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# AN EXAMPLE CONCERNING CLT AND LIL IN BANACH SPACE[1]

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Let $E$ be a separable Banach space with norm $||\bullet||$. Let $\{X_n\}$ be a sequence of $E$-valued independent, identically distributed random variables, and $S_n = X_1 + \cdots + X_n$. If $\{n^{-\frac{1}{2}}S_n\}$ converges in the sense of weak convergence of the corresponding measures in $E$, and $E$ is the real line, then it is well known that $\mathscr{E}[X_1] = 0$ and $\mathscr{E}[||X_1||^2] < \infty$; consequently, the Hartman–Wintner law of the iterated logarithm also holds. We give an example here, with $E = C[0, 1]$, such that the above convergence does *not* imply $\mathscr{E}[||X_1||^2] < \infty$, nor does it imply the law of the iterated logarithm.

**1. Introduction.** Let $E$, $\{X_n\}$ and $S_n$ be as above. For convenience we will say that $X_1$ satisfies the central limit theorem (CLT) if $\{n^{-\frac{1}{2}}S_n\}$ converges in the sense of weak convergence of the corresponding measures; the limit measure must necessarily be Gaussian (possibly degenerate). If the space $E$ satisfies the property that $\exists\, \alpha > 0$ such that $\forall\, x_1, x_2, \cdots, x_n \in E$; $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n$ independent Rademacher random variables (i.e., $\varepsilon_j = 1$ or $-1$ with probability $\frac{1}{2}$), $\mathscr{E}[||\varepsilon_1 x_1 + \cdots + \varepsilon_n x_n||^2] \geqq \alpha \sum_{j=1}^n ||x_j||^2$, then it is shown in [1] that CLT $\Rightarrow$ $\mathscr{E}[X_1] = 0$ and $\mathscr{E}[||X_1||^2] < \infty$, where $\mathscr{E}[X_1]$ is taken in the Bochner sense. It is also shown in [1] that in general CLT $\Rightarrow P[||X_1|| > \lambda] = O(\lambda^{-2})$. We give an example below to show that in general CLT $\not\Rightarrow \mathscr{E}[||X_1||^2] < \infty$.

In the real-valued case CLT and the Hartman–Wintner law of the iterated logarithm (LIL) are equivalent. The following formulation of the LIL in $E$ is due to Kuelbs [3]. An $E$-valued random variable $X$ is said to satisfy the LIL if for $X_1, X_2, \cdots$ independent copies of $X$ we have a limit set $K \subset E$ such that

$$(1.1) \qquad P\left\{\omega : \lim_n d\left(\frac{S_n(\omega)}{a_n}, K\right) = 0\right\} = 1$$

and

$$(1.2) \qquad P\left\{\omega : C\left(\left\{\frac{S_n(\omega)}{a_n}, n \geqq 1\right\}\right) = K\right\} = 1,$$

where $a_n = (2n \log\log n)^{\frac{1}{2}}$, $d(x, A) = \inf_{y \in A} ||x - y||$, and $C(\{x_n, n \geqq 1\})$ = set ot strong limit points of the sequence $\{x_n, n \geqq 1\}$ in $E$. We will show that our example does not satisfy such a LIL even though it satisfies the CLT. It should also be remarked that Kuelbs [3] has shown that LIL $\not\Rightarrow$ CLT.

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**2. The example.** Let $\xi_1, \xi_2, \cdots$ be independent, identically distributed, *real-valued*, *symmetric* random variables such that

$$P[|\xi_1| > \lambda] = \frac{c}{\lambda^2(\log \lambda)^2}, \quad \lambda \geqq 2, \qquad = 1, \quad 0 \leqq \lambda \leqq 2, \tag{2.1}$$

so that $c = 4(\log 2)^2$. Let $\mathscr{E}[\xi_1^2] = \alpha$, which is finite. Let $\{\varphi_j, j \geqq 1\}$ be a sequence of *nonnegative* functions in $C[0, 1]$ (with the sup norm) such that

$$\varphi_j(t)^2 = 0 \quad \text{for} \quad t \notin (2^{-j-1}, 2^{-j})$$
$$= j^{-1} \quad \text{for} \quad t = 3 \cdot 2^{-j-2}, \quad \text{and}$$
linear in between.

