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RECURRENCE OF STATIONARY SEQUENCES[1]

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Let $\{X_n\}_{-\infty}^{+\infty}$ be a stationary sequence of random variables, with common distribution $\pi(dx)$. If the initial value X_0 is repeated with probability one (e.g. when $\pi(dx)$ is discrete), then the "shifted" sequence $\{X_{n+N}\}_{-\infty}^{\infty}$ is also stationary where $N = N(\omega)$ is the first $n > 0$ for which $X_n(\omega) = X_0(\omega)$. Surprisingly, this may even occur when $\pi(dx)$ is continuous and $\{X_n\}$ is ergodic (although not when $\{X_n\}$ is ϕ-mixing). For Markov sequences, we also give other conditions which prohibit the a.s. recurrence of X_0.

For recurrent sequences, we show that when X_0 is "conditionally discrete," the invariant σ-field for the $\{X_{n+N}\}$ process coincides (up to null sets) with $X_0 \vee \mathscr{A}$, the σ-field generated by X_0 and the invariant sets for $\{X_n\}$. Finally, we find an expression for $E(N \mid X_0 \vee \mathscr{A})$ which reduces to Kac's recurrence formula when X_0 is an indicator function.

0. Introduction. Suppose $\{X_n\}$, $u \in \mathbb{Z}$ (the integers), is a stationary random sequence, and let N be the first time $n \geqq 1$ that $X_n = X_0$ or $N = \infty$ if there is no such n. If the sequence is independent and identically distributed (i.i.d.), then it is easy to see that the finiteness of N is completely determined by the common distribution π as follows: if $X_0 = x$ is an atom of π, then $N < \infty$; otherwise $N = \infty$ (all with the exception of a set of probability zero).

The purpose of this paper is to discuss the return time N for an arbitrary stationary sequence; specifically we allow π to be continuous. We begin with some more precise definitions. Let X be a random variable on a probability space $(\Omega, \mathscr{F}, P)$, and $T : \Omega \to \Omega$ an automorphism, i.e. a bijective, bimeasurable, measure-preserving transformation. Our primary interest is in real-valued random variables, but we will allow X to take its values in a measurable state space $(E, \mathscr{E})$, with $\mathscr{E}$ *separable* (see Example 1 below). The distribution of X is denoted by $\pi : \pi(\Gamma) = P(X \in \Gamma)$, $\Gamma \in \mathscr{E}$. Now, for $n \in \mathbb{Z}$, let $X_n = X \circ T^n$ ($T^0 =$ identity), noting that every stationary sequence may be realized in this manner. The *return time* N is defined by

$$N(\omega) = \min\{n \geqq 1 : X_n(\omega) = X_0(\omega)\} \tag{1}$$

with the usual convention that $N(\omega) = \infty$ if the set in (1) is empty. The transformation $T^N : \Omega \to \Omega$ is defined as $T^N(\omega) = T^{N(\omega)}(\omega)$ when $N(\omega) < \infty$, $T^N(\omega) = \omega$ otherwise. It follows from Neveu [6] (also see [3]) that, when $N < \infty$ a.s., T^N preserves P-measure. We note that the discreteness of π is sufficient to yield $N < \infty$ a.s., by the ergodic theorem, but it is not necessary.

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Similar questions for (dependent) stationary sequences with π discrete were first studied by Kac [4] and by Ryll-Nardzewski [9] (for point processes) whereas Breiman [1] considers the first time the sequence returns to a Borel set B of positive π-measure given that it started in B—this amounts to taking the sequence $I_B(X_n)$.

When $N < \infty$ a.s. and T is ergodic, it is known [3] that

$$E(N \mid X) = 1/\pi(\{X\}) . \tag{2}$$

An immediate consequence is that π is discrete iff $E(N \mid X) < \infty$ a.s.; another is Kac's formula for the mean return time to a set of positive measure. Next, since T^N preserves measure, one may study its ergodic structure. Because X itself is a T^N-invariant function, T^N is never ergodic, except in the trivial case when X is constant. Indeed, when X(or π) is discrete, and T is ergodic, the T^N-invariant σ-field in Ω, denoted $\mathscr{A}_N$, coincides (up to null sets) with $\sigma(X)$, the σ-field generated by X.

