E(\chi((j-1)2^{-k} \leq \tau < j2^{-k}) E((\varphi_\nu(j2^{-k} + h_n') - \varphi_\nu(j2^{-k}))^{2m} \mid \mathscr{F}_\tau))\,. \tag{51}$$

To evaluate the last expression we use Lemma 13, we write

$$\mathscr{C}'_{2m}(\nu) = \{\mathbf{u}_{2m} : |u_j| \leq \nu,\ 1 \leq j \leq 2m\}$$

and

$$\mathscr{I}_{2m}(a, b) = \{\mathbf{s}_{2m} : 0 \leq a \leq s_j \leq b,\ 1 \leq j \leq 2m\}\,.$$

With this notation we have

$$\begin{aligned}
&\chi((j-1)2^{-k} \leq \tau < j2^{-k}) E((\varphi_\nu(j2^{-k} + h_n') - \varphi_\nu(j2^{-k}))^{2m} \mid \mathscr{F}_\tau) \\
&= \left(\frac{1}{2\pi}\right)^{2m} \int_{\mathscr{C}'_{2m}(\nu)} \Big(\int_{\mathscr{I}_{2m}(j2^{-k}, j2^{-k}+h_n')} \chi((j-1)2^{-k} \leq \tau < j2^{-k}) \\
&\qquad \times E(\exp(i \textstyle\sum_{l=1}^{2m} u_l X(s_l)) \mid \mathscr{F}_\tau)\, d\mathbf{s}_{2m}\Big)\, d\mathbf{u}_{2m} \\
&= \left(\frac{1}{2\pi}\right)^{2m} \int_{\mathscr{C}'_{2m}(\nu)} \Big(\int_{\mathscr{I}_{2m}(j2^{-k}, j2^{-k}+h_n')} \chi((j-1)2^{-k} \leq \tau < j2^{-k}) \\
&\qquad \times \exp(-\tfrac{1}{2} \textstyle\sum_{1\leq l, l' \leq 2m} u_l u_{l'} \rho(s_l - \tau, s_{l'} - \tau) \\
&\qquad + i \textstyle\sum_{l=1}^{2m} u_l X_p(s_l \mid \tau))\, d\mathbf{s}_{2m}\Big)\, d\mathbf{u}_{2m}\,,
\end{aligned}$$

and hence

$$|\chi((j-1)2^{-k} \leqq \tau < j2^{-k})E((\varphi_\nu(j2^{-k}+h_n')-\varphi_\nu(j2^{-k}))^{2m} \mid \mathscr{F}_\tau)|$$
$$\leqq \left(\frac{1}{2\pi}\right)^{2m} \int_{\mathscr{C}'_{2m}(\nu)} \left(\int_{\mathscr{S}_{2m}(j2^{-k}, j2^{-k}+h_n')} \chi((j-1)2^{-k} \leqq \tau < j2^{-k})\right.$$
$$\left.\times \exp(-\tfrac{1}{2}\textstyle\sum_{1\leqq l,l'\leqq 2m} u_l u_{l'} \rho(s_l-\tau, s_{l'}-\tau))\, d\mathbf{s}_{2m}\right) d\mathbf{u}_{2m}$$
$$= \left(\frac{1}{2\pi}\right)^{2m} \int_{\mathscr{C}'_{2m}(\nu)} \left(\int_{\mathscr{S}_{2m}(j2^{-k}-\tau, j2^{-k}+h_n'-\tau)} \chi((j-1)2^{-k} \leqq \tau < j2^{-k})\right.$$
$$\left.\times \exp(-\tfrac{1}{2}\textstyle\sum_{1\leqq l,l'\leqq 2m} u_l u_{l'} \rho(s_l, s_{l'}))\, d\mathbf{s}_{2m}\right) d\mathbf{u}_{2m}$$
$$\leqq \left(\frac{1}{2\pi}\right)^{2m} \int_{\mathscr{C}'_{2m}(\nu)} \left(\int_{\mathscr{S}_{2m}(0, 2^{-k}+h_n')} \chi((j-1)2^{-k} \leqq \tau < j2^{-k})\right.$$
$$\left.\times \exp(-\tfrac{1}{2}\textstyle\sum_{1\leqq l,l'\leqq 2m} u_l u_{l'} \rho(s_l, s_{l'}))\, d\mathbf{s}_{2m}\right) d\mathbf{u}_{2m}$$
$$\leqq \left(\frac{1}{2\pi}\right)^{2m} \left(\int\int_{\mathscr{S}_{2m}(0, 2^{-k}+h_n')} \chi((j-1)^{-k} \leqq \tau < j2^{-k}\right.$$
$$\left.\times \exp(-\tfrac{1}{2}\textstyle\sum_{1\leqq l,l'\leqq 2m} u_l u_{l'} \rho(s_l, s_{l'}))\, d\mathbf{s}_{2m}\right) d\mathbf{u}_{2m}$$
$$= (2m)! \left(\frac{1}{2\pi}\right)^{2m} \int_{\mathscr{S}_{2m}(0, 2^{-k}+h_n')} \left(\int \chi((j-1)2^{-k} \leqq \tau < j2^{-k})\right.$$
$$\left.\times \exp(-\tfrac{1}{2}\textstyle\sum_{1\leqq l,l'\leqq 2m} u_l u_{l'} \rho(s_l, s_{l'}))\, d\mathbf{u}_{2m}\right) d\mathbf{s}_{2m}$$

