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ON WEAK CONVERGENCE OF EXTREMAL PROCESSES

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Lamperti in 1964 showed that the convergence of the marginals of an extremal process generated by independent and identically distributed random variables implies the full weak convergence in the Skorohod J_1-topology. This result is generalized to the kth extremal process and to random variables which need not be identically distributed. The proof here is based on the weak convergence of a certain point-process (which counts the number of up-crossings of the variables) to a two-dimensional nonhomogeneous Poisson process.

1. Introduction. Let $\{X_i\}$ be a sequence of independent random variables defined on some probability space $(\Omega, \mathscr{F}, P)$ and let $X_{ni} = (X_i - a_n)/b_n$, where a_n and $b_n > 0$ are norming constants. For each pair of positive integers k, n define the kth extremal process $m_n{}^k = \{m_n{}^k(t) : t \geqq 0\}$ by

$$m_n{}^k(t) = k\text{th largest among } \{X_{n1}, \cdots, X_{n[nt]}\}$$

if $1 \leqq k \leqq [nt]$ and $m_n{}^k(t) = X_{n1}$ if $k > [nt]$. Let $I_n(t, x) = \#\{X_{ni} > x : i = 1, 2, \cdots, [nt]\}$.

Suppose that there exists a family of distribution functions $\{G_t : t > 0\}$ such that as $n \to \infty$

$$\mathscr{L}(m_n{}^1(t)) \to G_t \qquad t > 0 . \tag{1.1}$$

In [8] we have shown that there exists a two-dimensional nonhomogeneous Poisson process I such that

$$I_n \to I$$

and

$$m_n{}^k \to m^k$$

in the sense of convergence of all the finite-dimensional laws (fdl), where $m^k(t) = \min\{x : I(t, x) \leqq k - 1\}$. In case G_1 is continuous the parameter set of I is $T = \{(t, x) : t \geqq 0, x > {}_*x_t\}$, where ${}_*x_t = \sup\{x : G_t(x) = 0\}$, ${}_*x_0 = \lim_{t \downarrow 0} {}_*x_t$ exists (possibly $-\infty$) and $G_0(x) \equiv 1$ $(x > {}_*x_0)$. We should mention here that (without loss of generality) $G_t(x)$ is either of the form $G_1(t^\theta x)$ $(\theta \neq 0)$ or of the form $G_1(x - c \log t)$ $(c \geqq 0)$.

Let $\Rightarrow$ denote weak convergence with respect to the Skorohod J_1-topology. Our main result is the following.

Theorem 1.1. *Suppose* (1.1) *holds with a continuous G_1 and with G_t which are*

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not identical. Then for all $0 < a < b$

$$m_n{}^k \Rightarrow m^k \quad \text{in} \quad D[a, b]\,. \tag{1.2}$$

In case $\theta < 0$, (1.2) *holds for all* $0 \leqq a < b$.

The case $k = 1$ with identically distributed $\{X_i\}$ was treated by Lamperti [3]. His main idea was to show tightness by proving that $\limsup_{n\to\infty} P\{\Delta_n(c) > \varepsilon\} = 0$ for all $\varepsilon > 0$ (here Δ_n is a certain modulus of continuity of $m_n{}^1$). Our proof is based on the fact that $I_n \Rightarrow I$ in the Skorohod J_1-topology (extended to the plane). Other uses of the two-dimensional Poisson process in connection with extremal process have been made in Pickands [4], Weissman [7] and Resnick [5].

The proof of the theorem appears in Section 2. We end this section with some definitions. For given $b, M, \delta > 0$ let $U = U(b, M, \delta) = \{(t, x) : 0 \leqq t \leqq b, K_t \leqq x \leqq M\}$, where $K_t = \max\{-M, {}_*x_t + \delta\}$. We define $D(U)$ to be the space of all integer-valued functions $z : U \mapsto R^1$ which are finite, right-continuous in each argument with left-hand limits, nondecreasing in t and nonincreasing in x. Let Λ^2 be the group of all transformations γ from U onto U of the form $\gamma(t,x) = (\gamma_1(t), \gamma_2(x))$ where each γ_i $(i = 1, 2)$ is continuous and strictly increasing. We introduce the (extended) Skorohod J_1-topology on $D(U)$ by defining the "Skorohod" distance between z and y in $D(U)$ to be

$$d^2(z, y) = \inf\{\max(\|z - y\gamma\|, \|\gamma\|) : \gamma \in \Lambda^2\}\,, \tag{1.3}$$

where

$$\begin{aligned} \|z - y\gamma\| &= \sup\{|z(u) - y(\gamma(u))|_1 : u \in U\}\,, \\ \|\gamma\| &= \sup\{|\gamma(u) - u|_2 : u \in U\} \end{aligned} \tag{1.4}$$

and $|\cdot|_i$ is the standard norm on R^i $(i = 1, 2)$.

The space of finite right-continuous functions with left-hand limits, defined on $[a, b]$, is $D[a, b]$.

Let Λ^1 be the group of transformations γ_1 from $[0, b]$ onto $[0, b]$ which are continuous and strictly increasing. Then the "Skorohod" distance d^1 on $D[0, b]$ (which determines the J_1-topology) is obtained by replacing in (1.3) and (1.4), 2 by 1 and U by $[0, b]$.

