The Annals of Probability
1976, Vol. 4, No. 3, 453–455

ON THE RECURRENCE PROPERTY OF GAUSSIAN TAYLOR SERIES[1]

BY J. S. HWANG

Université de Montréal

We prove that a Gaussian Taylor series has the recurrence property on any rectifiable Jordan domain such that the intersection of its boundary with the unit circle is of positive measure.

Let $\{Z_n\}$ be a sequence of complex random variables which are independent normally distributed with expection zero and unit variance. As was introduced by J. P. Kahane ([2], page 125), a power series is called a Gaussian Taylor series if it can be written as

$$F(z) = \sum_0^\infty a_n Z_n z^n\,, \tag{1}$$

where $a_n > 0$, $\limsup_{n\to\infty} a_n^{1/n} = 1$, and z is a complex variable. Since $Z_n = O(\log n)^{\frac{1}{2}}$ a.s. (almost surely) ([2], page 121, Proposition 3), it follows that a.s. (1) admits the unit circle $C = \{z : |z| = 1\}$ as a natural boundary ([2], page 32, Theorem 1).

Let D be the unit disk and let E be a subset of D. We say that F has the recurrence property on E if we have a.s. $\liminf |F(z) - w| = 0$, as $|z| \to 1$, $z \in E$, for each complex number w. With the help of this definition, a theorem of Zygmund ([2], page 127, Theorem 1) can be stated as follows:

THEOREM A. *Let F be defined by* (1); *if* $\sum_0^\infty a_n^2 = \infty$, *then F has the recurrence property on the whole disk, i.e.* $E = D$.

Naturally, we may ask on what kind of subset of D, F has the recurrence property? Kahane has answered this question under some additional conditions on the coefficients a_n ([2], Theorems 2, page 127, and 6, page 132], namely,

THEOREM B. *If* $\sum_0^\infty a_n^2 = \infty$ *and* $a_n = O(1/n^{\frac{1}{2}})$, *then F has the recurrence property on any radius.*

THEOREM C. *If* $\sum_0^\infty a_n^2 = \infty$ *and* $a_n = o(n/\log n)^{\frac{1}{2}}$, *then F has the recurrence property on any circular set* (*i.e. a union of circles tending to* C).

Without those additional restrictions on a_n, in our present note, we shall prove the following generalization of Zygmund's theorem.

THEOREM. *If* $\sum_0^\infty a_n^2 = \infty$, *then F has the recurrence property on any subregion R of D, bounded by a rectifiable Jordan curve J such that the measure* $|J \cap C| > 0$.

Received March 20, 1975; revised June 26, 1975.

[1] I am indebted to Professors A. Joffe and R. Duncan for their support.

AMS 1970 *subject classifications.* Primary 60G15; Secondary 60G50.

Key words and phrases. Recurrence property, Gaussian Taylor series.

PROOF. The proof of the above result depends mainly on the following six well-known theorems: Fatou ([1], Theorem 2.1), Carathéodory ([1], Theorem 3.2), Riesz ([1], Theorem 3.3), Nevanlinna ([1], page 41 or [3], page 204), Lindelöf ([1], Theorem 2.3), and Paley–Zygmund ([2], page 45, Theorem 1).

Let $z = z(w)$ be a conformal mapping from $D_w = \{w : |w| < 1\}$ onto R and let $G(w) = F(z(w))$. Clearly, F has the recurrence property on R if and only if G has the recurrence property on D_w. Suppose on the contrary that G has no recurrence property on D_w, then there exists a disk $D(a)$ with center at a such that with positive probability the range of G has no common point with $D(a)$. It follows that with a positive probability, the function $H(w) = 1/(G(w) - a)$ is bounded and therefore by Fatou's theorem $H(w)$ as well as $G(w)$ has a radial limit along almost every radius.

According to the theorems of Carathéodory and Riesz, the mapping $z(w)$ can be extended to be homeomorphic on the boundary and the derivative $z'(e^{i\theta})$ exists almost everywhere. This allows us to define the following two sets:

Let A be the set of all points p of $J \cap C$ such that $G(w)$ has a radial limit along the radius ending at $w(p)$, where $w(z)$ is the inverse of $z(w)$.

