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POTENTIALS OF MARKOV PROCESSES WITHOUT DUALITY[1]

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The potential of a natural additive functional of a transient standard process is represented as a potential of a measure without the usual assumption of strong duality for the process. The balayage on a Borel set, B, of the potential of an additive functional or bounded function is represented as the potential of a measure supported by the closure of B.

1. Recently, Chung (1973) showed that the equilibrium principle can be proved for the potential theory of a Markov process without the usual strong duality. In this paper, we extend Chung's methods to show that the potential of a natural additive functional can be represented as the potential of a measure, again without a strong duality assumption. This is an extension of a result proved by Revuz (1970) under strong duality conditions.

The balayage of the potential of a natural additive functional and the balayage of the potential of a function are also shown to be potentials of measures. The methods used in this paper depend on an analytic condition, (A) below, also used by Chung (1973).

2. Let $X = (\Omega, \mathscr{F}, \mathscr{F}_t, X_t, \theta_t, P^x)$ be a standard Markov process with lifetime ζ on a locally compact, countable base space $(E, \mathscr{E})$. Let X be transient in the sense that it last exits from any compact set in finite time a.s. Our notation will follow Blumenthal and Getoor (1968) unless specifically noted.

We assume that the potential operator for the process, defined by

$$Uf(x) = E^x\{\int_0^\infty f(X_t)\,dt\},$$

f nonnegative and measurable, satisfies the following condition:

(A) (i) There is a sigma-finite measure, ξ, and a nonnegative measurable function, $u(x, y)$ such that

$$U(x, C) = \int_C u(x, y)\xi(dy) \qquad \text{for any} \quad C \in \mathscr{E};$$

(ii) $y \to u(x, y)^{-1}$ is finite and continuous for all $x \in E$, where $u(x, y)^{-1} = 1/u(x, y)$;

(iii) $u(x, y) = \infty$ only if $x = y$.

With the potential kernel density, $u(x, y)$, we define the *potential of a measure*, μ:

$$U_\mu(x) = \int_E u(x, y)\mu(dy).$$

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We also define: An *additive functional* $A = (A_t)$, $t \geqq 0$, is a family of functions from Ω to $[0, \infty]$ such that

(a) almost surely $t \to A_t$ is nondecreasing, right continuous, continuous at ζ, and $A_0 = 0$;
(b) for each t, A_t is measurable with respect to $\mathscr{F}_t$;
(c) for t and $s \geqq 0$, $A_{s+t} = A_t + A_s \circ \theta_t$, almost surely.

If (b) is not satisfied, but (a) and (c) are, then (A_t) is a *nonadapted* additive functional.

An additive functional is *natural* if $t \to A_t$ and $t \to X_t$ have no common discontinuities, almost surely.

A *natural potential* is a finite excessive function f such that for any sequence of optional times T_n increasing to $T \geqq \zeta$, almost surely

$$\lim_{n\to\infty} P_{T_n} f(x) = 0 .$$

The potential of an additive functional (A_t) and its *potential kernel* $U_A(x, dy)$ are defined by

$$U_A 1(\cdot) = E^{(\cdot)}\{A_\infty\} \qquad \text{and} \qquad U_A f(x) = E^x\{\textstyle\int_0^\infty f(X_s)\, dA_s\} .$$

Meyer (1965) has shown that for a standard process any natural potential is the potential of a unique natural additive functional. A proof in terms of the notation used here is found in Blumenthal and Getoor (1968, notes for IV. 4).

We formally define the measure developed by Revuz (1970):

DEFINITION. Let $A = (A_t)$ be a natural additive functional, with finite potential. There is a measure associated with A which we shall call Revuz measure and denote m_A and define by

$$m_A(f) = \lim_{t\to 0} (1/t) E^\xi\{\textstyle\int_0^t f(X_s)\, dA_s\} ,$$

for f bounded and continuous and where ξ is the reference measure used in condition (A).

The fact that m_A is a measure will become clear from the proof of Theorem 1.

3. THEOREM 1. *Let X be a transient standard process satisfying* (A). *Let (A_t) be a natural additive functional with finite natural potential $U_A 1$ and Revuz measure m_A.*

Then for any nonnegative or bounded and measurable f

$$U_A f(x) = \textstyle\int u(x, y) f(y) m_A(dy) ,$$

and m_A is the only such measure.

