The Annals of Probability
1976, Vol. 4, No. 3, 456-463

FLUCTUATIONS OF SEQUENCES WHICH CONVERGE IN DISTRIBUTION

BY HOLGER ROOTZÉN

University of Lund and University of North Carolina

A sequence $\{Y_n\}_{n=1}^{\infty}$ of random variables with values in a metric space is mixing with limiting distribution G if $P(\{Y_n \in A\} \mid B) \to G(A)$ for all G-continuity sets A and all events B that have positive probability. It is shown that if $\{Y_n\}$ is mixing with limiting distribution G and if the support of G is separable, then the range $\{Y_n(\omega); n \geqq 1\}$ is dense in the support of G almost surely. A theorem that, under rather general conditions, establishes mixing for the summation processes based on a martingale is given, and as an application it is shown that, under certain conditions, the range of the periodogram is dense in R^+ almost surely.

1. Introduction. Distributional limit theorems for sequences of random variables (rv's) in a metric space are usually based on some kind of asymptotic independence, and thus one would expect that a typical realization of such a sequence would fluctuate rather wildy. In the present paper this is made precise in the following way. Let $\{Y_n\}$ be a sequence of rv's on $(\Omega, \mathscr{B}, P)$ that converges in distribution to the distribution G (e.g., $\{Y_n\}$ are processes based on normed sums or normed maxima of weakly dependent random variables). One measure of the fluctuations is the "size" of the range $\{Y_n(\omega); n \geqq 1\}$. If it is assumed that the support of G is separable then according to Theorem 2.2 below, *if* $\{Y_n\}$ *is mixing with limiting distribution* G,[1] *then the range of* $\{Y_n(\omega)\}$ *is dense in the support of* G *for almost all* ω, and thus $Y_n(\omega)$ fluctuates strongly. We note that the hypothesis of mixing is satisfied in many cases of interest (Rootzén (1974)), and that a subsequence of a mixing sequence also is mixing. Further it is interesting that although "mixing with limiting distribution function G" looks like a distributional limit property, it actually implies a strong law.

The behavior of the realizations of a sequence of random variables is of special importance in connection with time series analysis, and as an example the above result is extended to the periodogram, which leads to a theorem that includes as special cases results earlier obtained by e.g., Grenander (1951) and Olshen (1967). Here the tool is Theorem 2.4 which establishes mixing for the summation process based on a martingale, for a large class of martingales.

The main theorems are given in Section 2, and in Section 3 they are applied to the periodogram. All proofs are postponed to Section 4.

2. Main theorems. Let $\{Y_n\}_{n=1}^{\infty}$ be a sequence of measurable mappings from

Received June 3, 1975; revised September 22, 1975.

[1] See Section 2 for a definition (Definition 2.1).

AMS 1970 *subject classifications*. 60F05, 60G17, 60G35.

Key words and phrases. Fluctuations, convergence in distribution, mixing in the sense of Rényi, martingales, periodogram.

$(\Omega, \mathscr{B}, P)$ to a metric space (S, ρ) with its Borel σ-algebra $\mathscr{S}$, i.e., the Y_n's are rv's in $(S, \mathscr{S})$. The sequence $\{Y_n\}_1^\infty$ converges in distribution to a probability distribution G on $(S, \mathscr{S})$ (notation: $Y_n \to_d G$) if $P(Y_n \in A) \to G(A)$ for all G-continuity sets $A \in S$.

DEFINITION 2.1. The sequence $\{Y_n\}_1^\infty$ is mixing with limiting distribution G if

$$P(\{Y_n \in A\} \mid B) \to G(A)$$

for every G-continuity set $A \in S$ and every $B \in \mathscr{B}$ with $P(B) > 0$.

This important concept is due to Rényi and it is known that e.g., sequences of normed sums or normed maxima which converge in distribution also are mixing under rather general conditions.

THEOREM 2.2. *If the sequence $\{Y_n\}_1^\infty$ of* rv'*s is mixing with limiting distribution G and if the support of G is separable, then the range $\{Y_n(\omega); n \geqq 1\}$ is dense in the support of G, for ω outside a null set.*[2]

Now consider the special case where $\{Y_n\}$ is a sequence of real random variables. If we note that subsequences of mixing sequences are mixing and apply Theorem 2.2 to the countably many subsequences $\{Y_n, Y_{n+1}, \cdots\}$ for $n = 1, 2, \cdots$ then the corollary below easily follows. (It is also a direct consequence of Theorem 1 of Fischler (1967).)