Let B be the set of all points p of $J \cap C$ such that the inverse $w(z)$ is conformal at p from the interior of R.

Clearly, by what we have proved, we can see that $|A| = |J \cap C|$. It remains to prove that $|B| = |J \cap C|$. To do this, we first observe that the Riesz theorem ([1], pages 50–52) has shown that the derivative $z'(w)$ belongs to the Hardy class H^1 and the boundary function $z(e^{i\theta})$ is absolutely continuous. By virtue of Nevanlinna's theorem, we find that $z'(w)$ has angular limits almost everywhere on the unit circle C_w in w-plane. Let w_0 be a point on C_w for which the angular limit $z'(w_0)$ exists and is different from zero. We shall prove that $z(w)$ is conformal at w_0 from the interior of C_w. Let $w_1(t)$ and $w_2(t)$ be two analytic arcs lying in the interior of C_w and ending at w_0 such that

$$\lim_{t\to 0} w_j(t) = w_0 \qquad \text{and} \qquad \lim_{t\to 0} w_j'(t) = w_j' \neq 0\,, \qquad j = 1, 2\,.$$

Let α be the angle subtended by these two arcs at the point w_0, then we have

$$\cos\alpha = \lim_{t\to 0} (w_1'(t), w_2'(t))/|w_1'(t)w_2'(t)| = (w_1', w_2')/|w_1'w_2'|\,,$$

where (a, b) is the inner product.

Let β be the corresponding angle subtended by their images $z(w_1(t))$ and $z(w_2(t))$ at the point $z(w_0)$. Then by applying the following relation,

$$\lim_{t\to 0} z'(w_1(t)) = \lim_{t\to 0} z'(w_2(t)) = z'(w_0) \neq 0\,,$$

we conclude that

$$\begin{aligned}\cos\beta &= \lim_{t\to 0} (z'(w_1(t))w_1'(t), z'(w_2(t))w_2'(t))/|z'(w_1(t))w_1'(t)z'(w_2(t))w_2'(t)| \\ &= (z'(w_0)w_1', z'(w_0)w_2')/|z'(w_0)w_1'z'(w_0)w_2'| = \cos\alpha\,.\end{aligned}$$

This shows that $z(w)$ is conformal at w_0 from the interior of C_w.

On the other hand, from the absolute continuity of $z(e^{i\theta})$, we can see that the image of a set of measure zero is of measure zero. It follows that the set of all points p for which the inverse $w(z)$ is not conformal at p from the interior of R is of measure zero. We thus establish the equality $|B| = |J \cap C|$ and therefore $|A \cap B| = |J \cap C|$.

We now consider a point $e^{i\theta} \in A \cap B$. Let $r(\theta)$ be a local piece of the radius $\{re^{i\theta} : 1 - \delta \leqq r < 1\}$ for small enough $\delta > 0$. Then by the definition of B, the image $w(r(\theta))$ lies within a triangle in D_w with one vertex at $w(e^{i\theta})$. From the definition of A, the function G as well as H has a radial limit along the radius ending at $w(e^{i\theta})$. Owing to the Lindelöf theorem, we can see both of G and H have angular limits at $w(e^{i\theta})$. It follows that F has a radial limit along $r(\theta)$. This shows that with a positive probability, the function F has a radial limit along almost every radius ending at $J \cap C$. Since $\sum_0^\infty a_n^2 = \infty$ and $|J \cap C| > 0$, this contradicts the theorem of Paley–Zygmund. The proof is complete.

Acknowledgment. I am indebted to Professor Zygmund for many valuable discussions in this work when he visited in Montreal, and I wish to show my thanks to the referee for many valuable comments and corrections.

REFERENCES

[1] COLLINGWOOD, E. F. and LOHWATER, A. J. (1966). *The Theory of Cluster Sets.* Cambridge Univ. Press.

[2] KAHANE, J. P. (1968). Some random series of functions, D.C. Heath, Lexington, Mass.

[3] NEVANLINNA, R. (1970). *Analytic Functions.* Springer-Verlag, New York.

CENTRE DE RECHERCHES MATHÉMATIQUES
UNIVERSITÉ DE MONTRÉAL
C.P. 6128, MONTRÉAL 101
QUÉBEC, CANADA