Define

$$X(t) = \sum_{j=1}^{\infty} \xi_j \varphi_j(t), \qquad t \in [0, 1], \tag{2.2}$$

which is well defined since the $\varphi_j$'s have disjoint supports. We claim that if $X_1, X_2, \cdots$ are taken to be independent copies of $X$, then this sequence constitutes our example. The following lemmas prove this claim.

LEMMA 2.1. *The series* $\sum_{j=1}^{\infty} \xi_j \varphi_j$ *converges in norm in* $C[0, 1]$ a.s.

PROOF. Since the $\varphi_j$'s have disjoint supports, it suffices to check that (see Example 4.3 in [2])

$$\sum_j P[|\xi_j| > a||\varphi_j||^{-1}] < \infty, \qquad \forall a > 0. \tag{2.3}$$

Since $||\varphi_j|| = j^{-\frac{1}{2}}$, by (2.1) we have

$$P[|\xi_j| > a||\varphi_j||^{-1}] \sim \frac{4c}{a^2 j(\log j)^2}$$

as $j \to \infty$, and the lemma follows.

LEMMA 2.2. *If* $X$ *is given by* (2.2), *then* $\mathscr{E}[||X||^2] = \infty$.

PROOF. By Corollary 3.5 [2] it suffices to check that

$$\sum_{j=1}^{\infty} \int_{\Lambda_j} \xi_j^2 ||\varphi_j||^2 \, dP = \infty, \qquad \forall a > 0, \tag{2.4}$$

where $\Lambda_j = [|\xi_j| \, ||\varphi_j|| > a]$. Now

$$\int_{\Lambda_j} \xi_j^2 ||\varphi_j||^2 \, dP = j^{-1} \int_{\Lambda_j} \xi_j^2 \, dP$$
$$= a^2 P[|\xi_1| > a j^{\frac{1}{2}}] + 2j^{-1} \int_{a j^{\frac{1}{2}}}^{\infty} x P[|\xi_1| > x] \, dx$$

by integration by parts. Therefore $\exists \, c_1 > 0$ such that for all $j$ sufficiently large the $j$th term in the sum in (2.4) dominates $c_1/j \log j$. This proves the lemma.

LEMMA 2.3. $X$, *given by* (2.2), *satisfies the* CLT.

PROOF. First observe that the Theorem 4.1 [2] we have

$$\mathscr{E}[||X||^{2-\varepsilon}] < \infty, \qquad \text{for } 0 < \varepsilon \leqq 2. \tag{2.5}$$

We will need this only for $\varepsilon = 1$. Let $\{\xi_j^{(k)}; k \geqq 1, j \geqq 1\}$ be independent real-valued random variables each having the same distribution as $\xi_1$. Define

$$(2.6) \qquad Z_n = \sum_{j=1}^{\infty} n^{-\frac{1}{2}}(\xi_j^{(1)} + \cdots + \xi_j^{(n)})\varphi_j .$$

We will show that $\{Z_n\}$ is a *tight* sequence. For $\delta > 0$ and $x \in C[0, 1]$, define

$$(2.7) \qquad ||x||_\delta = \sup_{|s-t|\leqq\delta} |x(s) - x(t)| .$$

Note that $||\cdot||_\delta$ is a pseudo-norm on $C[0, 1]$.

