The Annals of Probability
1976, Vol. 4, No. 3, 464–469

CONVERGENCE OF SUBMARTINGALES IN BANACH LATTICES

BY JERZY SZULGA AND WOJBOR A. WOYCZYŃSKI

Wroclaw University and the Institute of Mathematics, Polish Academy of Sciences

We discuss analogues of Doob's convergence theorem for submartingales with values in Banach lattices with the Radon–Nikodym property.

1. Introduction. In this note we make a few observations concerning submartingales taking values in certain Banach lattices. As far as martingales in Banach spaces are concerned, there is the following fundamental characterization of those Banach spaces in which the analogue of the Doob's convergence theorem holds true:

THEOREM 1.1 (Chatterji (1968)). *For a separable Banach space $\mathscr{X}$ the following three conditions are equivalent*:

(a) *for each $\mathscr{X}$-valued martingale $(M_n, n \in \mathbb{N})$ satisfying the Doob's condition* $\sup_{n\in\mathbb{N}} E||M_n|| < \infty$ *there exists an $M_\infty \in L^1(\mathscr{X})$ such that $M_n \to M_\infty$* a.s.;

(b) *for each $\mathscr{X}$-valued martingale $(M_n, n \in \mathbb{N})$ such that $\sup_{n\in\mathbb{N}} E||M_n||^p < \infty$ for some $1 < p < \infty$, there exists an $M_\infty \in L^p(\mathscr{X})$ such that $M_n \to M_\infty$ in $L^p(\mathscr{X})$*;

(c) *$\mathscr{X}$ has the Radon–Nikodym property*; i.e. *any vector measure with values in $\mathscr{X}$ and finite total variation which is absolutely continuous with respect to a scalar measure has with respect to it a Bochner integrable density.*

For other, especially geometric, properties of a Banach space that are equivalent to the Radon–Nikodym property, see Davis (1973–1974).

At the same time we have also the following:

THEOREM 1.2 (Neveu (1972), page 63). *If $(X_n, n \in \mathbb{N})$ is a real-valued submartingale and* $\sup_{n\in\mathbb{N}} EX_n^+ < \infty$ *then there exists an $X_\infty \in L^1(\mathbb{R})$ such that $X_n \to X_\infty$* a.s.

What we are interested in is how the above Theorem 1.2 (which can also be dually formulated for supermartingales) carries over to the case of martingales with values in Banach lattices (to be defined below). In what we did we were encouraged by the fact that recently Schwartz (1973) extensively developed the theory of supermartingales that have measures as their values, and applied it efficiently to the disintegration of measures. His model fits into our general framework.

2. Definitions, notation, facts. A vector lattice $(\mathscr{X}, \leqq)$ is said to be a Banach lattice if it is equipped with monotone ($|x| \leqq |y|$ implies $||x|| \leqq ||y||$) and complete norm. As usual, if $x \in \mathscr{X}$ then $x^+ \equiv \sup(x, 0)$, $x^- \equiv \sup(-x, 0)$,

Received on December 9, 1974.

AMS 1970 *subject classifications.* Primary 60G45; Secondary 60B99.

Key words and phrases. Submartingale, Banach lattice, Radon–Nikodym property.

$|x| \equiv x^+ + x^-$. $\mathscr{X}^*$, the norm dual of $\mathscr{X}$, is also a Banach lattice under the natural ordering, and by $\mathscr{X}_+$ and $\mathscr{X}_+^*$ we denote nonnegative cones in $\mathscr{X}$ and $\mathscr{X}^*$, respectively.

The set $A \subset \mathscr{X}$ is said to be order bounded if there exists $x_0 \in \mathscr{X}$ such that for all $y \in A$, $|y| \leqq x_0$, and the linear operator T from a Banach space $\mathscr{Y}$ into a Banach lattice $\mathscr{X}$ is called lattice bounded if it maps the unit ball of $\mathscr{Y}$ into an order bounded subset of $\mathscr{X}$. In general, and in many concrete cases, lattice bounded operators were thoroughly investigated and completely described in the book by Kantorovich, Vulikh and Pinsker (1950, VIII.4–VIII.6). In particular, they show that an an operator $T = (a_{ik}) : l^q \to l^p$, $q > 1$, $p \geqq 1$ is lattice bounded if, and only if

$$\textstyle\sum_{i=1}^{\infty} [\sum_{k=1}^{\infty} |a_{ik}|^{q/(q-1)}]^{p(q-1)/q} < \infty .$$

Also recently, the papers by Garling (1974), Nielsen (1973) and others raised interest in such operators in connection with absolutely summing and radonifying operators, although neglecting the results of the above mentioned book.