COROLLARY 2.3. *If the sequence $\{Y_n\}$ is mixing with limiting distribution function G then almost surely*

$$\limsup_{n\to\infty} Y_n \geqq \sup(\operatorname{supp} G)\,,$$
$$\liminf_{n\to\infty} Y_n \leqq \inf(\operatorname{supp} G)\,.$$

Kesten (1970) has rather extensively studied the range of normed sums of independent, identically distributed rv's. Let us see what Theorem 2.2 can give in that context. Firstly let $Y_n = (X_1 + \cdots + X_n)/n^{\frac{1}{2}}$ where the X_i's are independent and identically distributed with zero means and finite variances. Then $\{Y_n\}$ is mixing with limiting normal distribution function and thus according to Theorem 2.2 the range of $\{Y_n\}$ is dense in R almost surely, which also is one of Kesten's results. Secondly, let instead the X_i's belong to the domain of partial attraction of a stable law and consider the sums $X_1 + \cdots + X_n$, normed with the proper constants. Then Theorem 2.2 gives results not contained in Kesten's paper, and in particular throws some light on problem (2), page 1174, of that paper.

Of course, in many cases much more precise results about e.g., $\limsup_{n\to\infty} Y_n$ than those of Corollary 2.3 are known. It is however intersting to note that it is not possible to find asymptotic bounds for Y_n using only the hypothesis of Theorem 2.2.

Recently rather general central limit theorems for martingales have been

[2] We say that the set A is dense in the set B if $B \subset Cl(A)$.

proved by Brown (1971), Scott (1973) and McLeish (1974). The following theorem gives the mixing version of their functional limit theorem.

Let $\{S_n\}_1^\infty$ be a martingale on $(\Omega, \mathscr{B}, P)$ with $S_n = \sum_{k=1}^n X_k$ for $n \geqq 1$, $E(X_1) = 0$ and $E(S_n^2) = s_n^2 < \infty$ where $s_n^2 \uparrow \infty$ as $n \to \infty$.

Define for $n \geqq 1$ and $\omega \in \Omega$ random functions $Y_n(\cdot, \omega)$ on $[0, 1]$ by

$$Y_n(t, \omega) = s_n^{-1} S_k(\omega) \quad \text{for} \quad t \in [0, 1] \quad \text{such that} \quad s_k^2 \leqq s_n^2 t \leqq s_{k+1}^2 ,$$
$$k = 0, \cdots, n \quad (S_0 = s_0 = 0) .$$

Then $Y_n(\cdot, \omega)$ belongs to $D = D(0, 1)$, the space of real functions on $[0, 1]$ which are right continuous and have left-hand limits. Further let W denote Wiener measure on $D(0, 1)$. In the sequel we will use the metric d_0 on $D(0, 1)$ defined on page 112 of Billingsley (1968). Under d_0 the space $D(0, 1)$ is separable and complete.

THEOREM 2.4. *If $\{S_n\}_1^\infty$ is a martingale as above, the random elements $\{Y_n\}_1^\infty$ in $(D(0, 1), d_0)$ are defined by* (2.1) *and furthermore*

$$s_n^{-2} \sum_{k=1}^n X_k^2 \to_P 1 \qquad n \to \infty$$
$$s_n^{-2} \sum_{k=1}^n X_k^2 I(|X_k| > \varepsilon s_n) \to_P 0 \qquad n \to \infty$$

for all $\varepsilon > 0$ then $\{Y_n\}_1^\infty$ is mixing with limiting distribution W.

COROLLARY 2.5. *If $\{Y_n\}_1^\infty$ satisfies the conditions of Theorem* 2 *and N' is an infinite set of integers, then for ω outside a null set the set of functions $\{Y_n(\cdot, \omega); n \in N'\}$ is dense in the set of continuous functions on* $[0, 1]$ *that vanish at zero.*

One advantage of using mixing to prove strong limit theorems is that sequences that differ from a mixing sequence by a sequence tending to zero in probability are mixing as well; see Lemma 2.6 below. Thus for example if the normed sums of a stationary process can be approximated in probability by a normed martingale then the results of Theorem 2.4 and Corollary 2.5 carry over to the stationary process. The usual method of extending strong limit theorems is to make approxiximations that hold almost surely. (See e.g., Heyde (1973) for extensions of the law of the iterated logarithm.) Since convergence almost surely is a stronger requirement than convergence in probability it is necessary to impose more restrictive conditions when using that method than when using the method of the present paper.