In view of formula (2), the question was raised in [3] of whether it was even *possible* to have T ergodic, N a.s. finite, and π *continuous*. We begin Section 1 with some examples answering this affirmatively, and then show that for processes satisfying certain mixing conditions, and for various types of Markov processes, it is impossible to have N a.s. finite and π continuous. Section 2 contains some generalizations of the results discussed above. Specifically, equation (2) is extended to the case in which T need not be ergodic nor π discrete. Finally, we discuss the structure of $\mathscr{A}_N$ with no restriction on T; for stationary Markov processes, there are no further requirements on X either, but in general we must still assume that X is "conditionally discrete" as explained below.

1. Recurrence, mixing, and the Markov case. First, we give two examples which show the possibility of having T ergodic, $N < \infty$ a.s., and π continuous. If the ergodicity requirement is dropped, it is trivial to construct examples, e.g., using a rational rotation on the circle; on the other hand, if the σ-field $\mathscr{E}$ in the state space is not separable, one may easily give pathological examples satisfying all three conditions. (We would like to thank John Walsh for the remark on separability and for a conversation concerning Theorem 5(b) below.) The state space in our example will be the unit circle with its usual Borel σ-field. Notice that (2) necessitates $E(N \mid X) = \infty$ a.s.

EXAMPLE 1. Let X_0 be uniformly distributed on the unit circle, and θ be a number such that $\theta/2\pi$ is irrational. Next let $\{Y_n\}$ be an i.i.d. sequence with $P\{Y_n = +\theta\} = P\{Y_n = -\theta\} = \frac{1}{2}$, and $S_n = Y_1 + Y_2 + \cdots + Y_n (S_0 \equiv 0)$ the corresponding random walk. Now define $X_n = X_0 e^{iS_n}$ for $n \geqq 0$. This is a stationary Markov process having transition function $P(e^{ix}, \Gamma) = \frac{1}{2} I_\Gamma(e^{i(x+\theta)}) + \frac{1}{2} I_\Gamma(e^{i(x-\theta)})$, where Γ is a circular Borel set. Using the customary function space representation, one may construct an automorphism T as described in Section 0, but we shall not pursue this point. Clearly π is continuous and, because the random walk S_n is recurrent, N is a.s. finite. We defer the proof of ergodicity for the moment—see below.

EXAMPLE 2. Let $\Omega = \{0, 1, 2\}^{\mathbb{N}}$, where $\mathbb{N} = \{1, 2, 3, \cdots\}$ and let P be the product measure on Ω, each coordinate measure being uniform over $\{0, 1, 2\}$. Let $\Omega_0 \subseteqq \Omega$ be the set of sequences which contain infinitely many zeros, ones and twos; clearly $P(\Omega_0) = 1$. Define T on Ω_0 as follows:

$$T(\omega_1, \omega_2, \cdots, \omega_k, \omega_{k+1}, \cdots) = (0, 0, \cdots, 0, \omega_k + 1, \omega_{k+1}, \cdots),$$

where k is the first integer for which $\omega_k \neq 2$. The transformation T (called the "adding machine" and well-known to ergodic theorists) is measure-preserving and ergodic.

Finally, let $X(\omega) = \sum_{k=1}^{\infty} \delta(0, \omega_k)2^{-k}$, where δ is the Kronecker symbol. This clearly has a continuous distribution and one checks that $N < \infty$ on Ω_0.

(We learned of this example through an associate editor of this *Annals*.)

We will now show that, in various situations, it is impossible to have N a.s. finite and π continuous. Recall that a stationary sequence $\{X_n\}$ is *ϕ-mixing* if, for $A \in \mathscr{F}_n$ and $B \in \mathscr{F}'_{n+k}$,

$$|P(B \cap A) - P(A)P(B)| \leqq \phi(k)P(A)$$

where $\phi(k) \to 0$ as $k \to \infty$; here $\mathscr{F}_n$ is the σ-field generated by all X_m, $m \leqq n$, and $\mathscr{F}'_{n+k}$ is that generated by all X_m, $m \geqq n + k + 1$.

(3) THEOREM. *If $\{X_n\}$ is ϕ-mixing and X_0 has a continuous distribution, then $P\{N < \infty\} < 1$.*

PROOF. One can check that there is no loss of generality in assuming X_0 is uniformly distributed on $[0, 1]$. Let

$$A_k = \{X_n = X_0 \text{ for some } n,\ n_k < n \leqq n_{k+1}\},$$

where the sequence n_k is chosen so that $\sum \phi(n_k) < \infty$. If $P(N < \infty) = 1$, it follows (from the fact that T^N preserves measure) that $P(A_k \text{ i.o.}) = 1$ ("i.o." means "infinitely often").