By a lemma of Hoffmann-Jørgensen (Lemma 3.4, [2]) we have

$$(2.8) \qquad \begin{aligned} &P[|\xi_j^{(1)} + \cdots + \xi_j^{(n)}| > \lambda n^{\frac{1}{2}}] \\ &\leqq P[\max_{1\leqq k\leqq n} |\xi_j^{(k)}| > \lambda n^{\frac{1}{2}}/3] + 4P[|\xi_j^{(1)} + \cdots + \xi_j^{(n)}| > \lambda n^{\frac{1}{2}}/3]^2 \\ &\leqq nP[|\xi_1| > \lambda n^{\frac{1}{2}}/3] + 4 \cdot 81 \cdot \alpha^2 \cdot \lambda^{-4} , \end{aligned}$$

where Chebychev's inequality is used for the second estimate. Using (2.1), we see that there exist $c_2 > 0$ and $\lambda_0 < \infty$ such that we have

$$(2.9) \qquad P[|\xi_j^{(1)} + \cdots + \xi_j^{(n)}| > \lambda n^{\frac{1}{2}}] \leqq \frac{c_2}{\lambda^2(\log \lambda)^2} , \qquad \forall\, n \geqq 1, \forall\, \lambda \geqq \lambda_0 .$$

We will now apply the comparison theorem in [2] (Theorem 5.3 (5.9)). Let $n$ and $\delta$ be fixed. For the linear space in that theorem we take $C[0, 1]$ with $||\cdot||_\delta$ as pseudo-norm (the results in [2] hold for pseudo-norms as well, without any modification in arguments), and we indicate parenthetically what replaces the corresponding quantities in our present context, $\varphi(x)\,(= x)$, $\eta_j\,(= \xi_j)$, $\xi_j\,(= (\xi_j^{(1)} + \cdots + \xi_j^{(n)})n^{-\frac{1}{2}})$, $a\,(= 2)$, $b\,(= 1)$, $\alpha\,(= c_2^{-1})$, $x_0\,(= \lambda_0)$, $u_k\,(= \varphi_k)$. We thus conclude that for $n \geqq 1$

$$(2.10) \qquad \mathscr{E}[||Z_n||_\delta] \leqq (\tfrac{1}{2}\lambda_0 c_2 + c_2)\mathscr{E}[||X_1||_\delta] = c_3\mathscr{E}[||X_1||_\delta] ,$$

say. The important thing is that $c_3$ does not depend on $n$ or $\delta$. Since $||X_1||_\delta \leqq 2||X_1||$, and a.s. $||X_1||_\delta \to 0$ as $\delta \to 0$, using (2.5) by the dominated convergence theorem we have $\mathscr{E}[||X_1||_\delta] \to 0$ as $\delta \to 0$. Therefore by (2.10) we conclude that $\mathscr{E}[||Z_n||_\delta] \to 0$ as $\delta \to 0$ *uniformly* in $n$. This shows that $\{Z_n\}$ is a *tight* sequence since the finite dimensional distributions of $\{Z_n\}$ converge by the finite dimensional CLT. Hence CLT holds for $X$.

Finally we show that $X$ does not satisfy the LIL. The following two lemmas suffice for this.

LEMMA 2.4. *Let $X$ be a* $C[0, 1]$*-valued random variable with mean* 0 *and a continuous covariance* $\rho(s, t) = \mathscr{E}[X(s)X(t)]$. *If $X$ satisfies the* LIL *and $K$ is the set occurring in* (1.1) *and* (1.2), *then* $x \in K \Rightarrow ||x|| \leqq ||\mathscr{E}[X(t)^2]^{\frac{1}{2}}||$.

PROOF. By the Hartman-Wintner LIL there exists a set $\Omega_1$ of probability 1 such that

$$\Omega_1 = \Omega_0 \cap \left\{\omega : \limsup_n \frac{S_n(t, \omega)}{a_n} = a(t) , \right.$$
$$\left. \liminf_n \frac{S_n(t, \omega)}{a_n} = -a(t), \quad \forall\, t \text{ rational} \in [0, 1]\right\} ,$$

where $\Omega_0$ is the intersection of the two sets in (1.1) and (1.2), $a_n = (2n \log\log n)^{\frac{1}{2}}$, $a(t) = \mathscr{E}[X(t)^2]^{\frac{1}{2}}$, and $S_n = X_1 + \cdots + X_n$; $X_1, X_2, \cdots$ being independent copies of $X$. If $x \in K$ and $\omega \in \Omega_1$, then $\exists\, n' \nearrow \infty$ such that

$$\frac{S_{n'}(\omega)}{a_{n'}} \to x \qquad \text{as} \quad n' \to \infty\,.$$

Therefore $\forall\, t$ rational $\in [0, 1]$, $|x(t)| \leqq a(t)$, and the lemma follows.