Now, let $(\Omega, \mathscr{A}, P)(\omega \in \Omega)$ be a probability space, and let $(\mathscr{A}_n, n \in \mathbb{N})$ be an increasing sequence of sub-σ-algebras of $\mathscr{A}$. We say that the sequence $(X_n, n \in \mathbb{N})$ of random vectors in $\mathscr{X}$ is a submartingale [martingale, supermartingale] if $X_n \in L^1(\Omega, \mathscr{A}_n, P; \mathscr{X}) \equiv \{X: \Omega \to \mathscr{X}: X \text{ strongly } \mathscr{A}_n\text{-measurable}, \int \|X\|\, dP < \infty\}$ and $E^{\mathscr{A}_n}X_{n+1} \geqq X_n [E^{\mathscr{A}_n}X_{n+1} = X_n, E^{\mathscr{A}_n}X_{n+1} \leqq X_n]$ a.s. for all $n \in \mathbb{N}$. Without further comment we always assume that all Banach spaces in which random vectors take values are separable. For various elementary facts concerning vector lattices that are used freely throughout the note consult [4] or any other standard text on vector lattices.

3. Counterexamples. Doob's condition $\sup_{n\in\mathrm{N}} EX_n^+ < \infty$ for real-valued random variables has two analogues for Banach-lattice-valued random vectors, namely: order boundedness of $(E(X_n^+), n \in \mathbb{N})$ and $\sup_{n\in\mathrm{N}} E\|X_n^+\| < \infty$. Both boil down to Doob's condition in the real case. However, as we shall see below, in general neither is sufficient to assure the a.s. convergence of a submartingale $(X_n, n \in \mathbb{N})$.

It is not difficult to check that for both real and vector submartingales the set $(E(X_n^+), n \in \mathbb{N})$ is order bounded if and only if $(E|X_n|, n \in \mathbb{N})$ is such. However, even for vector-valued martingales it might happen that $\sup_{n\in\mathrm{N}} E\|X_n^+\| < \infty$ and still $\sup_{n\in\mathrm{N}} E\|X_n^-\| = \infty$, so that it will not be surprising that the condition $\sup_{n\in\mathrm{N}} E\|X_n^+\| < \infty$ does not, in general, imply the a.s. convergence of a submartingale (X_n) even in Banach lattices with the Radon–Nikodym property. On the other hand, the condition $\sup_{n\in\mathrm{N}} E\|X_n^+\| < \infty$ is stronger than order boundedness of $(E(X_n^+), n \in \mathbb{N})$ for any submartingale $(X_n, n \in \mathbb{N})$ with values in the Banach lattice $\mathscr{X}$ that does not contain c_0 isomorphically and that has the order continuous norm because in that case any norm bounded increasing sequence in $\mathscr{X}$ has a least upper bound. The latter result is due to Tzafriri and may be found in [8] (Theorem 14).

It is not hard to see that if $\mathscr{X}$ is a Banach lattice with the Radon–Nikodym property then in order to produce examples of

(e) a martingale $(M_n, \mathscr{A}_n, n \in \mathbb{N})$ in $\mathscr{X}$ such that $\sup_{n\in\mathbb{N}} E||M_n^+|| < \infty$ and at the same time $\sup_{n\in\mathbb{N}} E||M_n^-|| = \infty$ and $\sup_{n\in\mathbb{N}} E||M_n|| = \infty$, and

(ee) a submartingale $(X_n, \mathscr{A}, n \in \mathbb{N})$ such that $\sup_{n\in\mathbb{N}} E||X_n^+|| < \infty$ and (X_n) is divergent a.s., it is sufficient to find

(eee) an a.s. divergent sequence $(Y_n, n \in \mathbb{N})$ of nonnegative independent random vectors in $L^1(\mathscr{X})$ such that both $\sup_{n\in\mathbb{N}} E||Y_n|| < \infty$ and

$$\sup_{n\in\mathbb{N}} ||\textstyle\sum_{i=0}^{n} EY_i|| < \infty .$$

Indeed, given such a sequence (Y_n), it is enough to take

$$\mathscr{A}_n = \sigma(Y_0, \cdots, Y_n)\,, \qquad A_0 = 0\,,$$
$$A_n = \textstyle\sum_{i=0}^{n-1} Y_i\,, \qquad M_n = -A_{n+1} + EA_{n+1}\,,$$
$$X_n = M_n + A_n = EA_{n+1} - Y_n\,, \quad n \geqq 1\,.$$

Then the sequence (EA_n) converges because $\mathscr{X}$ does not contain isomorphically c_0 (because it has the Radon–Nikodym property, cf. Davis (1973–1974)), and since in every such Banach lattice monotone norm-bounded sequences are convergent (Tzafriri (1972), Theorem 14). On the other hand the sequence (A_n) diverges a.s. because the sequence (Y_n) itself diverges so that $\sup_{n\in\mathbb{N}} E||A_n|| = \infty$.