For expositions of weak convergence of processes with several parameters we refer the reader to Straf [6] and Bickel and Wichura [1]. For the general theory of weak convergence see Billingsley [2].

2. Weak convergence of I_n and $m_n{}^k$.

Before proving the main result we prove

THEOREM 2.1. *Suppose* (1.1) *holds with continuous* G_1 *and with* G_t *which are not identical. Then for all fixed* $b, M, \delta > 0$

$$I_n \Rightarrow I \quad \text{in} \quad D(U)\,. \tag{2.1}$$

PROOF. For each (t, x), $I(t, x)$ is Poisson with parameter $-\log G_t(x)$ which is continuous by our assumption. Thus, with probability 1, I has neither multiplicities in U nor points on the boundary of U (cf. Theorem 2 of [8]). Since all the fdl of I_n converge to those of I, a result due to Straf [6] (page 212) implies the full weak convergence, i.e. (2.1) holds. □

Let $z \in D(U)$ and let $s_z(t) \subset U$ be the set of its jump-points with abscissa $\leqq t((t_0, x_0)$ is a jump-point if $z(t_0, x_0-) - z(t_0-, x_0-) - z(t_0, x_0) + z(t_0-, x_0) \neq 0)$. The *k-max-path* of z is defined to be

$$(2.2) \qquad h_k(t \mid z) = k\text{th largest member of } s_z(t) \quad \text{if } \# s_z(t) \geqq k$$
$$= K_t \quad \text{if } \# s_z(t) < k\,.$$

Clearly $h_k(\cdot \mid z) \in D[0, b]$ for each $z \in D(U')$.

LEMMA. *The mapping h_k: $D(U) \mapsto D[0, b]$ is continuous.*

PROOF. Let $z_n, z \in D(U)$ and suppose $d^2(z_n, z) \to 0$. This means that for each $\varepsilon > 0$ there exists an n_ε such that $n > n_\varepsilon$ implies $d^2(z_n, z) < \varepsilon$. In particular, for $0 < \varepsilon < 1$, since z_n and z are integer-valued, there exists a $\gamma_n = (\gamma_{n1}, \gamma_{n2}) \in \Lambda^2$ such that if $n > n_\varepsilon$ then for all $u \in U$

$$z_n(\gamma_n(u)) = z(u)\,, \qquad \|\gamma_n\| < \varepsilon\,.$$

Thus $h_k(t \mid z_n \gamma_n) = h_k(t \mid z)$ $(0 \leqq t \leqq b)$ and

$$(2.3) \qquad |h_k(\gamma_{n1}(t) \mid z_n) - h_k(t \mid z)| \leqq \|\gamma_{n2}\| < \varepsilon\,.$$

Since $\|\gamma_{n1}\| < \varepsilon$, (2.3) implies $d^1(h_k(\cdot \mid z_n), h_k(\cdot \mid z)) < \varepsilon$. □

PROOF OF THEOREM 1.1. Applying the lemma and the *continuous mapping theorem* (5.1 in [2]) we get from (2.1)

$$(2.4) \qquad h_k(\cdot \mid I_n) \Rightarrow h_k(\cdot \mid I) \quad \text{in} \quad D[0, b]\,.$$

Notice that (2.2) depends on M and δ. For a given a $(0 < a < b)$ we consider the following events

$$(2.5) \qquad A_n(M, \delta) \equiv \{h_k(t \mid I_n) \neq m_n{}^k(t) \text{ for some } t \in [a, b]\}$$
$$\subseteqq \{I_n(b, M) > 0 \text{ or } I_n(a, K_a\} < k\} \equiv B_n(M, \delta)\,.$$

A similar relation holds with the n suppressed. The convergence of the fdl of I_n to those of I implies $P\{B_n(M, \delta)\} \to P\{B(M, \delta)\}$. For $x \to \infty$, $I(t, x) \to 0$ a.s. and for $x \downarrow {}_*x_t$, $I(t, x) \to \infty$ a.s. Thus by choosing a large M and a small $\delta > 0$, we can make $P\{B(M, \delta)\}$ arbitrarily small. Hence (2.4) and (2.5) imply (1.2) for all $0 < a < b$.

When $\theta \geqq 0$, $m^k(t) \to -\infty$ a.s. as $t \downarrow 0$ and thus $m^k(t)$ is unbounded in the neighborhood of 0. But for $\theta < 0$, $m^k(t) \to$ constant a.s. and thus (1.2) holds for $a = 0$. □

From (1.2) follows in particular that for each k, the sequence $\{m_n{}^k\}$ is tight in $D[a, b]$. In [8] we have shown that all the fdl of $(m_n{}^1, \cdots, m_n{}^k)$ converge. Thus we have (cf. problem 6, page 41 of [2])

COROLLARY. *Under the assumptions of Theorem* 1.1

$$(m_n^1, \cdots, m_n^k) \Rightarrow (m^1, \cdots, m^k) \quad \textit{in} \quad D^k[a, b]$$

for each fixed k *and* $0 < a < b$.

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