LEMMA 2.6. *If $\{Y_n\}_{n=1}^\infty$ and $\{Y_n'\}_{n=1}^\infty$ are sequences of random variables in the metric space (S, ρ) such that $\{Y_n'\}$ is mixing with limiting distribution G and such that $\rho(Y_n, Y_n') \to_P 0$ then $\{Y_n\}$ is mixing with limiting distribution G.*

3. **An application: the periodogram.** The results of the previous section easily carry over to various quantities from time series analysis, e.g., those considered by Heyde (1973) and Hannan (1973), but here we only consider one example, the periodogram, which contains some new problems.

Let $\{X_n\}_{n=-\infty}^\infty$ be a strictly stationary sequence with zero mean and covariance function $r_n = E(X_t X_{t+n}) = \int_{-\pi}^{\pi} e^{i\lambda n} f(\lambda)\, d\lambda$, where f is the spectral density function. The periodogram

$$I_n(\lambda) = \frac{1}{2\pi} \sum_{l=-n}^{n} \left(\frac{1}{n} \sum_{k=1}^{n-|l|} X_k X_{k+|l|}\right) e^{i\lambda l} = \frac{1}{2\pi} |n^{-\frac{1}{2}} \sum_{k=1}^{n} X_k e^{i\lambda k}|^2$$

is a natural estimate of $f(\lambda)$ based on a sample $X_1, \cdots, X_n$ of length n and it is also asymptotically unbiased, but unfortunately its variance does not tend to zero if $f(\lambda) > 0$. Under certain conditions the periodogram is asymptotically χ^2-distributed, and conceivably it could also converge in some stronger sense than "in distribution." However, it has often been observed that the periodgram computed for different values of n does not settle down as n increases. A number of authors, e.g., Grenander (1951) and Olshen (1967), have given some explanations for this. The best result in this direction is that of Olshen (1967), who showed that if X_n is an (infinite) moving average of independent variables and $I_n(\lambda)$ converges in probability, then it converges in probability to zero. However, using the methods of the previous section, we will obtain stronger results under weaker conditions. The conditions (3.1) and (3.2) of the theorem below correspond to the conditions (iii) and (iv) of Theorem 1 in Hannan (1973).

Let $\mathscr{M}_n$ be the σ-algebra generated by the variables up to X_n, i.e., $\mathscr{M}_n = \sigma(\cdots, X_{n-1}, X_n)$, and put $\mathscr{M}_{-\infty} = \bigcap_n \mathscr{M}_n$. Let $x_n^k = E(X_{n+k} \| \mathscr{M}_k) - E(X_{n+k} \| \mathscr{M}_{k-1})$ be the part of X_{n+k} stemming from the innovation at time k. Furthermore write $||X||$ for $E(|X|^2)^{\frac{1}{2}}$ and let $\chi_f^2(\cdot)$ be the distribution of the χ^2-distribution with f degrees of freedom.

THEOREM 3.1. *Let $\{X_n\}$ be a strictly stationary sequence with zero means satisfying any of the following conditions*

$$\sum_{k=0}^{\infty} ||x_k^0|| < \infty \quad \textit{and} \quad \mathscr{M}_{-\infty} \textit{ is trivial,} \tag{3.1}$$

or

$$\begin{gathered} X_n = \sum_{i=-\infty}^{\infty} a_i \varepsilon_{n-i} \textit{ where } \{\varepsilon_n\}_{-\infty}^{\infty} \textit{ is a stationary sequence} \\ \textit{of martingale differences with } \bigcap_n \sigma(\cdots, \varepsilon_{n-1}, \varepsilon_n) \textit{ trivial,} \\ \textit{and } \{a_n\}_{-\infty}^{\infty} \textit{ are real numbers with } \sum_{i=-\infty}^{\infty} a_i^2 < \infty, \textit{ and} \\ \textit{furthermore } g(x) = \sum_{k=-\infty}^{\infty} a_k e^{ikx} \textit{ is continuous is some} \\ \textit{neighbourhood of } \lambda\,. \end{gathered} \tag{3.2}$$

Then $\{I_n(\lambda)\}_{n=1}^{\infty}$ is mixing with limiting distribution function $\chi_2^2(\cdot/(f(\lambda)/2))$ if $\lambda \neq 0, \pm\pi$ and limiting distribution function $\chi_1^2(\cdot/f(\lambda))$ if $\lambda = 0, \pm\pi$. Furthermore, if $f(\lambda) > 0$ then for any infinite subset N' of $\{1, 2, \cdots\}$ the range $\{I_n(\lambda); n \in N'\}$ is dense in R^+ almost surely.