Let $\Gamma_i = [(i-1)/m, i/m)$, $1 \leqq i \leqq m$. We have, for m fixed,

$$\begin{aligned} P(A_k) &\leqq \textstyle\sum_{j=1}^{m} P\{X_0 \in \Gamma_j, X_n \in \Gamma_j \text{ for some } n,\ n_k < n \leqq n_{k+1}\} \\ &\leqq \textstyle\sum_{j=1}^{m} [P\{X_0 \in \Gamma_j\}P\{X_n \in \Gamma_j \text{ for some } n,\ n_k < n \leqq n_{k+1}\} \\ &\qquad + \phi(n_k)P\{X_0 \in \Gamma_j\}] \\ &\leqq \frac{1}{m} \textstyle\sum_{j=1}^{m} \frac{n_{k+1} - n_k}{m} + \phi(n_k) = \frac{n_{k+1} - n_k}{m} + \phi(n_k). \end{aligned}$$

Letting $m \to \infty$, we find $P(A_k) \leqq \phi(n_k)$, hence (Borel–Cantelli) $P(A_k \text{ i.o.}) = 0$, which contradicts $P(N < \infty) = 1$. □

In light of (3) it is interesting that the process of Example 1 satisfies the following mixing condition:

$$\text{(4)} \qquad \sup_{A \in \mathscr{F}'_n, B \in \mathscr{F}} |P(A \cap B) - P(A)P(B)| \to 0 \qquad \text{as } n \to \infty.$$

It is shown in [7] that (4) is equivalent to the triviality of $\mathscr{T}$, the tail σ-field of the process, i.e. $P(A) = 0$ or 1 for every $A \in \mathscr{T}$, and so implies ergodicity. We will prove that $\mathscr{T}$ is trivial in the appendix to Section 1.

If $\{X_n\}$ is an i.i.d. sequence with a continuous common distribution, the return time N is a.s. infinite. In contrast, a ϕ-mixing sequence may have $N < \infty$ on a set of probability arbitrarily close to 1. For example, let $0 < \alpha < 1$, and let μ be a continuous distribution on [0, 1]. Define a transition function of [0, 1] by $P(x, \Gamma) = \alpha I_\Gamma(x) + (1 - \alpha)\mu(\Gamma)$. The measure μ is invariant for this transition function, and one may verify that the corresponding stationary Markov process $\{X_n\}$ is ϕ-mixing, with $\phi(k) = \alpha^k$, and $P\{N < \infty\} = P\{N = 1\} = \alpha$.

Suppose now that $\{X_n\}$ is a stationary Markov process with state space $(E, \mathscr{E})$. We denote the transition function by $P(x, \Gamma)$, $x \in E$, $\Gamma \in \mathscr{E}$, and the initial (stationary) distribution by π. Our terminology for Markov process follows [7].

(5) THEOREM. *Under any of the conditions below, if* $P_\pi(N < \infty) = 1$, *then* π *is purely discrete*:

(a) *the Harris recurrence condition*;
(b) *the state space is indecomposable*;
(c) *the process is* π*-nonsingular* [7];
(d) $|P^n(x, \Gamma) - \pi(\Gamma)| \leqq a_n$, *where* $\sum a_n < \infty$.

PROOF. (a) First note that $P_\pi(N < \infty) = 1$ implies $P_\pi(X_n = X_0 \text{ i.o.}) = 1$ which implies $P_x\{X_n = x \text{ i.o.}\} = 1$ for π-a.e. x. Fix such an x, and define $f(y) = P_y\{X_n = x \text{ i.o.}\}$. This function is harmonic, hence ([7], page 22) constant by Harris recurrence. Since $f(x) = 1$, we have $f(y) \equiv 1$, and so conclude $P_\pi\{X_n = x \text{ i.o.}\} = 1$ for π-a.e. x. But then $\infty = \sum_{n=1}^\infty P_\pi\{X_n = x\} = \sum_1^\infty \pi(\{x\})$ for such x, and so $\pi(\{x\}) > 0$ for π-a.e. x, i.e. π is purely discrete.