PROOF. 1. Let f be a nonnegative, bounded, continuous function on E. Then the following calculation shows that

$$(1) \qquad U_A f(x) = \lim_{\varepsilon\to 0} E^x\{\textstyle\int_0^\infty f(X_t)(1/\varepsilon) \int_0^\infty 1_{(t,t+\varepsilon]}(s)\, dA_s\, dt\} .$$

The right side of the above is, by Fubini,

$$= \lim_{\varepsilon\to 0} E^x\{\textstyle\int_0^\infty (1/\varepsilon)[\int_{(s-\varepsilon)\vee 0}^s f(X_t)\, dt]\, dA_s\} .$$

However, the bracketed inner integral converges boundedly to $f(X_{s-})$, so that the whole integral converges to

$$E^x\{\int_0^\infty f(X_{s-})\,dA_s\} = U_A f(x)\,,$$

since A is natural.

2. On the other hand, the right side of (1) is

$$= \lim_{\varepsilon\to 0} \int_0^\infty E^x\{f(X_t)(1/\varepsilon) \int_0^\infty 1_{(t,t+\varepsilon]}(s)\,dA_s\}\,dt$$
$$= \lim_{\varepsilon\to 0} \int_0^\infty E^x\{f(X_t)E^{X_t}\{(1/\varepsilon)A_\varepsilon\}\}\,dt\,.$$

We let $\Psi_\varepsilon(y) = E^y\{(1/\varepsilon)A_\varepsilon\}$ and the above is

$$= \lim_{\varepsilon\to 0} U(f\cdot \Psi_\varepsilon)(x)\,.$$

Finally, we let $M_\varepsilon(dy) = \Psi_\varepsilon(y)\xi(dy)$ and we can conclude that

$$U_A f(x) = \lim_{\varepsilon\to 0} \int u(x, y)f(y)M_\varepsilon(dy)\,.$$

3. Fix an x and let $f(y) = u(x, y)^{-1}\varphi(y)$, where φ is continuous with compact support. Applying the above conclusion to f we have

$$\int_E \varphi(y)u(x, y)^{-1}U_A(x, dy) = \lim_{\varepsilon\to 0} \int \varphi(y)M_\varepsilon(dy)\,.$$

Therefore we may conclude that the M_ε converge vaguely to the measure

$$m_A(dy) = u(x, y)^{-1}U_A(x, dy)$$

and this is true for any x in E.

4. Let C be in $\mathscr{E}$ and not include x. Then by condition (A), $u(x, y)$ is finite for y in C so

$$\int_C u(x, y)m_A(dy) = U_A(x, C)\,. \tag{2}$$

We would like to show that (2) holds without restriction.

If $u(x, x) < \infty$, then (2) holds for this x and any C in $\mathscr{E}$. If $u(x, x) = \infty$, then

$$m_A(\{x\}) = u(x, x)^{-1}U_A(x, \{x\}) = 0\,. \tag{3}$$

Therefore, for $z \neq x$:

$$U_A(z, \{x\}) = u(z, x)m_A(\{x\}) = 0\,. \tag{4}$$

Furthermore, if $u(x, x) = \infty$, then x is not a holding point (Chung, 1973).

Let $S_m(\omega) = \min\{k2^{-m} \mid X(k2^{-m}, \omega) \neq x\}$ or ∞ if this set is empty. Then $P^x\{\lim_{m\to\infty} S_m = 0\} = 1$, since the process is right continuous and x is not a holding point. Therefore, since the S_m are countably-valued, we may apply the Markov property:

$$\begin{aligned} U_A(x, \{x\}) &= E^x\{\int_0^\infty 1_{\{x\}}(X_s)\,dA_s\} \\ &= \lim_{m\to\infty} E^x\{\int_{S_m}^\infty 1_{\{x\}}(X_s)\,dA_s\} \\ &= \lim_{m\to\infty} E^x\{U_A(X(S_m), \{x\})\} = 0\,. \end{aligned} \tag{5}$$

by (4). Then

$$\int_C u(x, y)m_A(dy) = \int_{C\setminus\{x\}} u(x, y)m_A(dy) \quad \text{by (3)}$$
$$= U_A(x, C\setminus\{x\}) \quad \text{by (2)}$$
$$= U_A(x, C) \quad \text{by (5)}.$$