The n-dimensional version is as follows.

THEOREM 3.2. *If $\{X_n\}$ satisfies* (3.1) *or* (3.2) *for $\lambda = \lambda_i$, $i = 1, \cdots, k$ and furthermore $\lambda_i \neq \pm\lambda_j$ for $i \neq j$ and $\lambda_i \neq 0, \pm\pi$ for $i = 1, \cdots, k$ then*

$$P(\bigcap_{i=1}^{k} \{I_n(\lambda_i) \leqq x_i\} \,|\, B) \to \prod_{i=1}^{k} \chi_2^2(x_i/(f(\lambda_i)/2)) \tag{3.3}$$

as $n \to \infty$ *for all* $(x_1, \cdots, x_k)$ *and* $B \in \mathscr{B}$ *such that* $P(B) > 0$. *If* $\lambda_i = 0, \pm\pi$ *for some* i *then the corresponding factor in the right-hand side of* (3.3) *changes to* $\chi_1^2(x_k/f(\lambda_i))$. *Furthermore, if* $f(\lambda_i) > 0$ *for* $i = 1, \cdots, k$ *then for any infinite subset* N' *of* $\{1, 2, \cdots\}$ *the range* $\{(I_n(\lambda_1), \cdots, I_n(\lambda_k)); n \in N'\}$ *is dense in* R_k^+ *almost surely.*

4. Proofs.

Proof of Theorem 2.2. Use the notation $B(x, y)$ for the open sphere in S with center x and radius y. The support of G (notation: supp G) is the set $\{s \in S; G(b(s, \varepsilon)) > 0, \forall \varepsilon > 0\}$. Choose points $\{r_i\}_1^\infty$ in S such that (i) $r_i \in \operatorname{supp} G$ for $i = 1, 2, \cdots$, (ii) $\{r_i\}$ is dense in supp G, and (iii) if x is an atom of G then $r_i = x$ for infinitely many i's. Choose positive numbers $\varepsilon_i \downarrow 0$ such that the sphere $B_i = B(r_i, \varepsilon_i)$ is a continuity set of G for each i. Let $A_i = \{\omega \mid Y_n \in B_i^*$ for all $n \geqq 1\}$. Then $P(\bigcup_{i=1}^\infty A_i) = 0$, since if not, $P(A_i) > 0$ for some i and we have by the definition of mixing that

$$P(Y_n \in B_i \mid A_i) \to G(B_i) > 0 \qquad n \to \infty$$

since $r_i \in \operatorname{supp} G$. This is a contradiction as $\{Y_n \in B_i\} \cap A_i = \emptyset$ for all n.

Let ω_0 be such that $\{Y_n(\omega_0); n \geqq 1\}$ is not dense in supp G. Then there exist a $\delta > 0$ and an $x \in \operatorname{supp} G$ such that the sphere $B(x, \delta)$ contains no point of $\{Y_n(\omega_0); n \geqq 1\}$. By (ii) and (iii) the sphere $B(x, \delta/2)$ contains infinitely many of the r_i's, and since $\varepsilon_i \downarrow 0$ there must be an i_0 such that $B_{i_0} \subset B(x, \delta)$. Since $Y_n(\omega_0) \in B(x, \delta)^* \subset B_{i_0}^*$ for $n = 1, 2, \cdots$ it follows that $\omega_0 \in A_{i_0}$. Thus

$$\{\omega \mid \{Y_n(\omega); n \geqq 1\} \text{ is not dense in } \operatorname{supp} G\} \subset \bigcup_{i=1}^\infty A_i,$$

which since $P(\bigcup_{i=1}^\infty A_i) = 0$ proves the theorem. □

Proof of Theorem 2.4. By using the Skorokhod representation as in Scott (1973) it can be shown that the conditional characteristic functions of linear combinations converge and by using Lemma A.1 of Rootzén (1974) and the Cramér–Wold theorem as in the proof below it follows that the finite-dimensional distributions are mixing. Since it is well known that $\{Y_n\}$ is tight, this proves the theorem. An alternative approach is given by McLeish (1974) who used a very nice technique to prove a central limit theorem for martingales and then also noted that his method easily can be applied to prove mixing. □