(b) As in (a), $P_x\{X_n = x \text{ i.o.}\} = 1$ π-a.e. For any such x, $P_y\{X_n = x$ for some $n \geqq 1\} > 0$ for every $y \in E$ by ([7], page 36), and so $P_\pi\{X_n = x \text{ i.o.}\} > 0$. One now shows $\pi(\{x\}) > 0$ as in the proof of (a).

(c) Fix x and let $f(y)$ be as in the proof of (a), so $f(x) = 1$. By nonsingularity there is an n such that $P^n(x, dy)$ is not purely singular. Since $f(x) = P^n f(x) = 1$, it follows that $f(y) > 0$ on a set of positive π-measure, whence $P_\pi\{X_n = x \text{ i.o.}\} > 0$. The proof is now completed as above. Notice that we only need the nonsingularity condition to hold for *almost* every x (cf. [7], page 42).

(d) Let Δ be the complement of the set of atoms of π. Then

$$\begin{aligned} P_\pi(X_n = X_0, X_0 \in \Delta) &= \int_\Delta P_x\{X_n = x\}\pi(dx) \\ &= \int_\Delta P^n(x, \{x\})\pi(dx) \\ &= \int_\Delta (\pi(\{x\}) + a_n)\pi(dx) \leqq a_n . \end{aligned}$$

By Borel–Cantelli, $P_\pi\{X_n = X_0 \text{ i.o.}, X_0 \in \Delta\} = 0$; if $P_\pi\{N < \infty\} = 1$, we must have $P_\pi\{X_0 \in \Delta\} = \pi(\Delta) = 0$. □

From (5) it follows that the Markov process in Example 1 cannot satisfy

Harris recurrence, indecomposability, or the exponential convergence condition. One verifies the first two of these directly by considering the orbits $\{e^{i(x+k\theta)}: k \in \mathbb{Z}\}$. The last is more difficult in view of the mixing condition (4).

Appendix. Let $\{X_n\}$, $n \geqq 0$, be a Markov process with state space $(E, \mathscr{E})$, initial distribution π, and transition function $P(x, \Gamma)$. The usual operator on bounded, $\mathscr{E}$-measurable functions f is given by

$$Pf(x) = \int P(x, dy)f(y) .$$

A sequence of functions g_n is called *space-time harmonic* if $Pg_{n+1} = g_n$ for every n. An easy adaptation of an argument in [7] proves: a necessary and sufficient condition for the tail σ-field $\mathscr{T}$ to be trivial relative to P_π is that every uniformly bounded space-time harmonic sequence $g_n(x)$ be constant π-a.e., i.e. for some constant c, $g_n(x) = c$ for every n and π-a.e. x.

Suppose $g_n(e^{ix})$ is a uniformly bounded space-time harmonic sequence for the process $\{X_n\}$ of Example 1; the defining relation becomes

$$g_n(e^{ix}) = \tfrac{1}{2}[g_{n+1}(e^{i(x+\theta)}) + g_{n+1}(e^{i(x-\theta)})] . \tag{6}$$

Write $c_m{}^n$ for the mth Fourier coefficient of g_n. Expanding (6) in Fourier series and equating coefficients (all of which is easily justified) we find $c_m{}^n = c_m{}^{n+1} \cos(m\theta)$ or $c_m{}^n = c_m/(\cos(m\theta))^{n-1}$, where $c_m = c_m{}^1$. Note, since $\theta/2\pi$ is irrational, $0 < |\cos(m\theta)| < 1$, for all $m \neq 0$. Thus, if $c_m \neq 0$, we have $|c_m{}^n| \to \infty$ $(n \to \infty)$.

Now g_n is real-valued, so $c_{-m}^n = c_m{}^n$. Let $M > 0$ be the first integer for which $c_m \neq 0$, if one exists. The Mth partial sum of the Fourier series for g_n is then $c_0 + 2c_M \cos(Mx)/(\cos(M\theta))^{n-1}$, and the Cesaro average of the first $M + 1$ partial sums equals $c_0 + 2c_M \cos(Mx)/(M + 1)(\cos(M\theta))^{n-1}$. It is well known that if a function is bounded in modulus by a constant, each of the corresponding Cesaro averages is bounded by the *same* constant. This is clearly impossible in the present situation unless $c_m = 0$ for every $m \neq 0$, i.e. $g_n(x) = c_0$ for every n and π-a.e. x.