where $\mathscr{S}_{2m}(a, b) = \{\mathbf{s}_{2m} : 0 \leqq a < s_1 < \cdots < s_{2m} < b\}$.

On integrating out the u_i's we obtain

$$|\chi((j-1)2^{-k} \leqq \tau < j2^{-k})E((\varphi_\nu(j2^{-k}+h_n')-\varphi_\nu(j2^{-k}))^{2m} \mid \mathscr{F}_\tau)|$$
$$\leqq (2m)! \left(\frac{1}{2\pi}\right)^{m} \chi((j-1)2^{-k} \leqq \tau < j2^{-k}) \int_{\mathscr{S}_{2m}(0, 2^{-k}+h_n')} |\rho(s_l, s_{l'})|\, d\mathbf{s}_{2m}$$
$$\leqq (2m)! \frac{(h_n')^{2m}}{(2\pi)^m} \chi((j-1)2^{-k} \leqq \tau < j2^{-k}) \int_{\mathscr{S}_{2m}(0, 2^{-k}(h_n')^{-1}+1)} |\rho(h_n's_l, h_n's_{l'})|\, d\mathbf{s}_{2m}\,.$$

Now the matrix $\boldsymbol{\lambda}(2m)$ may be obtained from the matrix $\boldsymbol{\rho}(2m) = (\rho(h_n's_l, h_n's_{l'}))$ by row and column operations and hence

$$|\chi((j-1)2^{-k} \leqq \tau < j2^{-k})E((\varphi_\nu(j2^{-k}+h_n')-\varphi_\nu(j2^{-k}))^{2m} \mid \mathscr{F}_\tau)|$$
$$\leqq (2m)! \frac{(h_n')^{2m}}{(2\pi)^m} \chi((j-1)2^{-k} \leqq \tau < j2^{-k}) \int_{\mathscr{S}_{2m}(0, 2^{-k}(h_n')^{-1}+1)} |\boldsymbol{\lambda}(2m)|^{-\frac{1}{2}}\, d\mathbf{s}_{2m}\,.$$

Substituting this inequality into (51) gives

$$E(\chi((j-1)2^{-k} \leqq \tau < j2^{-k})E((\varphi(j2^{-k}+h_n')-\varphi(j2^{-k}))^{2m} \mid \mathscr{F}_\tau))$$
$$\leqq (2m)! \frac{(h_n')^{2m}}{(2\pi)^m} E(\chi((j-1)2^{-k} \leqq \tau < j2^{-k})) \int_{\mathscr{S}_{2m}(0, 2^{-k}(h_n')+1)} |\boldsymbol{\lambda}(2m)|^{-\frac{1}{2}}\, d\mathbf{s}_{2m}$$