Now, (M_n) defined in such a way is a zero mean martingale such that $\sup_{n\in\mathbb{N}} E||M_n^+|| \leqq \sup_{n\in\mathbb{N}} ||EA_{n+1}|| < \infty$ (because $M_n^+ \leqq EA_{n+1}$); but at the same time, $E||M_n|| = E||A_{n+1} - EA_{n+1}|| \geqq E||A_{n+1}|| - ||EA_{n+1}||$ is unbounded. X_n is evidently a submartingale that is divergent a.s. and for which $\sup_{n\in\mathbb{N}} E||X_n^+|| \leqq \sup_{n\in\mathbb{N}} ||E(A_{n+1}^+)|| = \sup_{n\in\mathbb{N}} ||EA_{n+1}|| < \infty$.

Given below are examples of sequences (Y_n) of random vectors with values in certain classical Banach lattices that enjoy the property (eee).

EXAMPLE 3.1. Let $\mathscr{X} = l^p$, $p > 1$ (the reason why $p = 1$ is excluded appears in Corollary 4.1), $\Omega_i = [0, 1)$, $\mathscr{B}_i$ be all Borel subsets of $[0, 1)$ and λ_i be Lebesgue measure, $i \in \mathbb{N}$. Put $\Omega = \prod_{i\in\mathbb{N}} \Omega_i$, $\mathscr{A} = \prod_{i\in\mathbb{N}} \mathscr{B}_i$, $P = \prod_{i\in\mathbb{N}} \lambda_i$, $Y_0(\omega_0, \omega_1, \cdots) = 0$, and

$$Y_{2^{n-1}+k}(\omega_0, \omega_1, \cdots) = I_{[k/2^{n-1},(k+1)/2^{n-1})}(\omega_{2^{n-1}+k})e_{2^{n-1}+k}\,,$$

where $n = 1, 2, \cdots; k = 0, 1, \cdots, 2^{n-1} - 1$, I_A is the indicator function of the set A and $(e_n, n \in \mathbb{N})$ is the standard basis in l^p.

By definition $(Y_n, n \in \mathbb{N})$ are independent, nonnegative and in $L^1(\mathscr{X})$. $\sup_{n\in\mathbb{N}} E||Y_n|| \leqq 1$ and $\sup_{n\in\mathbb{N}} ||\sum_{i=1}^{n} EY_i|| < \infty$ because $EY_0 = 0$, $EY_{2^{n-1}+k} = 2^{1-n}e_{2^{n-1}+k}$, $n = 1, 2, \cdots, k = 0, 1, \cdots, 2^{n-1} - 1$, so that

$$\sup_{n\in\mathbb{N}} ||\textstyle\sum_{i=0}^{n} EY_i|| = ||\sum_{i=0}^{\infty} EY_i|| = (\sum_{i=0}^{\infty} 2^{i(1-p)})^{1/p}\,.$$

At the same time, however, (Y_n) is divergent for each $\omega \in \Omega$ because for each $\omega \in \Omega$ there exist sequences (n_i), $(n_i') \subset \mathbb{N}$ such that $||Y_{n_i}(\omega)|| = 1$ and $||Y_{n_i'}(\omega)|| = 0$.

4. Convergence of submartingales. Doob's decomposition of a real submartingale remains true in a Banach lattice. Namely, if $(X_n, \mathscr{A}_n, n \in \mathbb{N})$ is a submartingale with values in a Banach lattice $\mathscr{X}$ then $X_n = M_n + A_n$, where $(M_n, \mathscr{A}, n \in \mathbb{N})$ is a martingale and the sequence $(A_n, n \in \mathbb{N})$ is predictable (i.e. $A_n \in L^1(\mathscr{A}_{n-1}; \mathscr{X})$) and such that $A_0 = 0$, $0 \leqq A_n \uparrow$ a.s. (the decomposition is unique). As in the real case, to prove the above statement it is sufficient to put $A_0 = 0$, $M_0 = X_0$,

$$M_n = X_0 + \sum_{i=1}^n [X_i - E^{\mathscr{A}_{i-1}} X_i]\,, \qquad A_n = \sum_{i=1}^n E^{\mathscr{A}_{i-1}}[X_i - X_{i-1}]\,, \qquad n \geqq 1\,.$$