LEMMA 2.5. *$X$ defined by* (2.2) *does not satisfy the* LIL.

PROOF. It is clear that $X$ defined by (2.2) is symmetric and has a continuous covariance. Therefore by Lemma 2.4 it suffices to show that $\forall\, A > 0$

$$P[\|X_j\| > A\psi(j) \ \text{i.o.}] = 1\,, \tag{2.11}$$

where $\psi(j) = (j \log\log j)^{\frac{1}{2}}$, and $X_j$ are independent copies of $X$. Now

$$\begin{aligned} P[\|\textstyle\sum_{k=1}^{j} \xi_k \varphi_k\| > A\psi(j)] \\ &= P[\|2 \textstyle\sum_{k=1}^{j} \xi_k\varphi_k + \sum_{k=j+1}^{\infty} \xi_k\varphi_k - \sum_{k=j+1}^{\infty} \xi_k\varphi_k\| > 2A\psi(j)] \\ &\leqq P[\|X\| > A\psi(j)] + P[\|\textstyle\sum_{k=1}^{j} \xi_k\varphi_k - \sum_{k=j+1}^{\infty} \xi_k\varphi_k\| > A\psi(j)]\,, \end{aligned}$$

and using the symmetry and independence of the $\xi_j$'s the last two quantities are equal. Hence

$$P[\|X\| > A\psi(j)] \geqq \tfrac{1}{2} P[\|\textstyle\sum_{k=1}^{j} \xi_k\varphi_k\| > A\psi(j)]\,. \tag{2.12}$$

Since the $\varphi_k$'s have disjoint supports we have $\|\sum_{k=1}^{j} \xi_k\varphi_k\| = \max_{1\leqq k\leqq j} \|\xi_k\varphi_k\|$, hence

$$\begin{aligned} P[\|\textstyle\sum_{k=1}^{j} \xi_k\varphi_k\| > A\psi(j)] &= P[\max_{1\leqq k\leqq j} \|\xi_k\varphi_k\| > A\psi(j)] \\ &= \{1 - \textstyle\prod_{k=1}^{j} [1 - P(\|\xi_k\varphi_k\| > A\psi(j))]\} \\ &= \{1 - \textstyle\prod_{k=1}^{j} [1 - P(|\xi_1| > Ak^{\frac{1}{2}}\psi(j))]\}\,. \end{aligned} \tag{2.13}$$

Now using (2.1) and (2.12), for all $j$ sufficiently large,

$$\begin{aligned} P[\|X\| > A\psi(j)] &\geqq \frac{1}{2}\left\{1 - \textstyle\prod_{k=1}^{j}\left(1 - \dfrac{c}{A^2k\psi(j)^2(\log j)^2}\right)\right\} \\ &\geqq \tfrac{1}{2}\{1 - \exp[-c \textstyle\sum_{k=1}^{j} (A^2k\psi(j)^2(\log j)^2)^{-1}]\} \\ &\geqq \tfrac{1}{2}\{1 - \exp[-c_4(A^2 \log j\psi(j)^2)^{-1}]\}\,, \end{aligned} \tag{2.14}$$

for some $c_4 > 0$ and all $j$ sufficiently large. Since the last quantity behaves like $c_5(j \log j \log\log j)^{-1}$, it follows that

$$\textstyle\sum_{j=1}^{\infty} P[\|X_j\| > A\psi(j)] = \infty\,, \tag{2.15}$$

and (2.11) follows by the Borel–Cantelli lemma.

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