and on substituting this into (50) we obtain

$$E((\varphi(\tau+h_n')-\varphi(\tau))^{2m}) \leqq (2m)! \frac{(h_n')^{2m}}{(2\pi)^m} \lim_{k\to\infty} \int_{\mathscr{S}_{2m}(0, 2^{-k}(h_n')^{-1}+1)} |\boldsymbol{\lambda}(2m)|^{-\frac{1}{2}}\, d\mathbf{s}_{2m}$$
$$= (2m)! \frac{(h_n')^{2m}}{(2\pi)^m} \int_{\mathscr{S}_{2m}(1)} |\boldsymbol{\lambda}(2m)|^{-\frac{1}{2}}\, d\mathbf{s}_{2m}\,.$$

The last equality follows from an application of dominated convergence which may, for example, be justified by appealing to the inequality of (23). This completes the proof of the lemma.

Lemmas 4, 7 and 14 may now be combined and the method of Section 4 gives

THEOREM 2. *Suppose the stationary Gaussian process $X(t)$ has spectral density function f given by* (4) *and let τ be a stopping time for the process which is almost surely finite. Then for almost all ω*

$$\limsup_{h\downarrow 0} \frac{\varphi(x, \tau + h) - \varphi(x, \tau)}{h^{1-\alpha}(\log(-\log h))^{\alpha}} \leqq c_2 < \infty .$$

To obtain a lower bound for the Hausdorff measure of the zero set we follow Taylor and Wendel. We define

$$\tau_t = \min \{u : \varphi(0, u) = t\}$$

and as $\varphi(0, t)$ is $\mathscr{F}_t$-measurable, τ_t is a stopping time for the process $X(t)$. To show that τ_t is almost surely finite we proceed as follows. A trivial modification of the formulae given on page 295 of Berman [1] gives

$$E(\varphi(0, t)) = t/(2\pi)^{\frac{1}{2}}$$

and

$$E(\varphi(0, t)^2) = (2\pi)^{-1} \textstyle\int_0^t \int_0^t (1 - r^2)^{-\frac{1}{2}}\, du\, dv$$

where

$$r = r(u, v) = \textstyle\int_{-\infty}^{\infty} \cos(\lambda(u - v)) f(\lambda)\, d\lambda$$

where $f(\lambda)$ is the spectral density function of the process $X(t)$. This yields

$$V(\varphi(0, t)) = (2\pi)^{-1} \textstyle\int_0^t \int_0^t ((1 - r^2)^{-\frac{1}{2}} - 1)\, du\, dv$$

for the variance $V(\varphi(0, t))$ of $\varphi(0, t)$. With spectral density function $f(\lambda)$ given by (4) the Riemann–Lebesgue theorem tells us that $\lim_{s\to\infty} r(s) = 0$. This implies that $(1 - r^2)^{-\frac{1}{2}} - 1 \sim r^2/2$ for large $u - v$ and hence

$$V(\varphi(0, t)) = O(\textstyle\int_0^t \int_0^t r^2\, du\, dv) = o(t^2) .$$

Tchebychev's inequality then gives

$$\lim_{t\to\infty} P(|\varphi(0, t) - t/(2\pi)^{\frac{1}{2}}| \geqq t/3) = 0$$

which implies $P(\lim_{t\to\infty} \varphi(0, t) = \infty) = 1$. From this it follows that τ is almost surely finite. We may therefore apply Theorem 2 to obtain

$$\limsup_{h\downarrow 0} \frac{\varphi(0, \tau_t + h) - \varphi(0, \tau_t)}{h^{1-\alpha}(\log(-\log h))^{\alpha}} \leqq c_2 < \infty$$

almost surely. The remainder of the proof is then identical to that in [15], pages 176–177, and gives

THEOREM 3. *Suppose the stationary Gaussian process $X(t)$ has spectral density* *function $f(\lambda)$ given by (4). Then with probability one the set of zeros of the process has positive, possibly infinite Hausdorff measure with respect to the function*

$$\psi(h) = h^{1-\alpha}(\log(-\log h))^{\alpha}.$$

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