Proof of Theorem 3.1. First assume that (3.1) holds. Extending the ideas of Scott (1973) and Heyde (1973) we are going to approximate $\{X_t\}$ with a sequence of martingale differences $\{Y_t; \mathscr{M}_t\}$ where $Y_t = \sum_{u=0}^\infty x_u{}^t e^{i\lambda u}$ (which converges in mean square), by showing that $\|n^{-\frac{1}{2}} \sum_{t=1}^n X_t e^{i\lambda t} - n^{-\frac{1}{2}} \sum_{t=1}^n Y_t e^{i\lambda t}\| \to 0$ as $n \to \infty$. However, we have that

$$\begin{aligned} \sum_{t=1}^n Y_t e^{i\lambda t} &= \sum_{t=1}^n e^{i\lambda t} \sum_{u=0}^{t-1} x_u{}^{t-u} + \sum_{t=1}^n e^{i\lambda t} \sum_{u=n-t+1}^\infty x_u{}^t e^{i\lambda u} \\ &= \sum_{t=1}^n X_t e^{i\lambda t} - \sum_{t=1}^n E(X_t \,\|\, M_0) e^{i\lambda t} \qquad (4.4) \\ &\quad + \sum_{t=1}^n e^{i\lambda t} \sum_{u=n-t+1}^\infty x_u{}^t e^{i\lambda u}, \end{aligned}$$

where the first equality follows by changing the order of summation and the second by noting that

$$\sum_{u=0}^{t-1} x_u^{t-u} = E(X_t \,||\, \mathscr{M}_t) - E(X_t \,||\, \mathscr{M}_0) = X_t - E(X_t \,||\, \mathscr{M}_0)\,.$$

Since $E(X_t \,||\, \mathscr{M}_{-\infty}) = 0$ we have that $E(X_t \,||\, \mathscr{M}_0) = \sum_{u=0}^{\infty} x_{t+u}^{-u}$ and thus, using first that x_t^m and x_u^n are uncorrelated if $m \neq n$, then stationarity and finally (3.1), that

$$\begin{aligned} n^{-1}||\textstyle\sum_{t=1}^n E(X_t \,||\, \mathscr{M}_0)e^{i\lambda t}||^2 &= n^{-1} \textstyle\sum_{u=0}^{\infty} ||\sum_{t=1}^n x_{t+u}^{-u} e^{i\lambda t}||^2 \\ &\leqq n^{-1} \textstyle\sum_{u=0}^{\infty} (\sum_{t=1}^n ||x_{t+u}^0||)^2 \\ &\leqq (\textstyle\sum_{t=1}^{\infty} ||x_t^0||) n^{-1} \sum_{t=1}^n (\sum_{u=0}^{\infty} ||x_{t+u}^0||) \to 0 \end{aligned} \tag{4.5}$$

as $n \to \infty$. In a similar way

$$\begin{aligned} n^{-1}||\textstyle\sum_{t=1}^n e^{i\lambda t} \sum_{u=n-t+1}^{\infty} x_u^t e^{i\lambda u}||^2 &= n^{-1} \textstyle\sum_{t=1}^n ||\sum_{u=n-t+1}^{\infty} x_u^t e^{i\lambda u}||^2 \\ &\leqq n^{-1} \textstyle\sum_{t=1}^n (\sum_{u=n-t+1}^{\infty} ||x_u^0||)^2 \to 0 \end{aligned} \tag{4.6}$$

as $n \to \infty$. Together with (4.4)—(4.6) this implies that

$$||n^{-\frac{1}{2}} \textstyle\sum_{t=1}^n X_t^{i\lambda t} - n^{-\frac{1}{2}} \sum_{t=1}^n Y_t e^{i\lambda t}|| \to 0 \tag{4.7}$$

as $n \to \infty$. The hypothesis (3.1) implies that f is continuous and thus $n^{-1}||\sum_{t=1}^n X_t e^{i\lambda t}||^2 \to 2\pi f(\lambda)$ (see Olshen (1967), page 513) and furthermore $n^{-1}||\sum_{t=1}^n Y_t e^{i\lambda t}||^2 \to E(Y_0^2)$ as $n \to \infty$, and thus $E(Y_0^2) = 2\pi f(\lambda)$, which concludes the proof of the approximation.

