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# ON A RANDOM WALK BETWEEN A REFLECTING AND AN ABSORBING BARRIER

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The author shows that the formula given by B. Weesakul [3] for the absorption probability in a random walk between a reflecting and an absorbing barrier does not hold true in general.

The correct formula in the general case is given.

This paper refers to the problem considered by B. Weesakul [3]. In earlier papers, D. Fürst [1], [2] considered the same problem from a different point of view, by approximating the continuous case by the discrete. A comparison of these results leads to the necessity of a correction in [3], as recognized by Dr. Weesakul in a personal communication. Using the same notation and formula numbering as in [3], we note that:

The term, $\theta q\delta_{1,u}$, must be added to formula (5), where $\delta_{1,u}$ is the Kronecker delta. This missing term does not affect the calculation of $g(t/u)$ since (15) remains of the same form, namely

$$\varphi(\theta \mid u) = \frac{V(\theta)\theta_q \delta_{1,u} + U(\theta)}{V(\theta)} \equiv \frac{U_1(\theta)}{V(\theta)} . \tag{15}$$

In (17) we need $U_1(\theta_\nu)$, but this is equal to $U(\theta_\nu)$, $\theta_\nu$ being a root of $V(\theta)$. The estimation of the determinant $|D|$ as prescribed by Weesakul does not work for $u = 1$; in this case it is easy to see that the correct value for $|D|$ needed in (6) is

$$|D| = \theta_{pq} \frac{\lambda_1^{b-1} - \lambda_2^{b-1} - \theta p(\lambda_1^{b-2} - \lambda_2^{b-2})}{\lambda_1 - \lambda_2} .$$

Following (18) it is stated in [3] that $q^{\frac{1}{2}} \sin (b + 1)\alpha - p^{\frac{1}{2}} \sin b\alpha$ has $b$ distinct roots. A study of the function

$$f(\alpha) = [\sin (b + 1)\alpha]/\sin b\alpha$$

shows however that this is not always the case. Clearly the roots of $f(\alpha) = (p/q)^{\frac{1}{2}}$ that give distinct roots of $V(\delta)$ are in $[0, \pi)$ and if $(p/q)^{\frac{1}{2}} < 1 + 1/b$ there are $b$ distinct roots of this equation, so that formula (18) in Weesakul is correct in this case.

If $(p/q)^{\frac{1}{2}} > 1 + 1/b$ there are only $b - 1$ distinct roots $\alpha_\nu$ ($\nu = 2, \cdots, b$) that give distinct roots $\theta_\nu$ of $V(\theta)$. The remaining root of $V(\theta)$ is given by $\theta_1 = (2(pq)^{\frac{1}{2}} \cosh \alpha_1)^{-1}$, where $\alpha_1$ is the unique root of the equation $(p/q)^{\frac{1}{2}} = (\sinh (b + 1)\alpha)/\sinh b\alpha$.

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Actually because of the dependence of the roots of $V(\theta)$ it is easy to see that

$$\theta_1 = (-1)^m 2^{b-1} \frac{\prod_{\nu=2}^{b} \cos \alpha_\nu}{(p/q)^{(2m-b)/2}(pq)^{\frac{1}{2}}}$$

where $m = [(b+1)/2]$ (integer part of $(b+1)/2$).

The correct formula for $g(t/u)$ in the general case is then:

$$g(t \mid u) = 2^t p^{\frac{1}{2}(t-u)} q^{\frac{1}{2}(t+u)} \Big\{ \beta(\alpha_1) + \sum_{\nu=2}^{b} \cos^{t-1} \alpha_\nu \frac{[q^{\frac{1}{2}} \sin (b-u+1)\alpha_\nu - p^{\frac{1}{2}} \sin (b-u)\alpha_\nu] \sin \alpha_\nu}{[(b+1)q^{\frac{1}{2}} \cos (b+1)\alpha_\nu - bp^{\frac{1}{2}} \cos b\alpha_\nu]} \Big\}$$

where

$$\beta(\alpha_1) = -\cos^{t-1} \alpha_1 \frac{[q^{\frac{1}{2}} \sin (b-u+1)\alpha_1 - p^{\frac{1}{2}} \sin (b-u)\alpha_1] \sin \alpha_1}{[(b+1)q^{\frac{1}{2}} \cos (b+1)\alpha_1 - bp^{\frac{1}{2}} \cos b\alpha_1]} \quad \text{if } \left(\frac{p}{q}\right)^{\frac{1}{2}} < 1 + \frac{1}{b}$$

$$= \frac{2u}{b(b+1)(2b+1)} \quad \text{if } \left(\frac{p}{q}\right)^{\frac{1}{2}} = 1 + \frac{1}{b}$$

$$= \cosh^{t-1} \alpha_1 \frac{[q^{\frac{1}{2}} \sinh (b-u+1)\alpha_1 - p^{\frac{1}{2}} \sinh (b-u)\alpha_1] \sinh \alpha_1}{[(b+1)q^{\frac{1}{2}} \cosh (b+1)\alpha_1 - bp^{\frac{1}{2}} \cosh b\alpha_1]} \quad \text{if } \left(\frac{p}{q}\right)^{\frac{1}{2}} > 1 + \frac{1}{b}$$

and

$$\alpha_1 = \cos^{-1} \frac{1}{2(pq)^{\frac{1}{2}}\theta_1} \quad \text{if } \left(\frac{p}{q}\right)^{\frac{1}{2}} \leqq 1 + \frac{1}{b}$$

$$= \cosh^{-1} \frac{1}{2(pq)^{\frac{1}{2}}\theta_1} \quad \text{if } \left(\frac{p}{q}\right)^{\frac{1}{2}} > 1 + \frac{1}{b}.$$

$\theta_1$ is the smallest root in absolute value of $V(\theta)$.

## REFERENCES

[1] FÜRST, D. (1956). La rovina dei giocatori nel caso di riserva limitata. *Giorn. Ist. Ital. Attuari.* 63–83.

[2] FÜRST, D. (1957). Il caso limite del problema della rovina dei giocatori nell'ipotesi di riserva limitata. *Giorn. Ist. Ital. Attuari.* 120–143.

[3] WEESAKUL, B. (1961). The random walk between a reflecting and an absorbing barrier. *Ann. Math. Statist.* **32** 765–769.

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