2. Structure of the σ-field $\mathscr{A}_N$. In this section, we extend equation (2) and also indicate the structure of $\mathscr{A}_N$, the σ-field in $\mathscr{F}$ which is invariant under T^N. (These topics turn out to be closely related). In addition to the assumptions of Section 0, we now take $N < \infty$ a.s.

Let $\mathscr{G}$ be a sub σ-field of $\mathscr{F}$ and recall that a regular conditional probability (r.c.p.) given $\mathscr{G}$ is a Markov kernel $Q(\omega, A)$ which is a version of $P(A \mid \mathscr{G})$, i.e. $Q(\cdot, A)$ is $\mathscr{G}$-measurable for each $A \in \mathscr{F}$, $Q(\omega, \cdot)$ is a probability measure on $\mathscr{F}$ for each $\omega \in \Omega$, and $\int_G Q(\omega, A)P(d\omega) = P(AG)$ for each $A \in \mathscr{F}$, $G \in \mathscr{G}$. Since we will need the existence of r.c.p.'s for various sub σ-fields, we now impose the requirements that $\mathscr{F}$ be *separable* (i.e. generated by a countable subfamily) and that there is a compact subfamily C of $\mathscr{F}$ relative to which P has the "inner approximation property"

$$P(A) = \sup\{P(B): A \supset B \in C\} , \qquad A \in \mathscr{F} . \tag{7}$$

This implies the existence of an r.c.p. for *any* sub σ-field $\mathscr{G}$ of $\mathscr{F}$, as is fully explained in [5]; we will make no other use of (7).

Next we must lay the groundwork for the ergodic decomposition of T which will be used in reducing all problems to the ergodic case. An *atom* of a σ-field $\mathscr{G}$ is defined as an equivalence class for the relation $\omega \sim \omega'$ on Ω determined by $I_A(\omega) = I_A(\omega')$ for every $A \in \mathscr{G}$. It is well known that (i) every $\mathscr{G}$-measurable function is constant on the atoms of $\mathscr{G}$, and (ii) if $\mathscr{G}$ is separable, then its atoms are $\mathscr{G}$-measurable.

(8) LEMMA. *Let $Q(\omega, A)$ be an* r.c.p. *given a separable sub σ-field $\mathscr{G}$ of $\mathscr{F}$. Then*:

(a) *for almost every $\omega \in \Omega$, $Q(\omega, A) = I_A(\omega)$ for all $A \in \mathscr{G}$*;

(b) *if f is a $\mathscr{G}$-measurable function, then, for almost every $\omega \in \Omega$, f is equal to the constant $f(\omega)$ $Q(\omega, \cdot)$-a.s.*

PROOF. (a) Let $\mathscr{H}$ be a countable field which generates $\mathscr{G}$. Then there exists a P-null set N such that $Q(\omega, H) = I_H(\omega)$ for every $H \in \mathscr{H}$, $\omega \notin N$; this relation extends immediately to $\mathscr{G}$ since two measures agreeing on $\mathscr{H}$ must also agree on $\mathscr{G}$.

(b) Immediate, since $Q(\omega, \cdot)$ is concentrated on that atom of $\mathscr{G}$ which contains ω (by (a)), for almost every ω. ▯

Now we may discuss the ergodic decomposition of T. Proofs may be found in [8]. We write $\mathscr{A}$ for the σ-field in $\mathscr{F}$ of sets A which are strictly invariant under T: $T^{-1}A = A$. Finally, $\bar{\mathscr{F}}$ will denote the P-completion of $\mathscr{F}$ while $\bar{\mathscr{A}}$ is the augmentation of $\mathscr{A}$ *in* $\bar{\mathscr{F}}$, i.e. $\bar{\mathscr{A}}$ is the σ-field generated by $\mathscr{A}$ and P-null sets in $\bar{\mathscr{F}}$ (similar notation will be used for other sub σ-fields of $\mathscr{F}$).

(9) THEOREM. *There exists a separable σ-field $\mathscr{A}' \subseteq \mathscr{A}$ such that*

(a) $\bar{\mathscr{A}}' = \bar{\mathscr{A}}$;

(b) *$Q(\omega, A) = I_A(\omega)$ for all $A \in \mathscr{A}'$, and P-a.e. $\omega \in \Omega$, where Q is an* r.c.p. *given* $\mathscr{A}'$;

(c) *If $\mathscr{C}$ is any separable sub σ-field of $\mathscr{A}$ satisfying* (a) *and* (b), *and Q' an* r.c.p. *given $\mathscr{C}$, then, except for ω on a P-null set*

(i) *$Q'(\omega, \cdot)$ is preserved by T*

(ii) *$Q'(\omega, \cdot)$ is ergodic relative to T*, i.e. *$Q'(\omega, A) = 0$ or 1 for every $A \in \mathscr{A}$.*

Notice that Q, Q' above are also r.c.p.'s given $\mathscr{A}$.