The next step of the proof is to apply the Cramér-Wold technique to $n^{-\frac{1}{2}} \sum_{t=1}^n Y_t e^{i\lambda t}$. To avoid trivial complications assume that $\lambda \neq 0, \pm\pi$ and that $f(\lambda) > 0$. Let a and b be real numbers, not both zero. We are going to show that

$$n^{-1} \textstyle\sum_{t=1}^n Y_t^2(a \cos \lambda t + b \sin \lambda t)^2 \to (a^2 + b^2)\pi f(\lambda) \qquad n \to \infty \tag{4.8}$$

and

$$n^{-1} \textstyle\sum_{t=1}^n Y_t^2(a \cos \lambda t + b \sin \lambda t)^2 I(|Y_t(a \cos \lambda t + b \sin \lambda t)| > \varepsilon n^{\frac{1}{2}}) \to_P 0 \qquad n \to \infty\,, \ \forall\, \varepsilon > 0 \tag{4.9}$$

which by Theorem 2.4 implies that the sequence $\{an^{-\frac{1}{2}} \sum_{t=1}^n Y_t \cos \lambda t + bn^{-\frac{1}{2}} \sum_{t=1}^n Y_t \sin \lambda t\}$ is mixing with limiting distribution function $\Phi(\cdot/\{a^2\pi f(\lambda) + b^2\pi f(\lambda)\}^{\frac{1}{2}})$. However, putting $\theta = 2 \arctan(-b/a)$, (4.8) is seen to hold for arbitrary a and b if for $\theta \in [-\pi, \pi]$

$$n^{-1} \textstyle\sum_{t=1}^n (Y_t^2 - 2\pi f(\lambda)) + n^{-1} \sum_{t=1}^n (Y_t^2 - 2\pi f(\lambda)) \cos(2\lambda t + \theta) \to_P 0 \tag{4.10}$$

as $n \to \infty$. Since $\mathscr{M}_{-\infty}$ is trivial $\{X_n\}$ is ergodic, so the first term tends to zero a.s., and it only remains to prove that the second term tends to zero. Put $\zeta_t = Y_t^2 - 2\pi f(\lambda)$. Then ζ_t is integrable and $E\zeta_t = 0$ and thus for arbitrary $\varepsilon > 0$ it is possible to "truncate" so that $\zeta_t = \zeta_t' + \zeta_t''$ where $\{\zeta_t'\}$ is a bounded, ergodic and stationary sequence with zero mean while $E(|\zeta_t''|) < \varepsilon$. Then the second term in (4.10) equals

$$n^{-1} \textstyle\sum_{t=1}^n \zeta_t' \cos(2\lambda t + \theta) + n^{-1} \sum_{t=1}^n \zeta_t'' \cos(2\lambda t + \theta)\,. \tag{4.11}$$

Using the spectral representation and that $\mathscr{M}_{-\infty}$ is trivial, it is seen that $||n^{-1}\sum_{t=1}^{n}\zeta_t'\cos(2\lambda t+\theta)||\to 0$ as $n\to\infty$ and since $E|n^{-1}\sum_{t=1}^{n}\zeta_t''\cos(2\lambda t+\theta)|<\varepsilon$ this implies that the expression (4.11) tends to zero in probability and thus concludes the proof of (4.10). The proof of (4.13) is easy and is not given here.