The counterexamples of Section 3 show that, in general, the condition $\sup_{n \in \mathbb{N}} E||X_n^+|| < \infty$ for a submartingale $X_n = M_n + A_n$, does not imply its a.s. convergence. However, we do have

THEOREM 4.1. *For a separable Banach lattice the following three conditions are equivalent*:

(α) *for each $\mathscr{X}$-valued submartingale $(X_n = M_n + A_n, n \in \mathbb{N})$ satisfying the conditions $\sup_{n \in \mathbb{N}} E||X_n^+|| < \infty$ and $\sup_{n \in \mathbb{N}} E||M_n^-|| < \infty$ there exists an $X_\infty \in L^1(\mathscr{X})$ such that $X_n \to X_\infty$ a.s.*;

(β) *for each $\mathscr{X}$-valued submartingale $(X_n = M_n + A_n, n \in \mathbb{N})$ satisfying the conditions $\sup_{n \in \mathbb{N}} E||X_n^+||^p < \infty$ and $\sup_{n \in \mathbb{N}} E||M_n^-||^p < \infty$ for some $1 < p < \infty$ there exists an $X_\infty \in L^p(\mathscr{X})$ such that $X_n \to X_\infty$ in $L^p(\mathscr{X})$*;

(γ) *$\mathscr{X}$ has the Radon–Nikodym property.*

PROOF. $(\alpha) \Rightarrow (\gamma)$ $[(\beta) \Rightarrow (\gamma)]$. If $X_n = M_n$ then the conditions in (α) $[(\beta)]$ boil down to $\sup_{n \in \mathbb{N}} E||M_n|| < \infty$ $[\sup_{n \in \mathbb{N}} E||M_n||^p < \infty]$ and Theorem 1.1 gives the necessary implications.

$(\gamma) \Rightarrow (\alpha)$ $[(\gamma) \Rightarrow (\beta)]$. $X_n = M_n + A_n \geqq M_n$ a.s. so that $X_n^+ \geqq M_n^+$ a.s. and the monotonicity of the norm yields that

$$\sup_{n \in \mathbb{N}} E||M_n||^p \leqq 2^p(\sup_{n \in \mathbb{N}} E||X_n^+||^p + \sup_{n \in \mathbb{N}} E||M_n^-||^p)\,.$$

Hence, by Theorem 1.1 there exists $M_\infty \in L^1(\mathscr{X})$ $[M_\infty \in L^p(\mathscr{X})]$ such that $M_n \to M_\infty$ a.s. $[M_n \to M_\infty$ in $L^p(\mathscr{X})]$. Because $A_n = X_n - M_n$, we have that $A_n \leqq X_n^+ + M_n^-$ so that $\sup_{n \in \mathbb{N}} E||A_n||^p < \infty$, $1 \leqq p < \infty$. Utilizing again the monotonicity of the norm and the Lebesgue monotone convergence theorem we get that $E(\sup_{n \in \mathbb{N}} ||A_n||^p) < \infty$ so that $\sup_{n \in \mathbb{N}} ||A_n|| < \infty$ a.s. However, because the Radon–Nikodym property $\mathscr{X}$ does not contain isomorphic copies of c_0 as subspaces (cf., e.g., Davis (1973–1974)), and because in every Banach lattice with the latter property, norm bounded monotone sequences are convergent (Tzafriri (1972), Theorem 14), there exists a random vector A_∞ such that $A_n \to A_\infty$ a.s. Fatou's lemma yields that $E||A_\infty||^p \leqq \liminf_{n \in \mathbb{N}} E||A_n||^p \leqq \sup_{n \in \mathbb{N}} E||A_n||^p < \infty$ so that $A_\infty \in L^p(\mathscr{X})$, $1 \leqq p < \infty$, and letting $X_\infty = M_\infty + A_\infty$ concludes the proof.

In the next theorem we make weaker assumptions about the submartingale $(X_n, n \in \mathbb{N})$, namely that $(E(X_n^+), n \in \mathbb{N})$ is order bounded, but the convergence takes place only for a transformed submartingale.