Let now X be a random variable on $(\Omega, \mathscr{F}, P)$, and assume the return time N is a.s. finite. Without loss of generality we may take $Q(\omega, \cdot)$ to be ergodic for *every* $\omega \in \Omega$ where Q is as in (9). In what follows, we write $F(\omega, dx)$ for the conditional distribution of X given $\mathscr{A}'$, i.e. $F(\omega, dx) = Q(\omega, \{X \in dx\})$, and $X \vee \mathscr{A}$ for the σ-field generated by X and $\mathscr{A}$.

(10) THEOREM. $E(N \mid X \vee \mathscr{A})(\omega) = 1/F(\omega, \{X(\omega)\})$.

PROOF. When T is ergodic ($\mathscr{A}$ trivial), this result is Lemma (22) of [3], which we apply to the system $(\Omega, \mathscr{F}, Q(\omega', \cdot), T)$ for a fixed $\omega' \in \Omega$, obtaining for $Q(\omega', \cdot)$-a.e. ω,

$$E_{\omega'}(N \mid X)(\omega) = 1/F(\omega', \{X(\omega)\}) .$$

Here and below $E_{\omega'}$ indicates integration by the measure $Q(\omega', \cdot)$. We will show that, for ω' outside a P-null set,

$$1^\circ \qquad F(\omega', \{X(\omega)\}) = F(\omega, \{X(\omega)\}) \quad Q(\omega', \cdot)\text{-a.s.}$$
$$2^\circ \qquad E_{\omega'}(N \mid X)(\omega) = E(N \mid X \vee \mathscr{A}')(\omega) \quad Q(\omega', \cdot)\text{-a.s.},$$

and these together provide the result.

For every x, $F(\omega', \{x\})$ is an $\mathscr{A}'$-measurable function of ω', hence $F(\omega', \{x\}) = F(\omega, \{x\})$ for every $\omega \in A(\omega')$, where $A(\omega')$ denotes the atom of $\mathscr{A}'$ which contains ω'. Hence $F(\omega', \{X(\omega)\}) = F(\omega, \{X(\omega)\})$ for every $\omega \in A(\omega')$, and, since $Q(\omega', A(\omega')) = 1$, we have 1°.

To prove 2°, it suffices to show that each member has the same $E_{\omega'}$-integral over sets of the form $\{X \in \Gamma\} \cap A$, where $\Gamma \in \mathscr{E}$ and $A \in \mathscr{A}'$. The left member gives

$$\int_{\{X \in \Gamma\} \cap A} E_{\omega'}(N \mid X)(\omega) Q(\omega', d\omega) = I_A(\omega') E_{\omega'}(N I_\Gamma(X)) ,$$

while, on the right, we find $I_A(\omega') E_{\omega'}(I_\Gamma(X) E(N \mid X \vee \mathscr{A}'))$. Now, to see that these are equal for almost all ω', we note that both are $\mathscr{A}'$-measurable and have the same expectation for every $A \in \mathscr{A}'$. □

REMARK. As for EN itself, it can be shown that for discrete X, $EN = \sum_{x \in \mathbf{R}} P(B_x) \leqq \infty$ where $B_x = \bigcup_{n \in \mathbf{Z}} \{X_n = x\}$. Call X "conditionally discrete" if $F(\omega, dx)$ is a discrete distribution for almost every $\omega \in \Omega$. For such X, it follows at once from the invariance of the B_x's that

$$EN = E(\textstyle\sum_{x \in \mathbf{R}} I_{B_x}(\omega)) ,$$

the mean cardinality of the "range" $\{X_n(\omega)\}$. (Or take expectations of both sides of (10).) Of course, the order of integration cannot, in general, be reversed. In the ergodic case, EN is just the number of atoms of $P(X \in dx)$.

We now describe the σ-field $\mathscr{A}_N$ of sets $A \in \mathscr{F}$ such that $(T^N)^{-1}A = A$.