5. We show that m_A is the only such measure. Suppose μ is a measure with $\int u(\cdot, y)f(y)\mu(dy) = U_A f(\cdot) = \int u(\cdot, y)f(y)m_A(dy)$, for any f bounded and measurable. Fix x and let $f(y) = u(x, y)^{-1}1_C(y)$ for C in $\mathscr{E}$, with compact closure. If $u(x, x) < \infty$, then $\mu(C) = \int u(x, y)f(y)\mu(dy) = \int u(x, y)f(y)m_A(dy) = m_A(C)$. If $u(x, x) = \infty$, then $0 = U_A(x, \{x\}) = u(x, x)\mu(\{x\})$ so $\mu(\{x\}) = m_A(\{x\}) = 0$, and we still have $\mu(C) = m_A(C)$. Thus $\mu = m_A$.

6. Finally, we show that m_A is indeed the Revuz measure for A. Let f be a continuous and bounded function and define the new natural additive functional

$$B_t = \int_0^t f(X_s)\, dA_s .$$

Then $Um_B = U_B 1 = U_A f = \int u(\cdot, y)f(y)m_A(dy)$, so that $m_B(dy)$ and $f(y)m_A(dy)$ are the same measure by 5.

On the other hand, by following the steps in the proof for m_B we see that m_B is the vague limit of

$$(1/\varepsilon)E^y\{\int_0^\varepsilon f(X_s)\, dA_s\}\xi(dy)$$

so that

$$m_A(f) = m_B(E) = \lim_{\varepsilon\to 0} \int (1/\varepsilon)E^y\{\int_0^\varepsilon f(X_s)\, dA_s\}\xi(dy)$$
$$= \lim_{\varepsilon\to 0} E^\xi\{(1/\varepsilon) \int_0^\varepsilon f(X_s)\, dA_s\} .$$

The proof of Theorem 1 is complete.

COROLLARY. *Equilibrium measure in the sense of Chung* (1973) *is unique.*

PROOF. $P^{(\cdot)}\{T_B < \infty\}$ for B transient is a natural potential.

REMARK. Theorem 1 could be stated for any nonadapted additive functional $B = (B_t)$ if we define the potential of B by $U_B f(x) = E^x\{\int_0^\infty f(X_{s-})\, dB_s\}$, and require that $U_B 1$ be finite on E.

4. The balayage operator for a Borel set, B, is defined by

$$P_B f(x) = E^x\{f(X_T)\}$$

where $T = \inf\{t > 0 \mid X_t \in B\}$. We define the last exit from B, before time s by

$$l(s) = \sup\{t < s \mid X_s \in B\} .$$

THEOREM 2. *If $A = (A_t)$ is a natural additive functional with finite natural potential $U_A 1$ and B is a Borel set, then there is a measure, $m_A{}^B$, concentrated on $\bar{B}$, such that $P_B U_A 1 = Um_A{}^B$. Furthermore, $m_A{}^B(dy) = u(x, y)^{-1}\nu(x, dy)$, where*

$$\nu(x, C) = E^x\{\int_0^\infty 1_{\{l(s)>0\}} 1_C(X_{l(s)-})\, dA_s\} .$$

PROOF. We define $T(t) = t + T \circ \theta_t$ and following Getoor and Sharpe

(1973a, b) we define the raw balayage of (A_t) on B by

$$A_B(t) = A_{T(t)} - A_T .$$

$(A_B(t))$ is a nonadapted additive functional and by the remark following Theorem 1 we can apply that result. From the proof of Theorem 1, we see that the corresponding Revuz measure is given by

$$m_A{}^B(dy) = u(x, y)^{-1}U_{A_B}(x, dy) .$$

But $U_{A_B}(x, C) = E^x\{\int_0^\infty 1_C(X_{s-})\, dA_B(s)\}$, and by calculations of Getoor and Sharpe (1973a, b) the latter is

$$= E^x\{\int_0^\infty 1_C(\{X_{l(s)-})1_{\{l(s)>0\}}\, dA_s\} .$$

This is the measure $\nu(x, C)$ given in the statement of the theorem. Clearly, $\nu(x, \cdot)$ has all its mass concentrated on $\bar{B}$ and therefore $m_A{}^B$ does also. The proof is complete.

In particular, Theorem 2 yields the result that if h is a positive bounded function with bounded potential, Uh, then there is a measure μ_B concentrated on $\bar{B}$ such that

$$P_B Uh = U\mu_B .$$

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