THEOREM 4.2. *Let $\mathscr{X}$ be a separable Banach lattice, $\mathscr{Y}$ a separable Banach lattice with the Radon–Nikodym property, and $T: \mathscr{X} \to \mathscr{Y}$ a linear bounded positive operator such that its transpose $T^*: \mathscr{Y}^* \to \mathscr{X}^*$ is lattice bounded. If $(X_n, n \in \mathbb{N})$ is a submartingale with values in $\mathscr{X}$ such that $(E(X_n^+), n \in \mathbb{N})$ is order bounded then there exists a $Y_\infty \in L^1(\mathscr{Y})$ such that the submartingale $TX_n \to Y_\infty$* a.s.

PROOF. Let $X_n = M_n + A_n$ as before, and let $E(X_n^+) \leqq x_0 \in \mathscr{X}_+$ for all $n \in \mathbb{N}$. We show that under the above assumptions, $\sup_{n\in\mathbb{N}} E\|(TX_n)^+\| < \infty$ and $\sup_{n\in\mathbb{N}} E\|(TM_n)^-\| < \infty$; which in view of Theorem 4.1 would give the desired result because Doob's decomposition for the submartingale TX_n is $TM_n + TA_n$. Indeed,

$$\begin{aligned}\sup_{n\in\mathbb{N}} E\|(TX_n)^+\| &\leqq \sup_{n\in\mathbb{N}} E\|T(X_n^+)\| \\ &= \sup_{n\in\mathbb{N}} E(\sup_{\|y^*\|\leqq 1, 0\leqq y^*\in\mathscr{Y}^*} y^*T(X_n^+)) \\ &= \sup_{n\in\mathbb{N}} E(\sup_{\|y^*\|\leqq 1, 0\leqq y^*\in\mathscr{Y}^*} (T^*y^*)X_n^+) .\end{aligned}$$

However, the transpose T^* of a positive T is also positive, and thus the fact that T^* is lattice bounded implies the existence of an $x_0^* \in \mathscr{X}_+^*$ such that for each y^* with $\|y^*\| \leqq 1$, $|T^*y^*| \leqq x_0$. Thus we get that

$$\sup_{n\in\mathbb{N}} E\|(TX_n)^+\| \leqq E(x_0^* X_n^+) = x_0^* E(X_n^+) \leqq x_0^* x_0 < \infty .$$

Proceeding as above and utilizing the inequality

$$E(TM_n)^- = E(TM_n)^+ - E(TM_n) = E(TM_n)^+ - E(TM_0) \leqq E(TX_n)^+ - E(TM_0)$$

we get that

$$\sup_{n\in\mathbb{N}} E\|(TM_n)^-\| \leqq x_0^* x_0 + |x_0^* E(TM_0)| < \infty$$

because $M_0 \in L^1(\mathscr{X})$. This ends the proof.

Because l^1 is the Banach lattice with the Radon–Nikodym property (cf., e.g., Davis (1973–1974)) and because the operator $[Id(l^1, l^1)]^*$ is lattice bounded in l^∞ we obtain the following

COROLLARY 4.1. *If $(X_n, n \in \mathbb{N})$ is a submartingale with values in l^1 such that $(E(X_n^+), n \in \mathbb{N})$ is order bounded then there exists an $X_\infty \in L^1(l^1)$ such that $X_n \to X_\infty$* a.s.

REFERENCES

[1] CHATTERJI, S. D. (1968). Martingale convergence and the Radon-Nikodym theorem in Banach spaces. *Math. Scand.* **22** 21–41.

[2] DAVIS, W. J. (1973–1974). The Radon-Nikodym property. Séminaire Maurey-Schwartz, Espaces L^p, applications radonifiantes et géomètrie des espaces de Banach. Ecole Polytechnique, Paris. 0.1–0.12.

[3] GARLING, D. J. H. (1976). Lattice bounding, radonifying and summing mappings. *Proc. Cambridge Philos. Soc.* **77** 327–333.

[4] KANTOROVICH, L. V., VULIKH, B. Z. and PINSKER, A. G. (1950). Functional analysis in semiordered spaces (in Russian). Gosudarstv. Izdat. Tehn.-Teor. Lit. Moscow.

[5] NEVEU, J. (1972). *Martingales à Temps Discret.* Masson et Cie, Paris.

[6] NIELSEN, N. J. (1973). On Banach ideals determined by Banach lattices and their applications. Dissertationes Math. **109** 1–66.

[7] SCHWARTZ, L. (1973). Surmartingales régulières à valeurs mesures et désintégrations régulières d'une mesure. *J. Analyse Math.* **26** 1–168.

[8] TZAFRIRI, L. (1972). Reflexivity in Banach lattices and their subspaces. *J. Functional Anal.* **10** 1–18.

WROCLAW UNIVERSITY AND THE
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
GRUNWALDZKI 2/4
50384 WROCLAW, POLAND