(11) THEOREM. *For any random variable* X, *the following are equivalent*:

(a) $\mathscr{A}_N = \overline{\sigma(X)}$;
(b) (P_x, T^N) *is ergodic for* π-a.e. x.

If, in addition, X *is discrete, then* (a) *and* (b) *are equivalent to each of*

(c) $\mathscr{A} \subseteq \overline{\sigma(X)}$;
(d) $(P(\cdot \mid B), T)$ *is ergodic for all* x *for which* $P(X = x) > 0$.

PROOF. To prove (a) $\Rightarrow$ (b), apply Theorem (9c) to the system $(\Omega, \mathscr{F}, P, T^N)$. Suppose (b) holds for $A \in \mathscr{A}_N$; let $\Gamma = \{x : P_x(A) = 1\}$, noting that $P_x(A) = 0$

for $x \notin \Gamma$, except for a π-null set. We then have, for $B \in \mathscr{F}$,

$$P(A \cap B) = \int P_x(A \cap B)\pi(dx) = \int_\Gamma P_x(B)\pi(dx) = P(B, x \in \Gamma) ,$$

and hence $A = (X \in \Gamma)$ a.s., i.e. the symmetric difference has measure zero.

Now suppose that X is discrete. We will show that (c) $\Rightarrow$ (d) and (d) $\Rightarrow$ (b); since (a) $\Rightarrow$ (c) is trivial, we will then be done. (c) $\Rightarrow$ (d). Choose $A \in \mathscr{A}$ and x such that $P(X = x) > 0$. Since $\mathscr{A} \subseteq \overline{\sigma(X)}$, there exists a Γ such that $A = (X \in \Gamma)$ a.s. Now, if $x \in \Gamma$, $B_x \subseteq \bigcup_n (X_n \in \Gamma) = (X \in \Gamma)$ a.s.; if $x \notin \Gamma$, $B_x \subseteq \bigcup_n (X_n \in \Gamma^c) = (\bigcap_n (X_n \in \Gamma))^c = (X \in \Gamma^c)$ a.s. In either case, $P(A \mid B_x) = I_\Gamma(x)$ and (d) is proven. (d) $\Rightarrow$ (b). First, for any $\mathscr{F}$-measurable $Z \geqq 0$, if $P(X = x) > 0$,

$$(*) \qquad E(Z \mid B_x) = \frac{1}{E_x N} E_x \textstyle\sum_{k=0}^{n-1} Z \circ T^k .$$

To see this, write $E(Z; B_x) = \sum_{n=0}^{\infty} E(Z; X_{-n} = x, X_{-n+i} \neq x, i = 1, \cdots, n)$ and apply T^n in the nth term. The rest of the proof can be done along the lines suggested by Ryll-Nardzewski for the "point process" case (i.e. $X = I_\Gamma$), that is, using (*) and the characterization of ergodic measures as extreme points. We omit the details, although some care must be taken since, in the point process case, $B_0 = B_1 = \Omega$ and the B_x's play no role.

(12) THEOREM. *If X is conditionally discrete, then* $\overline{\mathscr{A}}_N = \overline{X \vee \mathscr{A}}$.

NOTE. We will give an example below in which the conclusion of (12) holds, but with X not conditionally discrete, and also one with X conditionally discrete but not discrete. A general proof of (12) without any condition on X still eludes us; the following shows that it would suffice to prove (12) for ergodic systems.

PROOF. In the ergodic case "conditionally discrete" is just "discrete," so use Theorem 11, (c) $\Rightarrow$ (a). In general, for X conditionally discrete apply the result in the ergodic case to each system $(\Omega, \mathscr{F}, Q(\omega', \cdot), T)$: if Y is $\mathscr{A}_N$-measurable, it will be measurable in the $Q(\omega', \cdot)$-completion of $\sigma(X)$ for each ω', i.e., $Y = E_{\omega'}(Y \mid X)$, $Q(\omega', \cdot)$-a.s. Putting Y in place of N in 2° in the proof of (10), we conclude $Y = E(Y \mid X \vee \mathscr{A}')$ $Q(\omega', \cdot)$-a.s., and this for a.e. ω'. So Y is $\overline{X \vee \mathscr{A}'}$-measurable. Since $X \vee \mathscr{A} \subseteq \mathscr{A}_N$, the proof is concluded.