This establishes that $\{an^{-\frac{1}{2}}\sum_{t=1}^{n}Y_t\cos\lambda t+bn^{-\frac{1}{2}}\sum_{t=1}^{n}Y_t\sin\lambda t\}$ is mixing, i.e., that

$$P(\{an^{-\frac{1}{2}}\sum_{t=1}^{n}Y_t\cos\lambda t+bn^{-\frac{1}{2}}\sum_{t=1}^{n}Y_t\sin\lambda t\leqq x\}\,|\,B)\to\Phi(x/\{a^2\pi f(\lambda)+b^2\pi f(\lambda)\}^{\frac{1}{2}})$$

as $n\to\infty$ for all $B\in\mathscr{B}$ with $P(B)>0$ and all real a and b. But by the Cramér–Wold theorem this proves that the 2-dimensional vector $((2\pi n)^{-\frac{1}{2}}\sum_{t=1}^{n}Y_t\cos\lambda t,$ $(2\pi n)^{-\frac{1}{2}}\sum_{t=1}^{n}Y_t\sin\lambda t)\to_d N(0,0,f(\lambda)/2,f(\lambda)/2,0)$ on the probability space $(\Omega,\mathscr{B},P(\cdot\,|\,B))$, and thus because of (4.7), that $((2\pi n)^{-\frac{1}{2}}\sum_{t=1}^{n}X_t\cos\lambda t,$ $(2\pi n)^{-\frac{1}{2}}\sum_{t=1}^{n}X_t\sin\lambda t)\to_d N(0,0,f(\lambda)/2,f(\lambda)/2,0)$ on $(\Omega,\mathscr{B},P(\cdot\,|\,B))$. This implies that

$$P(\{I_n(\lambda)\leqq x\}\,|\,B)=P\left(\left\{\frac{1}{2\pi n}(\textstyle\sum_{t=1}^{n}X_t\cos\lambda t)^2+\frac{1}{2\pi n}(\sum_{t=1}^{n}X_t\sin\lambda t)^2\leqq x\right\}\Big|\,B\right)$$
$$\to\chi_2^2(x/(f(\lambda)/2))$$

and thus proves the first part of the theorem, i.e., that $\{I_n(\lambda)\}$ is mixing with the stated limiting distribution function. The second part then follows immediately by Theorem 2.2.

To prove the theorem under the hypothesis (3.2) note that the Cesàro sums of $g(\lambda)$ converges if g is continuous in some neighborhood of λ and apply Lemma 1 of Olshen (1967) to conclude that

$$||n^{-\frac{1}{2}}\sum_{t=1}^{n}X_te^{i\lambda t}-n^{-\frac{1}{2}}g(\lambda)\sum_{t=1}^{n}\varepsilon_te^{i\lambda t}||\to 0\qquad n\to\infty\,.$$

Then complete the proof in the same way as above. □

Finally Theorem 3.2 follows in the same way as above by using the Cramér–Wold theorem in $2k$ dimensions.

REFERENCES

[1] BILLINGSLEY, P. (1968). *Convergence of Probability Measures.* Wiley, New York.

[2] BROWN, B. M. (1971). Martingale central limit theorems. *Ann. Math. Statist.* **42** 59–66.

[3] FISCHLER, R. M. (1967). Borel-Cantelli type theorem for mixing sets. *Acta. Math. Acad. Sci. Hungar.* **18** 67–69.

[4] GRENANDER, U. (1951). On empirical spectral analysis of stochastic processes. *Ark. Mat.* **1** 503–531.

[5] HANNAN, E. J. (1973). Central limit theorems for time series regression. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **26** 157–170.

[6] HEYDE, C. C. (1973). Limit theorems for stationary processes via approximating martingales. Communication at the *Third Conference of Stochastic Processes and their Applications.* Sheffield, England.

[7] KESTEN, H. (1970). The limit points of a normalized random walk. *Ann. Math. Statist.* **41** 1173–1205.

[8] McLeish, D. L. (1974). Dependent central limit theorems and invariance principles. *Ann. Probability* **2** 620–628.

[9] Olshen, R. A. (1967). Asymptotic properties of the periodogram of a discrete stationary process. *J. Appl. Probability* **4** 508–528.

[10] Rényi, A. (1958). On mixing sequences of sets. *Acta Math. Acad. Sci. Hungar.* **9** 215–228.

[11] Rootzén, H. (1973). Some properties of convergence in distribution of sums and maxima. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **29** 295–307.

[12] Scott, D. J. (1973). Central limit theorems for martingales and for processes with stationary increments using a Skorokhod representation approach. *Advances in Appl. Probability* **5** 119–137.

[13] Strassen, V. (1967). Almost sure behaviour of sums of independent random variables. *Proc. Fifth Berkeley Symp. Math. Statist. Prob.* **2** 315–343.

Dept. of Mathematical Statistics
University of Lund
Box 725
S-22007 Lund, Sweden