(13) COROLLARY. *If* $E(N) < \infty$, $\overline{\mathscr{A}}_N = \overline{X \vee \mathscr{A}}$.

In fact, from the proof of (10) it is clear that if only the *conditional* expectation of N given X is finite, X is conditionally discrete, so that (12) applies.

REMARKS. (a) It can be shown that $X \vee \mathscr{A}$ and $\mathscr{A}_N$ have the same atoms whenever "points" $\{\omega\}$ are in $\mathscr{F}$. If, for example, $(\Omega, \mathscr{F})$ were a Blackwell space and these σ-fields were *separable*, one could then conclude they were equal, but $\mathscr{A}$ and $\mathscr{A}_N$ are typically *not* separable.

(b) Theorem (11) allows us to prove that $\overline{\mathscr{A}}_N = \overline{\sigma(X)}$ for any stationary

Markov process with $N < \infty$ a.s., in particular the process in Example 1. To see this, fix an x such that $P_x(X_n = x \text{ i.o.}) = 1$. Writing τ_k^x for the kth return to x (so $\tau_k^x = (T^N)^k$ P_x-a.s.), it is well known (see, e.g. [1], page 140) that the random ($\mathbb{R}^\infty$-valued) vectors $Z_k = (X_{\tau_k^x}, \cdots, X_{\tau_{k+1}^x - 1})$, $k \geqq 1$, are i.i.d. under P_x. However, a moment's reflection shows that any T^N-invariant random variable (for the process) is measurable over the tail σ-field of the $\{Z_k\}$ process, and hence constant P_x-a.s. Note that $\overline{\mathscr{A}_N} = \overline{\sigma(X)}$ is equivalent to $\overline{X \vee \mathscr{A}} = \bar{\mathscr{A}}_N$ since we always have $\mathscr{A} \subseteqq \overline{\sigma(X)}$—see [2], page 460.

EXAMPLE 3. We can "code" an integer-valued stationary sequence into one with the *same* recurrence structure but with a continuous initial distribution. For example, given independent, stationary sequences $\{Z_n\}$ and $\{Y_n\}$, the former $\mathbb{Z}$-valued and the latter with $P(Y_1 \in dx)$ continuous, consider the "coded" stationary sequence $\{Y_{Z_n(\omega)}(\omega)\}$, which repeats its initial value a.s. and has the same initial distribution as Y_0.

To pursue such examples, it will be convenient to take $\Omega = \prod_{-\infty}^{\infty} (\mathbb{R} \times \mathbb{Z})$ with the usual product σ-field. For $\omega = (\cdots, n_{-1}; y_0, n_0, y_1, n_1, \cdots) \in \Omega$, define $Z(\omega) = n_0$, $Y_n(\omega) = y_n$, and $X(\omega) = y_{n_0}$; moreover, define a bijective, bimeasurable transformation $T: \Omega \to \Omega$ by $T(\omega) = (\cdots, n_0; y_0, n_1, y_1, n_2, \cdots)$. In this way, $X \circ T^n = Y_{Z \circ T^n}$ for all n and $N \equiv \min(n \geqq 1 : X \circ T^n = X) \leqq \min(n \geqq 1 : Z \circ T^n = Z)$. For the probability P, we choose any one preserved by T and for which each of the Y_n's is independent of Z and has a continuous law. In particular, X then has a continuous law. (For instance, choose $P = \prod_{-\infty}^{\infty} (\mu \times \nu)$ where μ, ν are probabilities on $\mathbb{R}$, $\mathbb{Z}$ respectively and μ is continuous.)

Now clearly $N < \infty$ a.s. and notice that X, though not discrete, is "conditionally discrete." Indeed, $F(\omega, B) \equiv \sum_{n \in \mathbb{Z}} P(Z = n \mid \mathscr{A})(\omega) I_B(Y_n(\omega))$ is clearly a probability measure on $\mathscr{B}$ for each ω, $\mathscr{A}$-measurable (since the Y_n's are) for each $B \in \mathscr{B}$ and for any $A \in \mathscr{A}$, $B \in \mathscr{B}$: $E(F(\omega, B); A) = \sum_{n \in \mathbb{Z}} (Z = n, Y_n \in B, A) = P(X \in B, A)$. Thus $\overline{X \vee \mathscr{A}} = \bar{\mathscr{A}}_N$. Such systems, of course, are never ergodic due to the invariance of the Y_n's.

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